$\mathbf{R}$; (*b*) the set of all rational numbers on $\mathbf{R}$; (*c*) the disks $\{z \mid |z|<1\} \subset \mathbf{C}$ and $\{z \mid |z| \leqq 1\} \subset \mathbf{C}$.

12. **(Space $B[a, b]$)** Show that $B[a, b]$, $a<b$, is not separable. (Cf. 1.2-2.)

13. Show that a metric space X is separable if and only if X has a countable subset Y with the following property. For every $\varepsilon>0$ and every $x \in X$ there is a $y \in Y$ such that $d(x, y)<\varepsilon$.

14. **(Continuous mapping)** Show that a mapping $T: X \longrightarrow Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

15. Show that the image of an open set under a continuous mapping need not be open.

1.4 Convergence, Cauchy Sequence, Completeness

We know that sequences of real numbers play an important role in calculus, and it is the metric on $\mathbf{R}$ which enables us to define the basic concept of convergence of such a sequence. The same holds for sequences of complex numbers; in this case we have to use the metric on the complex plane. In an arbitrary metric space $X=(X, d)$ the situation is quite similar, that is, we may consider a sequence (x_n) of elements $x_1, x_2, \cdots$ of X and use the metric d to define convergence in a fashion analogous to that in calculus:

1.4-1 Definition (Convergence of a sequence, limit). A sequence (x_n) in a metric space $X=(X, d)$ is said to *converge* or to *be convergent* if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x)=0.$$

x is called the *limit* of (x_n) and we write

$$\lim_{n \to \infty} x_n = x$$

or, simply,

$$x_n \longrightarrow x.$$

We say that (x_n) *converges to* x or *has the limit* x. If (x_n) is not convergent, it is said to be *divergent.* ∎

How is the metric d being used in this definition? We see that d yields the sequence of real numbers $a_n = d(x_n, x)$ whose convergence defines that of (x_n). Hence if $x_n \longrightarrow x$, an $\varepsilon > 0$ being given, there is an $N = N(\varepsilon)$ such that all x_n with $n > N$ lie in the ε-neighborhood $B(x; \varepsilon)$ of x.

To avoid trivial misunderstandings, we note that the limit of a convergent sequence must be a point of the space X in 1.4-1. For instance, let X be the open interval $(0, 1)$ on $\mathbf{R}$ with the usual metric defined by $d(x, y) = |x - y|$. Then the sequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ is not convergent since 0, the point to which the sequence "wants to converge," is not in X. We shall return to this and similar situations later in the present section.

Let us first show that two familiar properties of a convergent sequence (uniqueness of the limit and boundedness) carry over from calculus to our present much more general setting.

We call a nonempty subset $M \subset X$ a *bounded set* if its *diameter*

$$\delta(M) = \sup_{x,y \in M} d(x, y)$$

is finite. And we call a sequence (x_n) in X a **bounded sequence** if the corresponding point set is a bounded subset of X.

Obviously, if M is bounded, then $M \subset B(x_0; r)$, where $x_0 \in X$ is any point and r is a (sufficiently large) real number, and conversely.

Our assertion is now as follows.

1.4-2 Lemma (Boundedness, limit). *Let $X = (X, d)$ be a metric space. Then:*

(a) *A convergent sequence in X is bounded and its limit is unique.*

(b) *If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then $d(x_n, y_n) \longrightarrow d(x, y)$.*

Proof. **(a)** Suppose that $x_n \longrightarrow x$. Then, taking $\varepsilon = 1$, we can find an N such that $d(x_n, x) < 1$ for all $n > N$. Hence by the triangle inequality (M4), Sec. 1.1, for all n we have $d(x_n, x) < 1 + a$ where

$$a = \max\{d(x_1, x), \cdots, d(x_N, x)\}.$$

This shows that (x_n) is bounded. Assuming that $x_n \longrightarrow x$ and $x_n \longrightarrow z$, we obtain from (M4)

$$0 \leqq d(x, z) \leqq d(x, x_n) + d(x_n, z) \longrightarrow 0+0$$

and the uniqueness $x = z$ of the limit follows from (M2).

(b) By (1), Sec. 1.1, we have

$$d(x_n, y_n) \leqq d(x_n, x) + d(x, y) + d(y, y_n).$$

Hence we obtain

$$d(x_n, y_n) - d(x, y) \leqq d(x_n, x) + d(y_n, y)$$

and a similar inequality by interchanging x_n and x as well as y_n and y and multiplying by -1. Together,

$$|d(x_n, y_n) - d(x, y)| \leqq d(x_n, x) + d(y_n, y) \longrightarrow 0$$

as $n \longrightarrow \infty$. ∎

We shall now define the concept of completeness of a metric space, which will be basic in our further work. We shall see that completeness does *not* follow from (M1) to (M4) in Sec. 1.1, since there are *incomplete* (not complete) metric spaces. In other words, completeness is an additional property which a metric space may or may not have. It has various consequences which make complete metric spaces "much nicer and simpler" than incomplete ones—what this means will become clearer and clearer as we proceed.

Let us first remember from calculus that a sequence (x_n) of real or complex numbers converges on the real line **R** or in the complex plane **C**, respectively, if and only if it satisfies the *Cauchy convergence criterion,* that is, if and only if for every given $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$|x_m - x_n| < \varepsilon \qquad \text{for all } m, n > N.$$

(A proof is included in A1.7; cf. Appendix 1.) Here $|x_m - x_n|$ is the distance $d(x_m, x_n)$ from x_m to x_n on the real line **R** or in the complex plane **C**. Hence we can write the inequality of the Cauchy criterion in the form

$$d(x_m, x_n) < \varepsilon \qquad (m, n > N).$$

And if a sequence (x_n) satisfies the condition of the Cauchy criterion, we may call it a *Cauchy sequence.* Then the Cauchy criterion simply says that a sequence of real or complex numbers converges on **R** or in **C** if and only if it is a Cauchy sequence. This refers to the situation in **R** or **C**. Unfortunately, in more general spaces the situation may be more complicated, and there may be Cauchy sequences which do not converge. Such a space is then lacking a property which is so important that it deserves a name, namely, completeness. This consideration motivates the following definition, which was first given by M. Fréchet (1906).

1.4-3 Definition (Cauchy sequence, completeness). A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy* (or *fundamental*) if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$(1) \qquad d(x_m, x_n) < \varepsilon \qquad \text{for every } m, n > N.$$

The space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X). ∎

Expressed in terms of completeness, the Cauchy convergence criterion implies the following.

1.4-4 Theorem (Real line, complex plane). *The real line and the complex plane are complete metric spaces.*

More generally, we now see directly from the definition that complete metric spaces are precisely those in which the Cauchy condition (1) continues to be necessary and sufficient for convergence.

Complete and incomplete metric spaces that are important in applications will be considered in the next section in a systematic fashion.

For the time being let us mention a few simple incomplete spaces which we can readily obtain. Omission of a point a from the real line yields the incomplete space $\mathbf{R} - \{a\}$. More drastically, by the omission of all irrational numbers we have the *rational line* **Q**, which is incomplete. An open interval (a, b) with the metric induced from **R** is another incomplete metric space, and so on.

It is clear from the definition that in an arbitrary metric space, condition (1) may no longer be sufficient for convergence since the space may be incomplete. A good understanding of the whole situation is important; so let us consider a simple example. We take $X=(0, 1]$, with the usual metric defined by $d(x, y)=|x-y|$, and the sequence (x_n), where $x_n=1/n$ and $n=1, 2, \cdots$. This is a Cauchy sequence, but it does not converge, because the point 0 (to which it "wants to converge") is not a point of X. This also illustrates that the concept of convergence is not an intrinsic property of the sequence itself but also depends on the space in which the sequence lies. In other words, a convergent sequence is not convergent "on its own" but it must converge to some point in the space.

Although condition (1) is no longer sufficient for convergence, it is worth noting that it continues to be necessary for convergence. In fact, we readily obtain the following result.

1.4-5 Theorem (Convergent sequence). *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof. If $x_n \longrightarrow x$, then for every $\varepsilon>0$ there is an $N=N(\varepsilon)$ such that

$$d(x_n, x)<\frac{\varepsilon}{2} \qquad \text{for all } n>N.$$

Hence by the triangle inequality we obtain for $m, n>N$

$$d(x_m, x_n) \leqq d(x_m, x)+d(x, x_n)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that (x_n) is Cauchy. ∎

We shall see that quite a number of basic results, for instance in the theory of linear operators, will depend on the completeness of the corresponding spaces. Completeness of the real line **R** is also the main reason why in calculus we use **R** rather than the *rational line* **Q** (the set of all rational numbers with the metric induced from **R**).

Let us continue and finish this section with three theorems that are related to convergence and completeness and will be needed later.

1.4-6 Theorem (Closure, closed set). *Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure as defined in the previous section. Then:*

(a) *$x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \longrightarrow x$.*

(b) *M is closed if and only if the situation $x_n \in M$, $x_n \longrightarrow x$ implies that $x \in M$.*

Proof. **(a)** Let $x \in \bar{M}$. If $x \in M$, a sequence of that type is $(x, x, \cdots)$. If $x \notin M$, it is a point of accumulation of M. Hence for each $n = 1, 2, \cdots$ the ball $B(x; 1/n)$ contains an $x_n \in M$, and $x_n \longrightarrow x$ because $1/n \longrightarrow 0$ as $n \longrightarrow \infty$.

Conversely, if (x_n) is in M and $x_n \longrightarrow x$, then $x \in M$ or every neighborhood of x contains points $x_n \neq x$, so that x is a point of accumulation of M. Hence $x \in \bar{M}$, by the definition of the closure.

(b) M is closed if and only if $M = \bar{M}$, so that (b) follows readily from (a). ∎

1.4-7 Theorem (Complete subspace). *A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.*

Proof. Let M be complete. By 1.4-6(a), for every $x \in \bar{M}$ there is a sequence (x_n) in M which converges to x. Since (x_n) is Cauchy by 1.4-5 and M is complete, (x_n) converges in M, the limit being unique by 1.4-2. Hence $x \in M$. This proves that M is closed because $x \in \bar{M}$ was arbitrary.

Conversely, let M be closed and (x_n) Cauchy in M. Then $x_n \longrightarrow x \in X$, which implies $x \in \bar{M}$ by 1.4-6(a), and $x \in M$ since $M = \bar{M}$ by assumption. Hence the arbitrary Cauchy sequence (x_n) converges in M, which proves completeness of M. ∎

This theorem is very useful, and we shall need it quite often. Example 1.5-3 in the next section includes the first application, which is typical.

The last of our present three theorems shows the importance of convergence of sequences in connection with the continuity of a mapping.

1.4-8 Theorem (Continuous mapping). *A mapping $T: X \longrightarrow Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point*

$x_0 \in X$ *if and only if*

$$x_n \longrightarrow x_0 \qquad \text{implies} \qquad Tx_n \longrightarrow Tx_0.$$

Proof. Assume T to be continuous at x_0; cf. Def. 1.3-3. Then for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(x, x_0) < \delta \qquad \text{implies} \qquad \tilde{d}(Tx, Tx_0) < \varepsilon.$$

Let $x_n \longrightarrow x_0$. Then there is an N such that for all $n > N$ we have

$$d(x_n, x_0) < \delta.$$

Hence for all $n > N$,

$$\tilde{d}(Tx_n, Tx_0) < \varepsilon.$$

By definition this means that $Tx_n \longrightarrow Tx_0$.

Conversely, we assume that

$$x_n \longrightarrow x_0 \qquad \text{implies} \qquad Tx_n \longrightarrow Tx_0$$

and prove that then T is continuous at x_0. Suppose this is false. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \neq x_0$ satisfying

$$d(x, x_0) < \delta \qquad \text{but} \qquad \tilde{d}(Tx, Tx_0) \geqq \varepsilon.$$

In particular, for $\delta = 1/n$ there is an x_n satisfying

$$d(x_n, x_0) < \frac{1}{n} \qquad \text{but} \qquad \tilde{d}(Tx_n, Tx_0) \geqq \varepsilon.$$

Clearly $x_n \longrightarrow x_0$ but (Tx_n) does not converge to Tx_0. This contradicts $Tx_n \longrightarrow Tx_0$ and proves the theorem. ∎

Problems

1. **(Subsequence)** If a sequence (x_n) in a metric space X is convergent and has limit x, show that every subsequence (x_{n_k}) of (x_n) is convergent and has the same limit x.