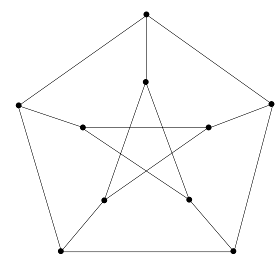
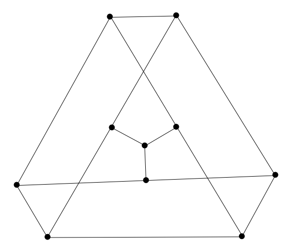
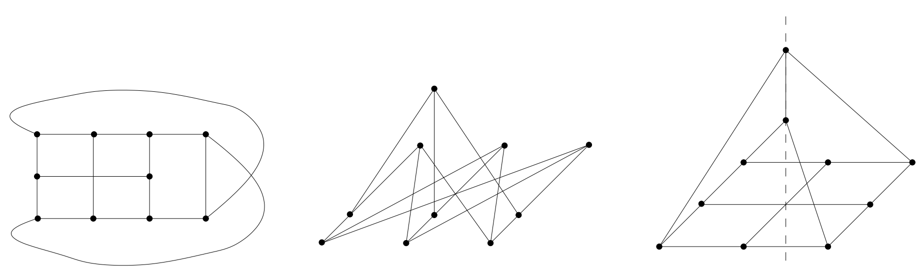
III. Strongly regular graphs

- definition of strongly regular graphs
- characterization with adjacency matrix
- classification (type I in II)
- Payley graphs
- Krein condition and Smith graphs
- more examples (Steiner and LS graphs)
- feasibility conditions and a table







Definition

Two similar regularity conditions are:

(a) any two adjacent vertices have exactly λ common neighbours,

(b) any two nonadjacent vertices have exactly μ common neighbours.

A regular graph is called **strongly regular** when it satisfies (a) and (b). Notation $\mathbf{SRG}(n,k,\lambda,\mu)$, where k is the valency of Γ and $n = |V\Gamma|$.

Strongly regular graphs can also be treated as extremal graphs and have been studied extensively.

Examples

5-cycle is SRG$(5, 2, 0, 1)$,
the Petersen graph is SRG$(10, 3, 0, 1)$.

What are the trivial examples?

K_n, $\quad m \cdot K_n$,

The **Cocktail Party graph $C(n)$**, i.e., the graph on $2n$ vertices of degree $2n-2$, is also strongly regular.

Lemma. *A strongly regular graph* Γ *is disconnected iff* $\mu = 0$.
If $\mu = 0$, *then each component of* Γ *is isomorphic to* K_{k+1} *and we have* $\lambda = k - 1$.

Corollary. *A complete multipartite graph is strongly regular iff its complement is a union of complete graphs of equal size.*

Homework: Determine all SRG with $\mu = k$.

Counting the edges between the neighbours and non-neighbours of a vertex in a connected strongly regular graph we obtain:

$$\mu(n-1-k) = k(k-\lambda-1),$$

i.e.,

$$n = 1 + k + \frac{k(k-\lambda-1)}{\mu}.$$

Lemma. *The complement of* $\mathrm{SRG}(n, k, \lambda, \mu)$ *is again strongly regular graph:*

$$\mathrm{SRG}(\overline{n}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (n, n-k-1, n-2k+\mu-2, n-2k+\lambda).$$

Let J be the all-one matrix of dim. $(n \times n)$.

A graph Γ on n vertices is strongly regular if and only if its adjacency matrix A satisfies

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

for some integers k, λ and μ.

Therefore, the valency k is an eigenvalue with multiplicity 1 and the nontrivial eigenvalues, denoted by σ and τ, are the roots of

$$x^2 - (\lambda - \mu)x + (\mu - k) = 0,$$

and hence $\lambda - \mu = \sigma + \tau$, $\mu - k = \sigma\tau$.

Theorem. *A connected regular graph with precisely three eigenvalues is strongly regular.*

Proof. Consider the following matrix polynomial:

$$M := \frac{(A-\sigma)(A-\tau)}{(k-\sigma)(k-\tau)}$$

If $A = A(\Gamma)$, where Γ is a connected k-regular graph with eigenvalues k, σ and τ, then all the eigenvalues of M are 0 or 1. But all the eigenvectors corresponding to σ and τ lie in Ker(A), so rank$M$=1 and $M\boldsymbol{j} = \boldsymbol{j}$,

hence $M = \dfrac{1}{n}J$. and $A^2 \in \mathrm{span}\{I, J, A\}$. ■

For a connected graph, i.e., $\mu \neq 0$, we have

$$n = \frac{(k-\sigma)(k-\tau)}{k+\sigma\tau}, \quad \lambda = k+\sigma+\tau+\sigma\tau, \quad \mu = k+\sigma\tau$$

and the multiplicities of σ and τ are

$$m_\sigma = \frac{(n-1)\tau + k}{\tau - \sigma} = \frac{(\tau+1)k(k-\tau)}{\mu(\tau-\sigma)}$$

and $m_\tau = n - 1 - m_\sigma$.

Multiplicities

Solve the system:

$$\begin{aligned} 1 + m_\sigma + m_\tau &= n \\ 1 \cdot k + m_\sigma \cdot \sigma + m_\tau \cdot \tau &= 0. \end{aligned}$$

to obtain

$$m_\sigma \text{ and } m_\tau = \frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}}\right).$$