Biplanes

A **biplane** or **biplane geometry** is a symmetric 2-design with $\lambda = 2$; that is, every set of two points is contained in two blocks ("lines"), while any two lines intersect in two points. They are similar to finite projective planes, except that rather than two points determining one line (and two lines determining one point), two points determine two lines (respectively, points). A biplane of order *n* is one whose blocks have $k = n + 2$ points; it has $v = 1 + (n + 2)(n + 1)/2$ points (since $r = k$).

The 18 known examples are listed below.

- (Trivial) The order 0 biplane has 2 points (and lines of size 2; a 2-(2,2,2) design); it is two points, with two blocks, each consisting of both points. Geometrically, it is the digon.
- The order 1 biplane has 4 points (and lines of size 3; a 2-(4,3,2) design); it is the complete design with $v = 4$ and $k = 3$. Geometrically, the points are the vertices of a tetrahedron and the blocks are its faces.
- The order 2 biplane is the complement of the Fano plane: it has 7 points (and lines of size 4; a 2-(7,4,2)), where the lines are given as the *complements* of the (3-point) lines in the Fano plane.
- The order 3 biplane has 11 points (and lines of size 5; a 2-(11,5,2)), and is also known as the **Paley biplane** after Raymond Paley; it is associated to the Paley digraph of order 11, which is constructed using the field with 11 elements, and is the Hadamard 2-design associated to the size 12 Hadamard matrix; see Paley construction I.

  Algebraically this corresponds to the exceptional embedding of the projective special linear group *PSL*(2,5) in *PSL*(2,11) – see projective linear group: action on *p* points for details.

- There are three biplanes of order 4 (and 16 points, lines of size 6; a 2-(16,6,2)). One is the Kummer configuration. These three designs are also Menon designs.
- There are four biplanes of order 7 (and 37 points, lines of size 9; a 2-(37,9,2)).
- There are five biplanes of order 9 (and 56 points, lines of size 11; a 2-(56,11,2)).
- Two biplanes are known of order 11 (and 79 points, lines of size 13; a 2-(79,13,2)).

Biplanes of orders 5, 6, 8 and 10 do not exist, as shown by the Bruck-Ryser-Chowla theorem.