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Some Properties of Hadamard Matrices

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Hadamard matrix - $H = (h_{ij})\ i, j = 1,2, \dots, n; 1 < n < \infty$

$$h_{ij} = \pm 1$$

$$\langle h_i \perp h_k \rangle = 0, i \neq k$$

$$h_i \equiv (h_{i1}, h_{i2}, \cdots, h_{in}) \in \mathbb{R}^n$$

$\boldsymbol{\mathcal{H}_n}$ - the class of all Hadamard matrices of order $n = 4k$.

Example: 8 × 8 Hadamard matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Practical use:

- **Error-correcting codes -** in early satellite transmissions.
  For example:
  1971 – 72, **Mariner 9's mission to Mars**, 54 billion bits of data had been transmitted;
  Flybys of the outer planets in the solar system.

- **Modern CDMA cellphones -** minimize interference with other transmissions to the base station.

- **New applications are everywhere about us such as in:**

  pattern recognition, neuroscience, optical communication and information hiding.

- **Compressive Sensing (Signal Reconstruction)**.
  D. J. Lum et al. Fast Hadamard transforms for compressive sensing. 2015

- **In Chemical Physics - Construction of the orthogonal set of molecular orbitals.**
  K. Balasubramanian. Molecular orbitals and Hadamard matrices 1993.

In 2002 V. Kvaratskhelia (in “Some inequalities related to Hadamard matrices”. *Functional Analysis and Its Applications*) considered the following characteristic:

$$\mathbb{R}^n \ is\ equipped\ with\ l_p - norm, 1 \leq p \leq \infty$$

$$\|x\|_p = \sqrt[p]{|x_1|^p + \cdots |x_n|^p}$$

$$\|x\|_\infty = max\{|x_1|, \ldots, |x_n|\}$$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

$$\varrho_{\,p,H} \equiv max\{\|h_1\|_p, \|h_1 + h_2\|_p, \cdots, \|h_1 + h_2 + \cdots + h_n\|_p\},$$

$$\boldsymbol{\alpha}_{\,\boldsymbol{p},\boldsymbol{\mathcal{H}_n}} \equiv \max_{\boldsymbol{H \in \mathcal{H}_n}} \boldsymbol{\varrho}_{\boldsymbol{p},\boldsymbol{H}}$$

$$\frac{1}{\sqrt{2}} \cdot n^{\left(\frac{1}{p}+\frac{1}{2}\right)} \leq \boldsymbol{\alpha}_{\boldsymbol{p},\boldsymbol{\mathcal{H}_n}} \leq n^{\left(\frac{1}{p}+\frac{1}{2}\right)}, 1 \leq p \leq 2, \qquad (1)$$

$$\boldsymbol{\alpha}_{\boldsymbol{p},\boldsymbol{\mathcal{H}_n}} = n, 2 \leq p \leq \infty\,. \qquad (2)$$

Naturally arises the question to estimate the minimum

$$\boldsymbol{\omega}_{\boldsymbol{p},\boldsymbol{\mathcal{H}_n}} \equiv \min_{\boldsymbol{H \in \mathcal{H}_n}} \boldsymbol{\varrho}_{\,\boldsymbol{p},\boldsymbol{H}}$$

Submitted paper (2018):

G. Giorgobiani, V. Kvaratskhelia. Maximum inequalities and their applications to Orthogonal and Hadamard matrices.

The following estimations are valid:

a) when $1 \leq \mathrm{p} < \infty$

$$\omega_{p,\mathcal{H}_n} \leq n^{\left(\frac{1}{p}+\frac{1}{2}\right)} \sqrt{7 \ln n} \ ;$$

b) when $\mathrm{p} = 2$

$$\omega_{2,\mathcal{H}_n} \leq \ n;$$

c) when $p = \infty$, for some absolute constant K

$$\omega_{\infty,\mathcal{H}_n} \leq K \sqrt{n} \, .$$

Case $2 < p < \infty$: the bound $n^{\left(\frac{1}{p}+\frac{1}{2}\right)} \sqrt{7 \ln n}$ is asymptoticly smaller then n of (2) and this is achieved for extremely large n-s ($if\ p = 25, n \geq 33;\ if\ p = 2.5, n > 2 \times 10^{11}$).

Case $p = \infty$: $\boldsymbol{\omega}_{\infty,\mathcal{H}_n} \ll \boldsymbol{\alpha}_{\infty,\mathcal{H}_n}$.

Algorithms

Sign-Algorithms – Spencer; Lovett & Meka: Partial Coloring Lemma (Herding algorithms of the Machin Learning)

Permutation-Algorithm – S. Chobanyan.

Generalization.
Complex Hadamard matrices

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \cdots & \cdots & \cdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix}$$

$$h_{ij} \in \mathbb{C}$$

$$|h_{ij}| = 1$$

$$HH^* = nI$$

$$H^* - \ conjugate\ transpose, I - \ identity$$

They are **Unitary** matrices after rescaling.

Example: rescaled **Fourier Matrix**, $n \geq 1$

$$H = \sqrt{n}[F_n]_{k,j}$$

$$[F_n]_{k,j} = \frac{1}{\sqrt{n}} e^{2\pi \boldsymbol{i}(k-1)(j-1)/n}, k, j = 1, \dots, n,$$

Unitary (complex) matrices are important in **Particle Physics**:

- **CKM (Cabibbo-Kobayashi-Maskawa) matrix,**
  appears in the coupling of quarks to $W^{\pm}$ bosons;

- **Reconstruction Problem of a unitary matrix** see e.g**.**

Auberson, G., Martin A., Mennessier G. "On the reconstruction of a unitary matrix from its moduli".

The CERN Theory Department:1990 - Report # CERN-TH-5809-90.

Applications of Complex Hadamard matrices (in 90-ies)

- **various branches of mathematics,**
- **quantum optics,**
- **high-energy physics**,
- **quantum teleportation**.

We plan to transfer our results for Real Hadamard matrices to the Complex case.

Thank you for your attention