Projective and affine planes

1 Projective planes

Consider incidence structures (whose elements are called points and lines) having the following properties:

(P1) Any two points are incident with a unique line.

(P2) Any two lines are incident with a unique point.

This class of structures contains some degenerate ones (containing a line incident with no, one or all points, or dually) which we do not want to consider. Slightly less degenerate is the following type: one line l is incident with all the points except one; each remaining line is incident with the point not on l and one other. (This includes a triangle, where any line can play the role of l). An axiom which excludes all of these is

(P3) There exist four points, no three incident with a common line.

A structure satisfying (P1), (P2) and (P3) is called a *projective plane*.

It is not hard to show that, for any finite projective plane, there is an integer $q > 1$ with the properties

(a) any line is incident with $q+1$ points;

(b) any point is incident with $q+1$ lines;

(c) there are q^2+q+1 points, and the same number of lines.

The integer q is called the *order* of the projective plane.

A projective plane of order q is a square 2-$(q^2+q+1, q+1, 1)$ design (a symmetric BIBD).

The smallest projective plane has order 2 (see Figure 1). It is known as the *Fano plane*. It is also, of course, the unique Steiner triple system of order 7.

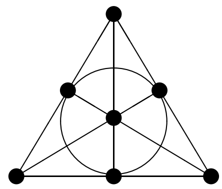

Figure 1: The Fano plane

2 Affine planes

Let l be a line of a projective plane Π of order q, and let Π^l be the incidence structure obtained by deleting l and all of its points from Π. Since each other line meetsΠ in a unique point, what remains is a 2-through a given point p of l are mutually disjoint in Π^l; there are q of them, so they cover all the points without overlapping.

Conversely, let A be a 2-$(q^2, q, 1)$ design. Choose a line l' and a point p' not on l'. Of the $q+1$ lines through p', q of them meet l', so just one is disjoint from l'. Thus *Euclid's parallel postulate* holds: the $q(q+1)$ lines fall into $q+1$ parallel classes, each containing q lines.

Now enlarge A by adjoining a set of $q+1$ points, in one-to-one correspondence with the parallel classes, and one new line l_∞. Each new point is incident with all the lines of the corresponding parallel class, and l_∞ is incident with all the new points. It is readily seen that the resulting structure is a projective plane of order q.

A 2-$(q^2, q, 1)$ design is called an *affine plane* of order q. We've seen that any projective plane with a distinguished line gives rise to an affine plane, and conversely. The construction of the projective plane is describes as *adding a line at infinity* to the affine plane.

In particular, there exists a projective plane of order q if and only if there exists an affine plane of order q.

An affine plane is a special kind of net. A *net* of order q and degree r is an incidence structure with q^2 points, q points on each line, r lines through each point, with two points on at most one line, satisfying Euclid's parallel postulate. Thus an affine plane of order q is a net of order q and degree $q+1$.

A net of order q and degree r exists if and only if $r-2$ mutually orthogonal Latin squares of order q exists. Thus, the existence of a projective (or affine) plane of order q is equivalent to that of a *complete set* of $q-1$ MOLS of order q.

3 Existence of projective planes

The definition of a projective plane is purely combinatorial, but all known constructions of finite projective planes are based on algebra.

The simplest, and most important, constructs a projective plane of prime power order q from the finite field $\mathrm{GF}(q)$.

Let V be a 3-dimensional vector space over $\mathrm{GF}(q)$. Let P be the set of 1-dimensional subspaces of V, and L the set of 2-dimensional subspaces; define an incidence relation between P and L by the rule that p is incident with l if $p \subseteq l$.

Since two 1-dimensional subspaces span a 2-dimensional subspace, and two 2-dimensional subspaces intersect in a 1-dimensional subspace, conditions (P1) and (P2) hold; moreover the four points (1-dimensional subspaces) spanned by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ satisfy (P3). So the structure is a projective plane, and its order is easily seen to be q. This plane is denoted by $\mathrm{PG}(2,q)$, and is called *desarguesian*, for reasons we will see shortly. (This is a special case of the construction of projective spaces of arbitrary dimension.)

Although, in general, different affine planes can be obtained from the same projective plane by deleting different lines, all the affine planes derived from the desarguesian projective plane are isomorphic. This plane is known as the *desarguesian affine plane* of order$q$, denoted $\mathrm{AG}(2,q)$. The net corresponding to the family of $q-1$ MOLS of order q constructed by the “finite field method” is the affine plane $\mathrm{AG}(2,q)$.

For prime orders, no other projective planes are known. For any composite prime power except 4 and 8, other projective planes are known. It has been shown that for each of the orders 2, 3, 4, 5, 7 and 8, it is known that only the desarguesian planes exist. It is also known that up to isomorphism there are just four projective planes of order 9.

For non-prime-powers, as we noted, no projective planes are known. However, only two non-existence results have been found, one general (the Bruck–Ryser theorem) and one specific (the result of a heroic computation by Lam *et al.*):

- If $n \equiv 1$ or 2 (mod 4), and n is not the sum of two squares of integers, then there is no projective plane of order n.

- There is no projective plane of order 10.

The Bruck–Ryser theorem shows that there are no projective planes of orders 6, 14, 21, 22, ...

4 Characterisations

The planes $\mathrm{PG}(2,q)$ can be recognised among finite projective planes by either Desargues' or Pappus' Theorem. These theorems assert that if the plane contains all the points and all but one of the lines in the appropriate figure (Figure 2 or 3 respectively), then it contains the remaining line as well. The assertion is that a finite projective plane is isomorphic to $\mathrm{PG}(2,q)$ if and only if it satisfies this incidence theorem.

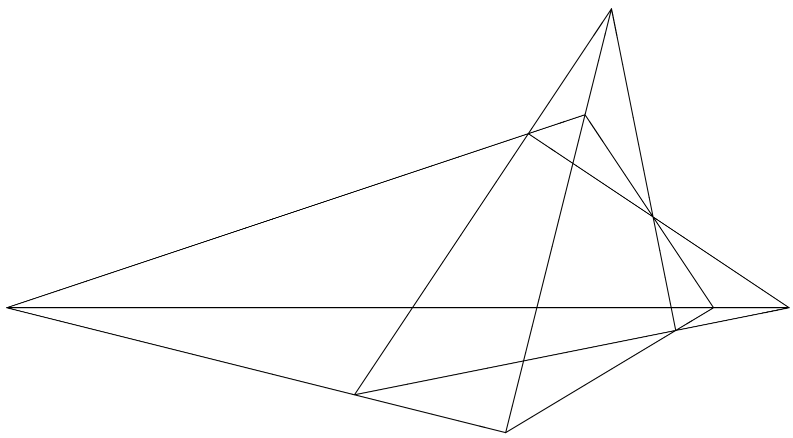

Figure 2: Desargues' Theorem

There are many other characterisations of these planes, involving either the existence of certain automorphisms, or algebraic properties of the coordinates. In particular, the *Ostrom–Wagner theorem* asserts that a finite projective plane is isomorphic to $\mathrm{PG}(2,q)$ for some q if and only if it has a doubly transitive automorphism group.

5 Configurations in projective planes

We now discuss briefly a few special types of configurations. The discussion is restricted to the planes $\mathrm{PG}(2,q)$; much less is known in other cases.

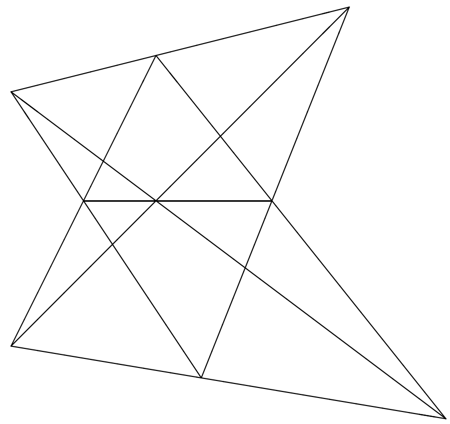

Figure 3: Pappus' Theorem

5.1 Ovals and hyperovals

An *oval* in a projective plane of order q is a set of points which meets every line in at most two points, and has a unique *tangent* (a line meeting it in one point only) at each of its points. The number of points in an oval is $q+1$. (For, if P is a point of the oval, then the q non-tangent lines through P each contain one further point of the oval.)

In $\mathrm{PG}(2,q)$, a *conic* is the set of zeros of a non-singular quadratic form. Any conic is an oval. If q is odd, the converse statement (that any oval is a conic) is a celebrated theorem of Segre. For q even, $q>4$, there are ovals which are not conics.

If q is even, the $q+1$ tangents to an oval have the remarkable property that they are all concurrent at a single point, the *nucleus* of the oval. An oval together with its nucleus is a set of $q+2$ points with the property that every line meets it in 0 or 2 points. Such a set is called a *hyperoval*.

5.2 Maximal arcs

A maximal arc in a projective plane of order q is a generalisation of a hyperoval: it is a set of points meeting every line in either 0 or k points, for some fixed k. If we exclude the degenerate case $k=q+1$ (when the set is the entire plane), it can be shown that k must divide q. Such a set is then called a *maximal k-arc*.

In $\mathrm{PG}(2,q)$, with q a power of 2, Denniston showed that there exist maximal

k-arcs for all divisors k of q. However, for q odd, Ball, Blokhuis and Mazzocca showed that there is no maximal k-arc in $\mathrm{PG}(2,q)$ for $1 < k < q$.

5.3 Blocking sets

In a projective plane, any two lines meet, and so if a set S of points contains a line, then it necessarily meets all the lines of the plane. A *blocking set* is a set S which meets all the lines but does not contain any line. The Fano plane has no blocking set, but any larger plane contains blocking sets.

A blocking set in a plane of order q must contain at least $q+\sqrt{q}+1$ points. If this bound is attained, then the blocking set must consist of the points of a subplane of order $\sqrt{q}$. In particular, if q is a square, then $\mathrm{PG}(2,q)$ contains a blocking set $\mathrm{PG}(2,\sqrt{q})$ meeting this bound. Better lower bounds are known for the size of a blocking set in $\mathrm{PG}(2,q)$ if q is not a square.

The complement of a blocking set is a blocking set, so we deduce that a blocking set contains at most $q^2-\sqrt{q}$ points. If we require that the blocking set is *minimal* (this means that, if a point is removed, the result is no longer a blocking set), then its size is at most $q\sqrt{q}+1$. This bound is also met in the plane $\mathrm{PG}(2,q)$ if q is a square; a set meeting the bound is a *unital*.

We see that the range of sizes even of minimal of blocking sets is very wide; there is no prospect of a classification of them.

References

[1] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford University Press,

Peter J. Cameron
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