1992

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Publication Details

Jennifer Seberry and Mieko Yamada, Hadamard matrices, Sequences, and Block Designs, Contemporary Design Theory – A Collection of Surveys, (D. J. Stinson and J. Dinitz, Eds.)), John Wiley and Sons, (1992), 431-560.

Hadamard matrices, Sequences, and Block Designs

Abstract

One hundred years ago, in 1893, Jacques Hadamard [31] found square matrices of orders 12 and 20, with entries ±1, which had all their rows (and columns) pairwise orthogonal. These matrices, $X = (X_{ij})$, satisfied the equality of the following inequality,

$|\det X|^2 \leq \Pi \Sigma |x_{ij}|^2$,

and so had maximal determinant among matrices with entries ±1. Hadamard actually asked the question of finding the maximal determinant of matrices with entries on the unit disc, but his name has become associated with the question concerning real matrices.

Disciplines

Physical Sciences and Mathematics

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Hadamard Matrices, Sequences, and Block Designs

Jennifer Seberry and Mieko Yamada

1 INTRODUCTION

One hundred years ago, in 1893, Jacques Hadamard [31] found square matrices of orders 12 and 20, with entries ± 1, which had all their rows (and columns) pairwise orthogonal. These matrices, $X = (x_{ij})$, satisfied the equality of the following inequality,

$$|\det X|^2 \le \prod_{i=1}^{n} \sum_{j=1}^{n} |x_{ij}|^2,$$

and so had maximal determinant among matrices with entries ± 1. Hadamard actually asked the question of finding the maximal determinant of matrices with entries on the unit disc, but his name has become associated with the question concerning real matrices.

Contemporary Design Theory: A Collection of Surveys, Edited by Jeffrey H. Dinitz and Douglas R. Stinson
ISBN 0-471-53141-3

Hadamard was not the first to study these matrices, for J. J. Sylvester in 1857, in his seminal paper, "Thoughts on inverse orthogonal matrices, simultaneous sign-successions and tesselated pavements in two or more colors with application to Newton's rule, ornamental tile work and the theory of numbers" [97], had found such matrices for all orders that are powers of two. Nevertheless, Hadamard showed that matrices with entries ± 1 and maximal determinant could exist only for orders 1, 2, and $4t$. The Hadamard conjecture states that "there exists an *Hadamard matrix*, or square matrix with every entry ± 1 and row (column) vectors pairwise orthogonal for these orders." This survey indicates the progress that has been made in the past 100 years.

Hadamard's inequality applies to matrices with entries from the unit circle. Matrices with entries ± 1, $\pm i$, and pairwise orthogonal rows (and columns) are called *complex Hadamard matrices* (note the scalar product is $a \cdot b = \sum a_i b_i^*$ for complex numbers). These matrices were first studied by R. J. Turyn [104]. We believe complex Hadamard matrices exist for every order $n \equiv 0 \pmod 2$. The truth of this conjecture would imply the truth of the Hadamard conjecture.

We begin by mentioning a few practical applications of Hadamard matrices. We note that it was M. Hall, Jr., L. Baumert, and S. Golomb [4] working with the U.S. Jet Propulsion Laboratories (JPL) who sparked the interest in Hadamard matrices in the past 30 years. In the 1960s the JPL was working toward building the *Mariner* and *Voyager* space probes to visit Mars and the other planets of the solar system. Those of us who saw early black-and-white pictures of the back of the moon remember that whole lines were missing. The black-and-white television pictures from the first landing on the moon were extremely poor quality. How many of us remember that the recent flyby of Neptune was by a space probe launched in the seventies? We take the high-quality color pictures of Jupiter, Saturn, Uranus, Neptune, and their moons for granted.

In brief, these high-quality color pictures are made by using three black-and-white pictures taken, in turn, through red, green, and blue filters. Each picture is then considered as a 1000×1000 matrix of black-and-white pixels. Each pixel is graded on a scale of 1 to 16, according to its greyness. So white is 1, and black is 16. These grades are then used to choose a codeword in an eight error correction code based on the Hadamard matrix of order 32. The codeword is transmitted to Earth, error corrected, the three black-and-white pictures are reconstructed, and then a computer is used to obtain the colored pictures.

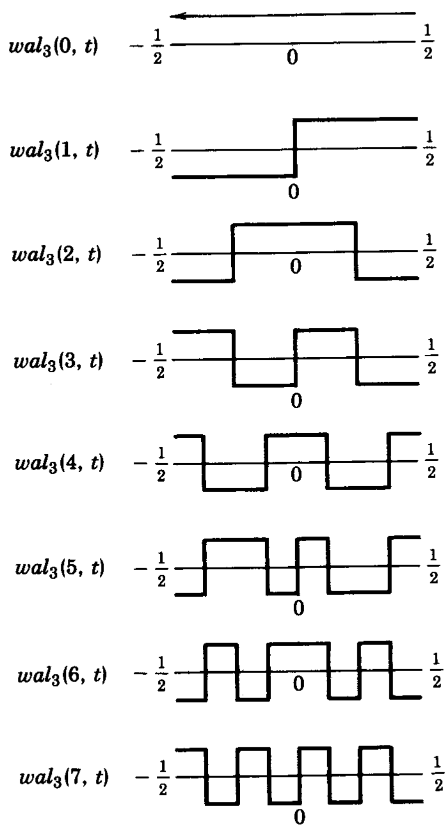
Hadamard matrices were used for these codewords for two reasons. First, error correction codes based on Hadamard matrices have maximal error correction capability for a given length of codeword. Second, the Hadamard matrices of powers of two are analogous to the Walsh functions, and thus all the computer processing can be accomplished using additions (which are very fast and easy to implement in computer hardware) rather than multiplications (which are far slower).

Sylvester's original construction for Hadamard matrices is equivalent to finding Walsh functions [48] which are the discrete analogue of Fourier Series.

Example 1.1. Let H be a Sylvester-Hadamard matrix (see Section 2) of order $8 = 2^3$.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

The Walsh function wal_3 generated by H is the following:



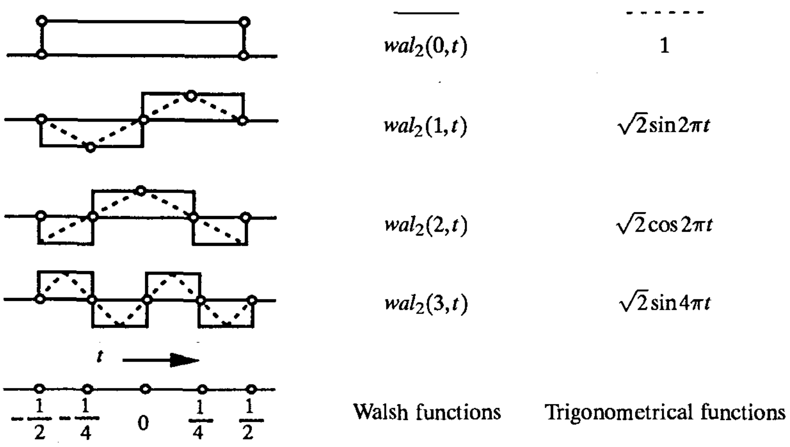


Figure 1.1. Walsh functions and trigonometrical functions.

The Walsh function wal_n is constructed in a similar way from the Sylvester-Hadamard matrix of order 2^n. The points of intersections of Walsh functions are identical with those of trigonometrical functions. See Figure 1.1.

As Figure 1.1 shows, by mapping $w(i,t) = wal_n(i,t)$ into the interval $[-\frac{1}{2},0]$, and then by extending the graph symmetrically into $[0,\frac{1}{2}]$, we get $w(2i,t)$, which is an even function. By operating similarly, we get $w(2i-1,t)$, an odd function.

Just as any curve can be written as an infinite Fourier series,

$$\sum_n a_n \sin nt + b_n \cos nt,$$

the curve can be written in terms of Walsh functions,

$$\sum_n a_n sal_n(i,t) + b_n cal_n(i,t) = \sum_n c_n wal_n(i,t),$$

where $sal_n(i,t)$ and $cal_n(i,t)$ are, respectively, even and odd components of the Walsh function $wal_n(i,t)$. The hardest curve to model with Fourier series is the step function $wal_2(0,t)$, and errors lead to the Gibbes phenomenon. Similarly, the hardest curve to model with Walsh functions is the basic $\sin 2\pi t$ or $\cos 2\pi t$ curve. Still, we see that we can transform each form to the other.

Many problems require Fourier transforms to be taken, but Fourier transforms require many multiplications that are slow and expensive to execute. On the other hand, the fast Walsh-Hadamard transform uses only additions and subtractions (addition of the complement) and so is used extensively to transform power sequency spectrum density, band compression of television signals or facsimile signals or image processing.

Walsh functions are easy to extend to higher dimensions (and higher dimensional Hadamard matrices) to model surfaces in three and higher dimensions—

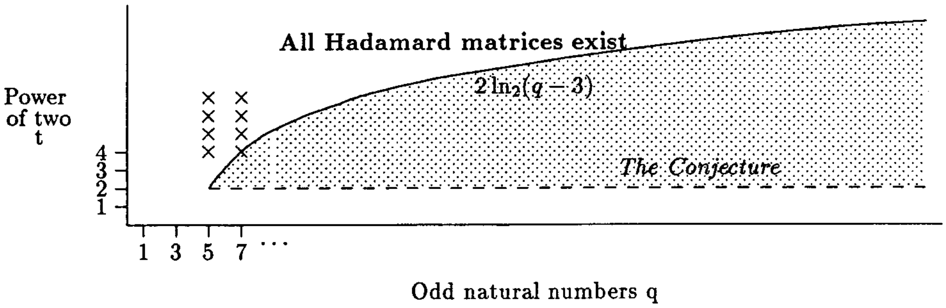


Figure 1.2. Hadamard matrices of order $2^t q$.

Fourier series are more difficult to extend. Walsh-Hadamard transforms in higher dimensions are also effected using only additions (and subtractions).

We now give an overview of construction methods for Hadamard matrices. Constructions for Hadamard matrices can be roughly classified into three types:

1. Multiplication theorems;
2. "Plug-in" methods;
3. Direct constructions.

In 1976, Jennifer Seberry Wallis, in her paper, "On the existence of Hadamard matrices" [121], showed that "given any odd natural number q, there exists a $t \approx 2\log_2(q-3)$ so that there is an Hadamard matrix of order $2^t q$ (and hence for all orders $2^s q$, $s \geq t$)." This is represented graphically in Figure 1.2.

In fact, as we show in our Appendix, Hadamard matrices are known to exist of order $2^2 q$ for most $q < 3000$ (we have results up to 40000 that are similar). In many other cases, Hadamard matrices of order $2^3 q$ or $2^4 q$ exist. A quick look at the Appendix shows most of the very difficult cases are for q (prime) $\equiv 3 \pmod 4$.

Hadamard's original construction for Hadamard matrices is a "multiplication theorem" as it uses the fact that the Kronecker product of Hadamard matrices of orders $2^a m$ and $2^b n$ is an Hadamard matrix of order $2^{a+b} mn$. Our graph shows that we would like to reduce this power of two. In his book, *Hadamard Matrices and Their Applications*, Agayan [1] shows how to multiply these Hadamard matrices to get an Hadamard matrix of order $2^{a+b-1} mn$ (which lowers the curve in our graph except for q prime).

Paley's 1933 "direct" construction [66], which gives Hadamard matrices of order $\Pi_{i,j}(p_i + 1)(2(q_j + 1))$, p_i (prime power) $\equiv 3 \pmod 4$, q_j (prime power) $\equiv 1 \pmod 4$, is extremely productive of Hadamard matrices, but we note again the proliferation of powers of two as more products are taken.

Many people do not realize that in the same issue of the *Journal of Mathematics and Physics* as Paley's paper appeared, J. A. Todd showed the equivalence of Hadamard matrices of order $4t$ and $(4t-1, 2t-1, t-1)$-SBIBD (see

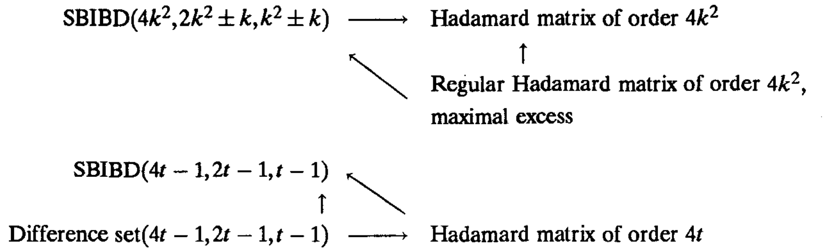


Figure 1.3. Relationship between SBIBD and Hadamard matrices.

Figure 1.3). This family of SBIBD, its complementary family $(4t-1, 2t, t)$-SBIBD, and the family $(4s^2, 2s^2 \pm s, s^2 \pm s)$-SBIBD are called *Hadamard designs*. The latter family satisfies the constraint $v = 4(k - \lambda)$, for $v = 4s^2$, $k = 2s^s \pm s$, and $\lambda = s^2 \pm s$, which appears in some constructions (e.g., Shrikhande [91]). Hadamard designs have the maximum number of one's in their incidence matrices among all incidence matrices of (v, k, λ)-SBIBD (see Tsuzuku [103]).

In 1944, J. Williamson [128], who coined the name *Hadamard matrices*, first constructed what have come to be called *Williamson matrices*, or with a small set of conditions, *Williamson type matrices*. These matrices are used to replace the variables of a formally orthogonal matrix. We say Williamson type matrices are "plugged in" to the second matrix. Other matrices that can be "plugged in" to arrays of variables are called *suitable matrices*. Generally the arrays into which suitable matrices are plugged are *orthogonal designs*, which have formally orthogonal rows (and columns) but may have variations such as Goethals-Seidel arrays, Wallis-Whiteman arrays, Spence arrays, generalized quarternion arrays, Agayan families, Kharaghani's methods, and regular s-sets of regular matrices that give new matrices. This is an extremely prolific method of construction. We will discuss methods that give matrices to "plug in" and matrices to "plug into."

As a general rule, if we want to check if an Hadamard matrix of a specific order $4pq$ exists, we would first check if there are Williamson type matrices of order p, q, pq; then we would check if there is an OD$(4t; t, t, t, t)$ for $t = q, p, pq$. This failing, we would check the "direct" constructions. Finally, we would use a "multiplication theorem." When we talk of "strength" of a construction, this reflects a personal preference.

Before we proceed to more detail, we will consider diagrammatically some of the linkages between conjectures that will arise in this survey: The conjecture implied is "the necessary conditions are sufficient for the existence of (say) Hadamard matrices" (see Figure 1.4). (A *weighing matrix* W has entries $0, \pm 1$, is square, and satisfies $WW^T = kI$.)

The hierarchy of conjectures for weighing matrices and ODs is more straightforward. Settling the OD conjecture in Table 1.1 would settle the weighing matrix conjecture to its left. This survey emphasizes those constructions, selected by us, which we believe show the most promise toward solving the Hadamard conjecture and which were found in the last 15 years.

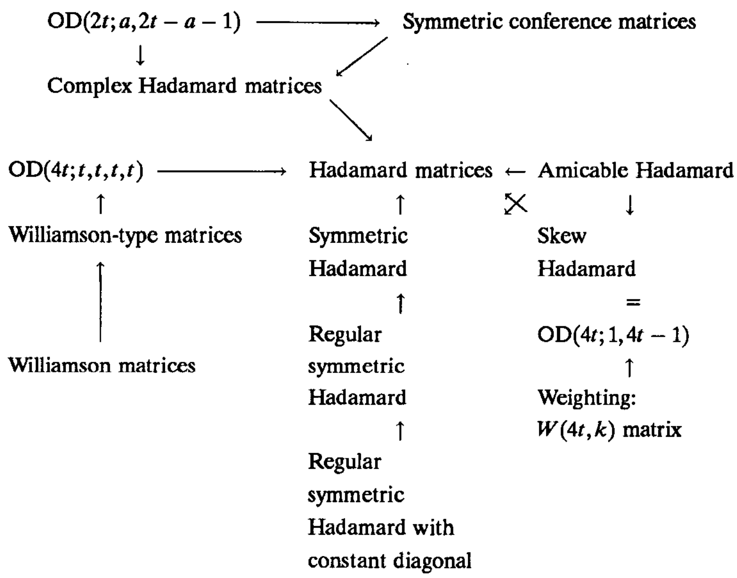


Figure 1.4. Conjecture: "The necessary conditions are sufficient for the existence of (say) Hadamard matrices."

TABLE 1.1 Weighing Matrix and OD Conjectures

	Matrices	OD's
Strongest	Skew-weighing	$OD(n;1,k)$
	Weighing $W(n;k)$, n odd	
	Weighing $W(2n,k)$, n odd	$OD(2n;a,b)$, n odd
	Weighing $W(4n,k)$, n odd	$OD(4n;a,b,c,d)$, n odd
Weakest	$W(2^s n,k)$, n odd, $s \geq 3$	$OD(2^s n;u_1,u_2,\ldots,u_s)$, n odd

2 HADAMARD MATRICES

A square matrix with elements ± 1 and order h, whose distinct row vectors are orthogonal is an *Hadamard matrix* of order h. The smallest examples are

$$[1], \quad \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix},$$

where we write $-$ for -1. These were first studied by J. J. Sylvester [97] who observed that if H is an Hadamard matrix, then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is also an Hadamard matrix. Indeed, using the matrix of order 2, we have

Lemma 2.1 (Sylvester [97]). *There is an Hadamard matrix of order 2^t for all integers t.*

We call matrices of order 2^t constructed by Sylvester's construction *Sylvester-Hadamard matrices*. We have seen that these matrices are naturally associated with the discrete orthogonal functions called *Walsh functions*. Using Sylvester's method, the first few Hadamard matrices obtained are

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}, \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ \hline 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{array}\right].$$

For these matrices, we count, row by row, the number of times the sign changes; for example, $1 - - 1$ changes sign twice. This gives

for the matrix of order 2 : $0, 1$;

for the matrix of order 4 : $0, 3, 1, 2$;

for the matrix of order 8 : $0, 7, 3, 4, 1, 6, 2, 5$.

Indeed, we will see that the set of the numbers of sign changes in the rows of a Sylvester-Hadamard matrix of order n is $\{0, 1, \dots, n-1\}$, corresponding to the number times the Walsh functions cross the x-axis.

In 1893, Jacques Hadamard [31] gave examples of Hadamard matrices for a few small orders and conjectured that they exist for every order divisible by 4. Some examples for order 12 are

$$
\left[\begin{array}{c|ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cline{2-12}
- & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - \\
- & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 \\
- & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - \\
- & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - \\
- & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - \\
- & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 \\
- & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 \\
- & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & 1
\end{array}\right],
$$

$$
\left[\begin{array}{c|ccccc c|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 \\ \cline{2-6}\cline{8-12}
1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - \\
 & & & & & & & & & & & \\
- & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\ \cline{2-6}\cline{8-12}
1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - \\
1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & - & 1 & - & - & - & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & - \\
1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & -
\end{array}\right],
$$

$$
\left[\begin{array}{ccc ccc ccc ccc}
1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - \\
 & & & & & & & & & & & \\
1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 \\
 & & & & & & & & & & & \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 \\
- & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 \\
- & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - \\
 & & & & & & & & & & & \\
1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 \\
- & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{cccc cccc cccc}
1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
- & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & - \\
- & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
- & 1 & - & 1 & - & - & - & 1 & - & - & - & - \\
 & & & & & & & & & & & \\
- & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - \\
- & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - \\
- & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 \\
 & & & & & & & & & & & \\
- & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & - & - & - & - & - & - & - & 1 & 1 & - \\
- & - & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1
\end{array}\right].
$$

We have given these matrices in full because, unfortunately, an earlier survey contains errors.

Two Hadamard matrices are said to be *Hadamard equivalent* (or just *equivalent*) if one can be obtained from the other by a sequence of operations of the following two types:

1. Permute rows (or columns).
2. Multiply any row (or column) by -1.

Although the Hadamard matrices of order 12 presented above appear to be different, it is possible to show that they are equivalent.

In fact, we know that there are 5 inequivalent matrices of order 16 [32], 3 of order 20 [33], 60 of order 24 [37, 47], 486 of order 28 [44], over 15 of order 32 (N. Ito, personal communication, 1989), and over 109 of order 36 [11].

An Hadamard matrix of order 20 is given in Figure 2.1. This figure is more easily described by calling the rows 0 to 19 and saying that the zeroth row is all ones, the first row has ones in positions

$$\{1,2,5,6,7,8,10,12,17,18\},$$

the second row has ones in positions

$$\{2,3,6,7,8,9,11,13,18,19\},$$

the third row has ones in positions

$$\{4,5,8,9,10,11,13,15,1,2\},$$

and so on.

This example illustrates the use of difference sets with the parameters $(4t-1, 2t-1, t-1)$ in the construction of Hadamard matrices. $\{1,2,5,6,7,8,10,12,17,18\}$ is a difference set with parameters $(19,9,4)$. For more information on difference sets, see the survey by Jungnickel in this volume [40].

Hadamard matrices can also be constructed using supplementary difference sets. The existence of supplementary difference sets in the abelian group $Z_3 \times Z_3$ and can be used to construct another Hadamard matrix of order 20 given in Figure 2.2.

We now recall some basic properties of Hadamard matrices:

Lemma 2.2. *Let H be an Hadamard matrix of order h. Then the following hold:*

1. $HH^T = hI_h$.
2. $|\det H| = h^{(1/2)h}$.
3. $HH^T = H^T H$.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
–	1	1	–	–	1	1	1	1	–	1	–	1	–	–	–	–	1	1	–
–	–	1	1	–	–	1	1	1	1	–	1	–	1	–	–	–	–	1	1
–	1	–	1	1	–	–	1	1	1	1	–	1	–	1	–	–	–	–	1
–	1	1	–	1	1	–	–	1	1	1	1	–	1	–	1	–	–	–	–
–	–	1	1	–	1	1	–	–	1	1	1	1	–	1	–	1	–	–	–
–	–	–	1	1	–	1	1	–	–	1	1	1	1	–	1	–	1	–	–
–	–	–	–	1	1	–	1	1	–	–	1	1	1	1	–	1	–	1	–
–	–	–	–	–	1	1	–	1	1	–	–	1	1	1	1	–	1	–	1
–	1	–	–	–	–	1	1	–	1	1	–	–	1	1	1	1	–	1	–
–	–	1	–	–	–	–	1	1	–	1	1	–	–	1	1	1	1	–	1
–	1	–	1	–	–	–	–	1	1	–	1	1	–	–	1	1	1	1	–
–	–	1	–	1	–	–	–	–	1	1	–	1	1	–	–	1	1	1	1
–	1	–	1	–	1	–	–	–	–	1	1	–	1	1	–	–	1	1	1
–	1	1	–	1	–	1	–	–	–	–	1	1	–	1	1	–	–	1	1
–	1	1	1	–	1	–	1	–	–	–	–	1	1	–	1	1	–	–	1
–	1	1	1	1	–	1	–	1	–	–	–	–	1	1	–	1	1	–	–
–	–	1	1	1	1	–	1	–	1	–	–	–	–	1	1	–	1	1	–
–	–	–	1	1	1	1	–	1	–	1	–	–	–	–	1	1	–	1	1
–	1	–	–	1	1	1	1	–	1	–	1	–	–	–	–	1	1	–	1

Figure 2.1. An Hadamard matrix of order 20.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	–	–	1	–	1	–	–	–	1	1	–	–	1	–	1	–
1	1	1	1	1	–	–	–	–	1	–	1	–	1	1	–	–	–	–	1
1	1	1	1	–	1	–	1	–	–	–	1	1	–	–	1	–	1	–	–
1	–	1	–	1	1	1	–	–	1	–	–	1	–	–	1	1	–	–	1
1	–	–	1	1	1	1	1	–	–	–	–	–	1	1	–	1	1	–	–
1	1	–	–	1	1	1	–	1	–	–	1	–	–	1	1	–	–	1	–
1	–	–	1	–	1	–	1	1	1	–	–	–	1	–	1	–	–	1	1
1	1	–	–	–	–	1	1	1	1	–	1	–	–	–	–	1	1	–	1
1	–	1	–	1	–	–	1	1	1	–	–	1	–	1	–	–	1	1	–
1	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	1	1
1	–	1	1	–	–	1	–	1	–	1	–	–	–	1	1	–	1	–	1
1	1	–	1	1	–	–	–	–	1	1	–	–	–	–	1	1	1	1	–
1	1	1	–	–	1	–	1	–	–	1	–	–	–	1	–	–	–	1	1
1	–	1	–	–	1	1	–	–	1	1	1	–	1	–	–	–	1	1	–
1	–	–	1	1	–	1	1	–	–	1	1	1	–	–	–	–	–	1	1
1	1	–	–	1	1	–	–	1	–	1	–	1	1	–	–	–	1	1	–
1	–	–	1	–	1	–	–	1	1	1	1	1	–	1	–	1	–	–	–
1	1	–	–	–	–	1	1	–	1	1	–	1	1	1	1	–	–	–	–
1	–	1	–	1	–	–	1	1	–	1	1	–	1	–	1	1	–	–	–

Figure 2.2. A second Hadamard matrix of order 20.

4. *Every Hadamard matrix is equivalent to an Hadamard matrix that has every element of its first row and column* $+1$ *(matrices of this latter form are called normalized).*
5. $h = 1, 2,$ *or* $4n$, *n an integer.*

6. *If H is a normalized Hadamard matrix of order $4n$, then every row (column) except the first has $2n$ minus ones and $2n$ plus ones in each row (column); further, n minus ones in any row (column) overlap with n minus ones in each other row (column).*

Definition 2.1. An Hadamard matrix H is said to be *regular* if the sum of all the elements in each row or column is a constant k. Hence $HJ = JH = kJ$, where J is the matrix of all ones.

Definition 2.2. If $M = (m_{ij})$ is a $m \times p$ matrix and $N = (n_{ij})$ is an $n \times q$ matrix, then the *Kronecker product* $M \times N$ is the $mn \times pq$ matrix given by

$$M \times N = \begin{bmatrix} m_{11}N & m_{12}N & \cdots & m_{1p}N \\ m_{21}N & m_{22}N & \cdots & m_{2p}N \\ \vdots & & & \vdots \\ m_{m1}N & m_{m2}N & \cdots & m_{mp}N \end{bmatrix}.$$

Example 2.1. Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Then

$$M \times N = \begin{bmatrix} N & N \\ N & -N \end{bmatrix} = \left[\begin{array}{cccc|cccc} -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hline -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \end{array}\right].$$

Lemma 2.3 (Hadamard [31]). *Let H_1 and H_2 be Hadamard matrices of orders h_1 and h_2. Then $H = H_1 \times H_2$ is an Hadamard matrix of order $h_1 h_2$.*

We now prove a stronger result than Hadamard's, first proved by Agayan and Sarukhanyan, and then strengthened by Seberry and Yamada [87] and

Agayan-Sarukhanyan [1]. These theorems have the advantage of reducing the power of two in the resulting Hadamard matrix.

Lemma 2.4 (The Multiplication Theorem of Agayan-Sarukhanyan [1]). *Let H_1 and H_2 be Hadamard matrices of orders $4h$ and $4k$. Then there is an Hadamard matrix of order $8hk$.*

Proof. Write the two Hadamard matrices as

$$H_1 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} K & L \\ M & N \end{bmatrix}.$$

We note that since $H_1H_1^T = 4hI$ and $H_2H_2^T = 4kI$, we have

$$PP^T + QQ^T = RR^T + SS^T = 2hI, \qquad PR^T + QS^T = O = RP^T + SQ^T;$$

$$KK^T + LL^T = MM^T + NN^T = 2kI, \qquad KM^T + LN^T = O = MK^T + NL^T.$$

The required Hadamard matrix of order $8hk$ is

$$\begin{bmatrix} \frac{1}{2}(P+Q)\times K + \frac{1}{2}(P-Q)\times M & \frac{1}{2}(P+Q)\times L + \frac{1}{2}(P-Q)\times N \\ \frac{1}{2}(R+S)\times K + \frac{1}{2}(R-S)\times M & \frac{1}{2}(R+S)\times L + \frac{1}{2}(R-S)\times N \end{bmatrix}$$

which can be verified by simple algebraic manipulation. □

Example 2.2. There are Hadamard matrices of orders 12 and 20. Sylvester's lemma guarantees the existence of an Hadamard matrix of order 240, while the Agayan-Sarukhanyan guarantees the existence of one of order 120.

This can also be strengthened.

Theorem 2.5 (Craigen-Seberry-Zhang [14]). *Suppose that there are Hadamard matrices of orders $4a, 4b, 4c, 4d$. Then there is an Hadamard matrix of order $16abcd$.*

So, for example, we can get an Hadamard matrix of order $16 \cdot 15 \cdot 15$ from this theorem.

3 THE STRONGEST HADAMARD CONSTRUCTION THEOREMS

For easy reference, we will now give the strongest construction theorems for Hadamard matrices. These theorems do not give all the known orders but give the vast majority of those known. We leave the proofs until our later book as well as details of when these conditions can be satisfied.

Theorem 3.1 (Paley [66]). *Let* $p \equiv 3 \pmod 4$ *be a prime power. Then there is an Hadamard matrix of order* $p + 1$.

Theorem 3.2 (Paley [66]). *Let* $p \equiv 1 \pmod 4$ *be a prime power. Then there is an Hadamard matrix of order* $2(p + 1)$.

Theorem 3.3 (Goethals-Seidel [25]). *Suppose that there is an Hadamard matrix of order h. Then there is a regular symmetric Hadamard matrix with constant diagonal of order* h^2.

Since Hadamard matrices are of order $h \equiv 0 \pmod 4$ and Hadamard's inequality studies matrices on the unit disc, it is natural to consider matrices with complex entries.

Definition 3.1. A matrix C of order $2n$ with elements ± 1, $\pm i$ that satisfies $CC^* = 2nI$ will be called a *complex Hadamard matrix.*

The strongest theorem using complex Hadamard matrices is the following "multiplication theorem":

Theorem 3.4 (Turyn [104]). *Suppose that there is a complex Hadamard matrix of order* $2n$ *and an Hadamard matrix of order* $4h$. *Then there is an Hadamard matrix of order* $8hn$.

This means that the complex Hadamard conjecture is intricately woven with the Hadamard conjecture.

Definition 3.2. X and Y are said to be *amicable matrices* if

$$XY^T = YX^T. \tag{1}$$

Now we look more precisely at definitions of matrices to "plug in."

Definition 3.3. Four circulant symmetric ± 1 matrices A, B, C, D of order w that satisfy

$$AA^T + BB^T + CC^T + DD^T = 4wI_w$$

will be called *Williamson matrices.* Four ± 1 matrices A, B, C, D of order w that satisfy both

$$XY^T = YX^T \qquad \text{for} \quad X, Y \in \{A, B, C, D\}$$

(that is, A, B, C, D are pairwise amicable), and

$$AA^T + BB^T + CC^T + DD^T = 4wI_w, \tag{2}$$

will be called *Williamson-type matrices.*

Analogously, eight circulant ± 1 matrices $A_1, A_2, \ldots, A_8$ of order w which are symmetric and which satisfy

$$\sum_{i=1}^{8} A_i A_i^T = 8wI_w$$

will be called *8-Williamson matrices.* Eight ± 1 amicable matrices $A_1, A_2, \ldots, A_8$ of order w which satisfy both

$$\sum_{i=1}^{8} A_i A_i^T = 8wI_w \qquad \text{and} \qquad A_j A_i^T = A_i A_j^T, \quad i, j = 1, \ldots, 8,$$

will be called *8-Williamson-type matrices.*

The most common structure matrices are "plugged into" is the orthogonal design, defined as follows:

Definition 3.4. An *orthogonal design of order n and type* $(s_1, \ldots, s_u)$, s_i positive integers, is an $n \times n$ matrix X, with entries $\{0, \pm x_1, \ldots, \pm x_u\}$ (the x_i commuting indeterminates) satisfying

$$XX^T = \left(\sum_{i=1}^{u} s_i x_i^2 \right) I_n. \tag{3}$$

We write this as $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$.

Alternatively, each row of X has s_i entries of the type $\pm x_i$, and the distinct rows are orthogonal under the euclidean inner product. We may view X as a matrix with entries in the field of fractions of the integral domain $Z[x_1, \ldots, x_u]$ (Z the rational integers), and if we let $f = (\Sigma_{i=1}^{u} s_i x_i^2)$, then X is an invertible matrix with inverse $(1/f)X^T$. Thus, $XX^T = fI_n$, and so our alternative definition that the row vectors are orthogonal applies equally well to the column vectors of X.

An orthogonal design with no zeros and in which each of the entries is replaced by $+1$ or -1 is an Hadamard matrix. A special orthogonal design, the $\mathrm{OD}(4t; t, t, t, t)$, is especially useful in the construction of Hadamard matrices. An OD(12;3,3,3,3) was first found by L. Baumert and M. Hall, Jr. [6], and an OD(20;5,5,5,5) by Welch (see below). $\mathrm{OD}(4t; t, t, t, t)$ are sometimes called *Baumert-Hall arrays.*

Another set of matrices of a very different kind can be obtained by partitioning a matrix as follows: Let M be a matrix of order tm. Then M can be expressed as a t^2 block M-structure when M is an orthogonal matrix:

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & & & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix},$$

where M_{ij} is of order m $(i,j = 1,2,\ldots,t)$.

Some orthogonal designs of special interest are the following:

1. The Williamson array—the OD(4; 1, 1, 1, 1):

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}$$ the right representation of the quaternions;

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$ the left representation of the quaternions.

2. The OD(8; 1, 1, 1, 1, 1, 1, 1, 1):

$$\left[\begin{array}{cccc|cccc} A & B & C & D & E & F & G & H \\ -B & A & D & -C & F & -E & -H & G \\ -C & -D & A & B & G & H & -E & -F \\ -D & C & -B & A & H & -G & F & -E \\ \hline -E & -F & -G & -H & A & B & C & D \\ -F & E & -H & G & -B & A & -D & C \\ -G & H & E & -F & -C & D & A & -B \\ -H & -G & F & E & -D & -C & B & A \end{array}\right].$$

3. The Baumert-Hall array—the OD(12;3,3,3,3):

$$A(x,y,z,w) =$$

$$\left[\begin{array}{cccc|cccc|cccc}
y & x & x & x & -z & z & w & y & -w & w & z & -y \\
-x & y & x & -x & w & -w & z & -y & -z & z & -w & -y \\
-x & -x & y & x & w & -y & -y & w & z & z & w & -z \\
-x & x & -x & y & -w & -w & -z & w & -z & -y & -y & -z \\
\hline
-y & -y & -z & -w & z & x & x & x & -w & -w & z & -y \\
-w & -w & -z & y & -x & z & x & -x & y & y & -z & -w \\
w & -w & w & -y & -x & -x & z & x & y & -z & -y & -z \\
-w & -z & w & -z & -x & x & -x & z & -y & y & -y & w \\
\hline
-y & y & -z & -w & -z & -z & w & y & w & x & x & x \\
z & -z & -y & -w & -y & -y & -w & -z & -x & w & x & -x \\
-z & -z & y & z & -y & -w & y & -w & -x & -x & w & x \\
z & -w & -w & z & y & -y & y & z & -x & x & -x & w
\end{array}\right];$$

or alternatively (using the Cooper-J.Wallis theorem [12]), the OD(12;3, 3,3,3) is

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
a & b & c & -b & a & d & -c & -d & a & -d & c & -b \\
c & a & b & a & d & -b & -d & a & -c & c & -b & -d \\
b & c & a & d & -b & a & a & -c & -d & -b & -d & c \\
\hline
b & -a & -d & a & b & c & -d & -b & c & c & -a & d \\
-a & -d & b & c & a & b & -b & c & -d & -a & d & c \\
-d & b & -a & b & c & a & c & -d & -b & d & c & -a \\
\hline
c & d & -a & d & b & -c & a & b & c & -b & d & a \\
d & -a & c & b & -c & d & c & a & b & d & a & -b \\
-a & c & d & -c & d & b & b & c & a & a & -b & d \\
\hline
d & -c & b & -c & a & -d & b & -d & -a & a & b & c \\
-c & b & d & a & -d & -c & -d & -a & b & c & a & b \\
b & d & -c & -d & -c & a & -a & b & -d & b & c & a
\end{array}\right]$$

4. The Plotkin array—the OD(24;3,3,3,3,3,3,3,3):
Let $A(x,y,z,w)$ be as in array 3, and let

$B = (x,y,z,w)$

$$
= \left[\begin{array}{cccc|cccc|cccc}
y & x & x & x & -w & w & z & y & -z & z & w & -y \\
-x & y & x & -x & -z & z & -w & -y & w & -w & z & -y \\
-x & -x & y & x & -y & -w & y & -w & -z & -z & w & z \\
-x & x & -x & y & w & w & -z & -w & -y & z & y & z \\
\hline
-w & -w & -z & -y & z & x & x & x & -y & -y & z & -w \\
y & y & -z & -w & -x & z & x & -x & -w & -w & -z & y \\
-w & w & -w & -y & -x & -x & z & x & z & y & y & z \\
z & -w & -w & z & -x & x & -x & z & y & -y & y & w \\
\hline
z & -z & y & -w & y & y & w & -z & w & x & x & x \\
y & -y & -z & -w & -z & -z & -w & -y & -x & w & x & -x \\
z & z & y & -z & w & -y & -y & w & -x & -x & w & x \\
-w & -z & w & -z & -y & y & -y & z & -x & x & -x & w
\end{array}\right],
$$

then $\begin{bmatrix} A(x_1,x_2,x_3,x_4) & B(x_5,x_6,x_7,x_8) \\ B(-x_5,x_6,x_7,x_8) & -A(-x_1,x_2,x_3,x_4) \end{bmatrix}$ is the required design.

5. The Welch array—the OD(20;5,5,5,5) constructed from 16-block circulant matrices is an M-structure:

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
-D & B & -C & -C & -B & C & A & -D & -D & -A & -B & -A & C & -C & -A & A & -B & -D & D & -B \\
-B & -D & B & -C & -C & -A & C & A & -D & -D & -A & -B & -A & C & -C & -B & A & -B & -D & D \\
-C & -B & -D & B & -C & -D & -A & C & A & -D & -C & -A & -B & -A & C & D & -B & A & -B & -D \\
-C & -C & -B & -D & B & -D & -D & -A & C & A & C & -C & -A & -B & -A & -D & D & -B & A & -B \\
B & -C & -C & -B & -D & A & -D & -D & -A & C & -A & C & -C & -A & -B & -B & -D & D & -B & A \\
\hline
-C & A & D & D & -A & -D & -B & -C & -C & B & -A & B & -D & D & B & -B & -A & -C & C & -A \\
-A & -C & A & D & D & B & -D & -B & -C & -C & B & -A & B & -D & D & -A & -B & -A & -C & C \\
D & -A & -C & A & D & -C & B & -D & -B & -C & D & B & -A & B & -D & C & -A & -B & -A & -C \\
D & D & -A & -C & A & -C & -C & B & -D & -B & -D & D & B & -A & B & -C & C & -A & -B & -A \\
A & D & D & -A & -C & -B & -C & -C & B & -D & B & -D & D & B & -A & -A & -C & C & -A & -B \\
\hline
B & -A & -C & C & -A & A & B & -D & D & B & -D & -B & C & C & B & -C & A & -D & -D & -A \\
-A & B & -A & -C & C & B & A & B & -D & D & B & -D & -B & C & C & -A & -C & A & -D & -D \\
C & -A & B & -A & -C & D & B & A & B & -D & C & B & -D & -B & C & -D & -A & -C & A & -D \\
-C & C & -A & B & -A & -D & D & B & A & B & C & C & B & -D & -B & -D & -D & -A & -C & A \\
-A & -C & C & -A & B & B & -D & D & B & A & -B & C & C & B & -D & A & -D & -D & -A & -C \\
\hline
-A & -B & -D & D & -B & B & -A & C & -C & -A & C & A & D & D & -A & -D & B & C & C & -B \\
-B & -A & -B & -D & D & -A & B & -A & C & -C & -A & C & A & D & D & -B & -D & B & C & C \\
D & -B & -A & -B & -D & -C & -A & B & -A & C & D & -A & C & A & D & C & -B & -D & B & C \\
-D & D & -B & -A & -B & C & -C & -A & B & -A & D & D & -A & C & A & C & C & -B & -D & B \\
-B & -D & D & -B & -A & -A & C & -C & -A & B & A & D & D & -A & C & B & C & C & -B & -D
\end{array}\right]
$$

6. The Ono-Sawade-Yamamoto array—the OD(36;9,9,9,9) constructed from 16 type one matrices is an M-structure and is given on the facing page.

a	a	a	b	c	d	−b	−d	−c	b	−a	a	b	c	−d	b	d	−c	c	−a	a	−b	c	d	b	−d	c	d	−a	a	b	−c	d	−b	d	c
a	a	a	d	b	c	−c	−b	−d	a	b	−a	−d	b	c	−c	b	d	a	c	−a	d	−b	c	c	b	−d	a	d	−a	d	b	−c	c	−b	d
a	a	a	c	d	b	−d	−c	−b	−a	a	b	c	−d	b	d	−c	b	−a	a	c	c	d	−b	−d	c	b	−a	a	d	−c	d	b	d	c	−b
−b	−d	−c	a	a	a	b	c	d	b	d	−c	b	−a	a	b	c	−d	b	−d	c	c	−a	a	−b	c	d	−b	d	c	d	−a	a	b	−c	d
−c	−b	−d	a	a	a	d	b	c	−c	b	d	a	b	−a	−d	b	c	c	b	−d	a	c	−a	d	−b	c	c	−b	d	a	d	−a	d	b	−c
−d	−c	−b	a	a	a	c	d	b	d	−c	b	−a	a	b	c	−d	b	−d	c	b	−a	a	c	c	d	−b	d	c	−b	−a	a	d	−c	d	b
b	c	d	−b	−d	−c	a	a	a	b	c	−d	b	d	−c	b	−a	a	−b	c	d	b	−d	c	c	−a	a	b	−c	d	−b	d	c	d	−a	a
d	b	c	−c	−b	−d	a	a	a	−d	b	c	−c	b	d	a	b	−a	d	−b	c	c	b	−d	a	c	−a	d	b	−c	c	−b	d	a	d	−a
c	d	b	−d	−c	−b	a	a	a	c	−d	b	d	−c	b	−a	a	b	c	d	−b	−d	c	b	−a	a	c	−c	d	b	d	c	−b	−a	a	d

−b	a	−a	−b	c	−d	−b	d	−c	a	a	a	b	−c	−d	−b	d	c	−d	−a	a	b	c	−d	−b	−d	−c	c	a	−a	b	c	d	−b	−d	c
−a	−b	a	−d	−b	c	−c	−b	d	a	a	a	−d	b	−c	c	−b	d	a	−d	−a	−d	b	c	−c	−b	−d	−a	c	a	d	b	c	c	−b	−d
a	−a	−b	c	−d	−b	d	−c	−b	a	a	a	−c	−d	b	d	c	−b	−a	a	−d	c	−d	b	−d	−c	−b		−a	c	c	d	b	−d	c	−b
−b	d	−c	−b	a	−a	−b	c	−d	−b	d	c	a	a	a	b	−c	−d	−b	−d	−c	−d	−a	a	b	c	−d	−b	−d	c	c	a	−a	b	c	d
−c	−b	d	−a	−b	a	−d	−b	c	c	−b	d	a	a	a	−d	b	−c	−c	−b	−d	a	−d	−a	−d	b	c	c	−b	−d	−a	c	a	d	b	c
d	−c	−b	a	−a	−b	c	−d	−b	d	c	−b	a	a	a	−c	−d	b	−d	−c	−b	−a	a	−d	c	−d	b	−d	c	−b	a	−a	c	c	d	b
−b	c	−d	−b	d	−c	−b	a	−a	b	−c	−d	−b	d	c	a	a	a	b	c	−d	−b	−d	−c	−d	−a	a	b	c	d	−b	−d	c	c	a	−a
−d	−b	c	−c	−b	d	−a	−b	a	−d	b	−c	c	−b	d	a	a	a	−d	b	c	−c	−b	−d	a	−d	−a	d	b	c	c	−b	−d	−a	c	a
c	−d	−b	d	−c	−b	a	−a	−b	−c	−d	b	d	c	−b	a	a	a	c	−d	b	−d	−c	−b	−a	a	−d	c	d	b	−d	c	−b	a	−a	c

−c	a	−a	−b	−c	d	b	−d	−c	d	a	−a	b	c	d	−b	d	−c	a	a	a	−b	c	−d	b	d	−c	−b	−a	a	−b	c	d	−b	−d	−c
−a	−c	a	d	−b	−c	−c	b	−d	−a	d	a	d	b	c	−c	−b	d	a	a	a	−d	−b	c	−c	b	d	a	−b	−a	d	−b	c	−c	−b	−d
a	−a	−c	−c	d	−b	−d	−c	b	a	−a	d	c	d	b	d	−c	−b	a	a	a	c	−d	−b	d	−c	b	−a	a	−b	c	d	−b	−d	−c	−b
b	−d	−c	−c	a	−a	−b	−c	d	−b	d	−c	d	a	−a	b	c	d	b	d	−c	a	a	a	−b	c	−d	−b	−d	−c	−b	−a	a	−b	c	d
−c	b	−d	−a	−c	a	d	−b	−c	−c	−b	d	−a	d	a	d	b	c	−c	b	d	a	a	a	−d	−b	c	−c	−b	−d	a	−b	−a	d	−b	c
−d	−c	b	a	−a	−c	−c	d	−b	d	−c	−b	a	−a	d	c	d	b	d	−c	b	a	a	a	c	−d	−b	−d	−c	−b	−a	a	−b	c	d	−b
−b	−c	d	b	−d	−c	−c	a	−a	b	c	d	−b	d	−c	d	a	−a	−b	c	−d	b	d	−c	a	a	a	−b	c	d	−b	−d	−c	−b	−a	a
d	−b	−c	−c	b	−d	−a	−c	a	d	b	c	−c	−b	d	−a	d	a	−d	−b	c	−c	b	d	a	a	a	d	−b	c	−c	−b	−d	a	−b	−a
−c	d	−b	−d	−c	b	a	−a	−c	c	d	b	d	−c	−b	a	−a	d	c	−d	−b	d	−c	b	a	a	a	c	d	−b	−d	−c	−b	−a	a	−b

−d	a	−a	b	−c	−d	−b	−d	c	−c	−a	a	b	−c	d	−b	−d	−c	b	a	−a	b	c	d	b	−d	−c	a	a	a	−b	−c	d	b	−d	c
−a	−d	a	−d	b	−c	c	−b	−d	a	−c	−a	d	b	−c	−c	−b	−d	−a	b	a	d	b	c	−c	b	−d	a	a	a	d	−b	−c	c	b	−d
a	−a	−d	−c	−d	b	−d	c	−b	−a	a	−c	−c	d	b	−d	−c	−b	a	−a	b	c	d	b	−d	−c	b	a	a	a	−c	d	−b	−d	c	b
−b	−d	c	−d	a	−a	b	−c	−d	−b	−d	−c	−c	−a	a	b	−c	d	b	−d	−c	b	a	−a	b	c	d	b	−d	c	a	a	a	−b	−c	d
c	−b	−d	−a	−d	a	−d	b	−c	−c	−b	−d	a	−c	−a	d	b	−c	−c	b	−d	−a	b	a	d	b	c	c	b	−d	a	a	a	d	−b	−c
−d	c	−b	a	−a	−d	−c	−d	b	−d	−c	−b	−a	a	−c	−c	d	b	−d	−c	b	a	−a	b	c	d	b	−d	c	b	a	a	a	−c	d	−b
b	−c	−d	−b	−d	c	−d	a	−a	b	−c	d	−b	−d	−c	−c	−a	a	b	c	d	b	−d	−c	b	a	−a	−b	−c	d	b	−d	c	a	a	a
−d	b	−c	c	−b	−d	−a	−d	a	d	b	−c	−c	−b	−d	a	−c	−a	d	b	c	−c	b	−d	−a	b	a	d	−b	−c	c	b	−d	a	a	a
−c	−d	b	−d	c	−b	a	−a	−d	−c	d	b	−d	−c	−b	−a	a	−c	c	d	b	−d	−c	b	a	−a	b	−c	d	−b	−d	c	b	a	a	a

7. The Goethals-Seidel array [27] (see also J. Wallis-Whiteman [113]):

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{bmatrix},$$

where A,B,C,D are circulant (type one) matrices satisfying (2) and R is the back diagonal (equivalent type two) $(0,1)$ matrix.

Definition 3.5. *Suitable matrices* of order w for an $\mathrm{OD}(n;s_1,s_2,\ldots,s_u)$ are u pairwise amicable (i.e., pairwise satisfy (1)) matrices, A_i, $i=1,\ldots,u$, that have entries $+1$ or -1 and that satisfy

$$\sum_{i=1}^{u} s_i A_i A_i^T = (\Sigma s_i) w I_w. \tag{4}$$

They are used in the following theorem:

Theorem 3.5 (Geramita-Seberry). *Suppose that there exists an* $\mathrm{OD}(\Sigma s_i;s_1,\ldots,s_u)$ *and u suitable matrices of order m. Then there is an Hadamard matrix of order* $(\Sigma s_i)m$.

If we generalize the definition of suitable matrices so that entries $0,+1,-1$ are allowed, then *weighing matrices* rather than Hadamard matrices could be constructed.

An overview of matrices to "plug in" and "plug into" is given in Table 3.1.

The most prolific method for constructing matrices to "plug into" uses T-matrices or T-sequences:

Definition 3.6 (T-matrices). A set of 4 *T-matrices*, T_i, $i=1,\ldots,4$ of order t are four circulant or type one matrices that have entries 0, $+1$ or -1 and that satisfy

1. $T_i * T_j = 0$, $i \neq j$ ($*$ denotes the Hadamard product);
2. $\sum_{i=1}^{4} T_i$ is a $(1,-1)$ matrix;
3. $\sum_{i=1}^{4} T_i T_i^T = tI_t$; and for r/v (5)
4. $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$, where t_i is the row [column] sum of T_i.

T-matrices are known (see Cohen, Rubie, Koukouvinos, Kounias, Seberry, Yamada [10] for a recent survey) (71 occurs in [58]) for many orders including the following:

TABLE 3.1 The Relationship Between Matrices to "Plug in" and Matrices to "Plug into"

	Matrices to "Plug in"	Matrices to "Plug into"
Hardest to find	Williamson Williamson-type	$OD(4t;t,t,t,t)$
	8-Williamson 8-Williamson-type	$OD(8t;t,t,t,t,t,t,t,t)$
Easiest to find	Suitable matrices	$OD(2^t n;u_1,u_2,\ldots,u_3)$
	4 circulant suitable matrices	Goethals-Seidel
	4 type one suitable matrices	J. Wallis-Whiteman
	Near suitable	"Bordered arrays"
	Regular s-sets M-structures Kharaghani matrices	Latin squares

$1,\ldots,72,74,\ldots,78,80,\ldots,82,84,\ldots,88,90,\ldots,96,98,\ldots,102,104,\ldots,106,108,110,$ $\ldots,112,114,\ldots,126,128,\ldots,130,132,\ldots,136,138,140,\ldots,148,150,152,\ldots,156,$ $158,\ldots,162,164,\ldots,166,168,\ldots,172,174,\ldots,178,180,182,184,\ldots,190,192,194,$ $\ldots,196$, 198, $200,\ldots,210,\ldots$. T-matrices of order t give Hadamard matrices of order $4t$.

Definition 3.7 (T-sequences). A set of four sequences $A = \{\{a_{11},\ldots,a_{1n}\}, \{a_{21},\ldots,a_{2n}\},\{a_{31},\ldots,a_{3n}\},\{a_{41},\ldots,a_{4n}\}\}$ of length n, with entries $0,1,-1$ so that exactly one of $\{a_{1j},a_{2j},a_{3j},a_{4j}\}$ is ± 1 (three are zero) for $j = 1,\ldots,n$ and with zero nonperiodic autocorrelation function, that is, $N_A(j) = 0$ for $j = 1,\ldots,n-1$, where

$$N_A(j) = \sum_{i=1}^{n-j}(a_{1i},a_{1,i+j} + a_{2i}a_{2,i+j} + a_{3i}a_{3,i+j} + a_{4i}a_{4,i+j}),$$

are called *T-sequences.*

T-matrices are a slightly weaker structure than T-sequences, being defined on finite abelian groups rather than the infinite cyclic group. They are known for a few important small orders, for example, 61 and 67 [36, 75] for which no T-sequences are yet known. Sequences are discussed extensively in Section 5. They are also known for even orders t for which no T-sequences of length t are known [53].

The following result, in a slightly different form, was also discovered by R. J. Turyn. It is the single, most useful method for constructing $OD(4n;n,n,n,n)$, that is, matrices to "plug into."

Theorem 3.6 (Cooper–J. Wallis [12]). *Suppose there exist circulant T-matrices (T-sequences) $X_i, i = 1, \ldots, 4$, of order n. Let a, b, c, d be commuting variables. Then*

$$\begin{aligned} A &= aX_1 + bX_2 + cX_3 + dX_4, \\ B &= -bX_1 + aX_2 + dX_3 - cX_4, \\ C &= -cX_1 - dX_2 + aX_3 + bX_4, \\ D &= -dX_1 + cX_2 - bX_3 + aX_4, \end{aligned}$$

can be used in the Goethal-Seidel (or J. Wallis-Whiteman) array to obtain an $\mathrm{OD}(4n; n, n, n, n)$ *and an Hadamard matrix of order* $4n$.

Corollary 3.7. *If there are T-matrices of order t, then there is an* $\mathrm{OD}(4t; t, t, t, t)$.

The results on T-matrices and T-sequences as applied to Hadamard matrices are given in Section 5.

The appropriate theorem for the construction of Hadamard matrices (it is implied by Williamson, Baumert-Hall, Welch, Cooper–J. Wallis, Turyn) is

Theorem 3.8. *Suppose that there exists an* $\mathrm{OD}(4t; t, t, t, t)$ *and four suitable matrices A, B, C, D of order w that satisfy*

$$AA^T + BB^T + CC^T + DD^T = 4wI_w.$$

Then there is an Hadamard matrix of order $4wt$.

Williamson matrices (which are discussed further in a later section) are suitable matrices for $\mathrm{OD}(4t; t, t, t, t)$, and as such, Williamson matrices are plugged into the OD.

Corollary 3.9. *If there are circulant T-matrices of order t and there are Williamson matrices of order w, there is an Hadamard matrix of order $4tw$. Alternatively, if there are an* $\mathrm{OD}(4t; t, t, t, t)$ *and Williamson matrices of order w, there is an Hadamard matrix of order $4tw$.*

We modify a construction of Turyn to obtain the first theorem which capitalized on M-structures. The $\mathrm{OD}(4s; u_1, \ldots, u_n)$ of the next theorem is an M-structure of which the Welch and Ono-Sawade-Yamamoto arrays are powerful examples.

Theorem 3.10 (Seberry-Yamada-Turyn [87, 108]). *Suppose that there are T-matrices of order t. Further suppose that there is an* $\mathrm{OD}(4s; u_1, \ldots, u_n)$ *constructed of 16 circulant (or type one) $s \times s$ blocks on the variables $x_1, \ldots, x_n$.*

Then there is an OD$(4st;tu_1,\ldots,tu_n)$. *In particular, if there is an* OD$(4s;s,s,s,s)$ *constructed of* 16 *circulant (or type one)* $s \times s$ *blocks, then there is an* OD$(4st;st,st,st,st)$.

Proof. We write the OD as (N_{ij}), $i,j = 1,2,3,4$, where each N_{ij} is circulant (or type one). Hence, we are considering the OD purely as an M-structure. Since we have an OD,

$$N_{i1}N_{j1}^T + N_{i2}N_{j2}^T + N_{i3}N_{j3}^T + N_{i4}N_{j4}^T = \begin{cases} \sum_{k=1}^4 u_k x_k^2 I_s, & i = j; \\ 0, & i \neq j. \end{cases}$$

Suppose that the T-matrices are T_1,T_2,T_3,T_4. Then form the matrices

$$\begin{aligned} A &= T_1 \times N_{11} + T_2 \times N_{21} + T_3 \times N_{31} + T_4 \times N_{41}, \\ B &= T_1 \times N_{12} + T_2 \times N_{22} + T_3 \times N_{32} + T_4 \times N_{42}, \\ C &= T_1 \times N_{13} + T_2 \times N_{23} + T_3 \times N_{33} + T_4 \times N_{43}, \\ D &= T_1 \times N_{14} + T_2 \times N_{24} + T_3 \times N_{34} + T_4 \times N_{44}. \end{aligned}$$

Now

$$AA^T + BB^T + CC^T + DD^T = t\sum_{k=1}^{4} u_k x_k^2 I_{st},$$

and since A,B,C,D are type one, they can be used in the J. Wallis-Whiteman generalization of the Goethals-Seidel array to obtain the result. □

Use the Welch and Ono-Sawade-Yamamoto arrays to see

Corollary 3.11. *Suppose that the T-matrices are of order t. Then there are orthogonal designs* OD$(20t;5t,5t,5t,5t)$ *and* OD$(36t;9t,9t,9t,9t)$.

Note that to prove the Hadamard conjecture "there is an Hadamard matrix of order $4t$ for all $t > 0$," it would be sufficient to prove:

Conjecture 3.12. *There exists an* OD$(4t;t,t,t,t)$ *for every positive integer* t.

The most encompassing theorem presently known, in that it gives a result for every odd q, is proved using a "plug in" technique:

Theorem 3.13 (Seberry [121]). *Let q be any odd natural number. Then there exists an integer $t \leq [2\log_2(q-3)] + 1$ so that there is an Hadamard matrix of order $2^t q$. (The best known bounds are $t \leq [\log_2(q-3)(q-7) - 1]$ for q (prime) $\equiv 3 \pmod 4$ and $t \leq [\log_2(q-1)(q-5)] + 1$ for p (prime) $\equiv 1 \pmod 4$.)*

The proof of this theorem allows a number of cases of interest and stronger results in some cases where q is not prime.

Corollary 3.14 (Seberry [121]). *Let q be any odd natural number. Then there exists a regular symmetric Hadamard matrix with constant diagonal of order $2^{2t}q^2$, $t \leq [2\log_2(q-3)]+1$.*

Corollary 3.15 (Seberry, unpublished).

1. *Let p and $p+2$ be twin prime powers. Then there exists a $t \leq [\log_2(p+3)(p-1)(p^2+2p-7)]-2$ so that there is an Hadamard matrix of order $2^t p(p+2)$.*
2. *Let $p+1$ be the order of a symmetric Hadamard matrix. Then there exists a $t \leq [\log_2(p-3)(p-7)]-2$ so that there is an Hadamard matrix of order $2^t p$.*

Corollary 3.16 [81]. *Let pq be an odd natural number. Suppose that all $\mathrm{OD}(2^s p; 2^r a, 2^r b, 2^r c)$ exist, $s \geq s_0$, $2^{s-r}p = a+b+c$. Then there exists an Hadamard matrix of order $2^t \cdot p \cdot q$, $s \leq t \leq [2\log_2((q-3)/p)]+r+1$. (The best-known bounds are $s \leq t \leq [\log_2((q-3)(q-7)/p)]-1+r$ for q (prime) $\equiv 3 \pmod 4$ and $st \leq [\log_2((q-1)(q-5)/p)]+r+1$ for q (prime) $\equiv 1 \pmod 4$.)*

Example 3.1. Often we can find better results than indicated by Theorem 3.13. Let $q = 3 \cdot 491$. We know there is an Hadamard matrix of order 12. Now, using the proof of Theorem 3.13, rather than the enunciation, we can find an Hadamard matrix of order $2^{15} \cdot 491$. So there is an Hadamard matrix of order $2^{16} \cdot 3 \cdot 19$ using the multiplication theorem. On the other hand, the proof of the corollory gives an Hadamard matrix of order $2^{13} \cdot 3 \cdot 491$ using the $\mathrm{OD}(2^{12} \cdot 3; 22, 3, 2^{12} \cdot 3 - 25)$.

Other similar results are known. The Appendix gives an indication of the smallest t for each odd natural number q for which an Hadamard matrix is known. A list of the construction methods used is given in Section A.3 of the Appendix.

Theorem 3.13 changes ideas for evaluating construction methods: We consider a method to be more powerful if it lowers the power of two for the resultant odd number. Thus, Agayan's theorem, which gives Hadamard matrices of order $8mn$ from Hadamard matrices of order $4m$ and $4n$, is more powerful than that of Hadamard, which gives a matrix of order $16mn$.

We now see another way to lower the power in a multiplication method. First, we introduce some notation.

Let $M = (M_{ij})$ and $N = (N_{gh})$ be orthogonal matrices or t^2 block M-structures of orders tm and tn, respectively, where M_{ij} is of order m $(i,j = 1,2,\ldots,t)$ and N_{gh} is of order n $(g,h = 1,2,\ldots,t)$.

We now define the operation $\bigcirc$ as the following:

$$M \bigcirc N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ \vdots & & & \vdots \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix},$$

where L_{ij} is of order of mn, and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

$i, j = 1, 2, \ldots, t$. We call this the *strong Kronecker* multiplication of two matrices. We note that the strong Kronecker product preserves orthogonality but not necessarily with entries in a useful form (i.e. equal to $0, \pm 1$).

Theorem 3.17. *Let A be an $\mathrm{OD}(tm; p_1, \ldots, p_u)$ with entries $x_1, \ldots, x_u$, and let B be an $\mathrm{OD}(tn; q_1, \ldots, q_s)$ with entries $y_1, \ldots, y_s$, then*

$$(A \bigcirc B)(A \bigcirc B)^T = \left(\sum_{j=1}^{u} p_j x_j^2 \right) \left(\sum_{j=1}^{s} q_j y_j^2 \right) I_{tmn}.$$

($A \bigcirc B$ is not an orthogonal design but an orthogonal matrix.) If A is a $W(tm, p)$ and B is a weighing matrix $W(tn, q)$, then $A \bigcirc B = C$ satisfies $CC^T = pqI_{tmn}$.

Hereafter, let $H = H_{ij}$ and $N = (N_{ij})$ of order $4h$ and $4n$, respectively, be 16 block M-structures. So

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix},$$

where

$$\sum_{j=1}^{4} H_{ij} H_{ij}^T = 4hI_h = \sum_{j=1}^{4} H_{ji} H_{ji}^T,$$

for $i = 1, 2, 3, 4$, and

$$\sum_{j=1}^{4} H_{ij} H_{kj}^T = 0 = \sum_{j=1}^{4} H_{ji}^T H_{jk},$$

for $i \neq k$, $i, k = 1, 2, 3, 4$, and similarly for N.

For ease of writing, we define $X_i = \frac{1}{2}(H_{i1} + H_{i2})$, $Y_i = \frac{1}{2}(H_{i1} - H_{i2})$, $Z_i = \frac{1}{2}(H_{i3} + H_{i4})$, and $W_i = \frac{1}{2}(H_{i3} - H_{i4})$, where $i = 1,2,3,4$. Then both $X_i \pm Y_i$ and $Z_i \pm W_i$ are $(1,-1)$ matrices with $X_i \wedge Y_i = 0$ and $Z_i \wedge W_i = 0$, where $\wedge$ is the Hadamard product.

Let

$$S = \begin{bmatrix} X_1 & -Y_1 & Z_1 & -W_1 \\ X_2 & -Y_2 & Z_2 & -W_2 \\ X_3 & -Y_3 & Z_3 & -W_3 \\ X_4 & -Y_4 & Z_4 & -W_4 \end{bmatrix}.$$

Obviously, S is a $(0,1,-1)$ matrix.

Write

$$R = \begin{bmatrix} Y_1 & X_1 & W_1 & Z_1 \\ Y_2 & X_2 & W_2 & Z_2 \\ Y_3 & X_3 & W_3 & Z_3 \\ Y_4 & X_4 & W_4 & Z_4 \end{bmatrix},$$

also a $(0,1,-1)$ matrix.

We note $S \pm R$ is a $(1,-1)$ matrix, $R \wedge S = 0$, and by the previous theorem,

$$SS^T = RR^T = 2hI_{4h}.$$

Lemma 3.18. *If there exists an Hadamard matrix of order* $4h$, *there exists an* $\mathrm{OD}(4h; 2h, 2h)$.

Proof. Form S and R as above. Now $H = S + R$. Note that $HH^T = SS^T + RR^T + SR^T + RS^T = 4hI_{4h}$ and $SS^T = RR^T = 2hI_{4h}$. Hence, $SR^T + RS^T = 0$. Let x and y be commuting variables; then $E = xS + yR$ is the required orthogonal design. □

In fact, exploiting the strong Kronecker product, Seberry and Zhang show

Lemma 3.19. *If there exist Hadamard matrices of order* $4h$ *and* $4n$, *there exists a* $W(4hn, 2hn)$. *If there exists an Hadamard matrix of order* $4h$, *there exists a* $W(4h, 2h)$ $(h > 1)$.

Theorem 3.20. *Suppose that* $4h$ *and* $4n$ *are the orders of Hadamard matrices; then there exist two disjoint amicable* $W(4hn, 2hn)$ *whose sum and difference are* $(1,-1)$ *matrices. Suppose that there exists an Hadamard matrix of order* $4h$; *then there exists disjoint amicable* $W(4h, 2h)$ *whose sum and difference are* $(1,-1)$ *matrices.*

We now proceed to use the idea of *orthogonal pairs* or ± 1 matrices, S and P of order n, satisfying

1. $SS^T + PP^T = 2nI_n$,
2. $SP^T = PS^T = 0$,

first introduced by R. Craigen [13] who showed

Lemma 3.21 (Craigen). *If there exist Hadamard matrices of order* $4p$ *and* $4q$, *then there exist two* $(1,-1)$ *matrices,* S *and* P *of order* $4pq$, *satisfying*

1. $SS^T + PP^T = 8pqI_{4pq}$,
2. $SP^T = PS^T = 0$.

Proof. By Theorem 3.20, there exist two $W(4pq, 2pq)$, X and Y, satisfying $X \wedge Y = 0$; $X \pm Y$ is a $(1,-1)$ matrix, and $XY^T = YX^T$. Let $S = X + Y, P = X - Y$. Then both S and P are $(1,-1)$ matrices of order $4pq$. Note that

$$SS^T + PP^T = 2(XX^T + YY^T) = 8pqI_{4pq}$$

and

$$SP^T = XX^T - YY^T = 0.$$

Similarly, $PS^T = 0$. So S and P are the required matrices. □

These results can be combined to give

Theorem 3.22 (Craigen-Seberry-Zhang [14]). *If there exist Hadamard matrices of order* $4m, 4n, 4p, 4q$, *then there exists an Hadamard matrix of order* $16mnpq$.

Proof. Let U, V be amicable $W(4mn, 2mn)$ constructed in Theorem 3.20. By Lemma 3.21, there exist two $(1,-1)$ matrices S and P of order $4pq$ satisfying conditions 1 and 2 in Lemma 3.21.

Let $H = U \times S + V \times P$. Then H is a $(1,-1)$ matrix, and

$$HH^T = UU^T \times SS^T + VV^T \times PP^T = 2mnI_{4mn}(SS^T + PP^T)$$

$$= 2mnI_{4mn} \times 8pqI_{4pq} = 16mnpqI_{16mnpq}.$$

Thus H is the required Hadamard matrix. □

The theorem gives an improvement and extension for the result of Agayan [1] that if there exist Hadamard matrices of order $4m$ and $4n$, then there exists an Hadamard matrix of order $8mn$, since using Agayan's theorem repeatedly on four Hadamard matrices of order $4m, 4n, 4p, 4q$ gives an Hadamard matrix of order $32mnpq$.

$$\begin{bmatrix} x & y \\ y & -x \end{bmatrix} \quad \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \quad \begin{bmatrix} a & b & b & d \\ -b & a & d & -b \\ -b & -d & a & b \\ -d & b & -b & a \end{bmatrix} \quad \begin{bmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{bmatrix}$$

(a) OD(2; 1, 1); (b) OD(4; 1, 1, 1, 1); (c) OD(4; 1, 1, 2); (d) OD(4; 1, 1).

Figure 4.1. Orthogonal designs.

Other similar results exist.

4 ORTHOGONAL DESIGNS AND ASYMPTOTIC EXISTENCE

The primary result regarding the asymptotic existence of Hadamard matrices is the theorem of Seberry Wallis (Theorem 4.11 of this section). In this section we outline the proof of this theorem. We begin this section with a discussion of orthogonal designs. These are key ingredients in the proof of the main theorem.

4.1. Orthogonal Designs

An orthogonal design is a generalization of an Hadamard matrix (see Definition 3.8). First we collect a few preliminary results and give some examples.

Example 4.1. Some small orthogonal designs are shown in Figure 4.1. Notice that Figure 4.1(b) is the Williamson array.

The following lemma gives some properties of orthogonal designs.

Lemma 4.1. *Let D be an orthogonal design* OD($n; u_1, u_2, \ldots, u_t$) *on the commuting variables* $x_1, x_2, \ldots, x_t$. *Then D can be written as*

$$D = x_1 A_1 + x_2 A_2 + \cdots + x_t A_t,$$

where, for each $i, j \in \{1, \ldots, t\}$,

1. A_i *is an* $n \times n$ *matrix with entries* $0, \pm 1$;
2. $A_i A_i^T = u_i I_n$;
3. $A_i A_j^T + A_j A_i^T = 0$, $i \neq j$.

We need one further basic result:

Lemma 4.2. *Let D be an orthogonal design* OD($n; u_1, u_2, \ldots, u_t$), *on the t commuting variables* $x_1, x_2, \ldots, x_t$. *Then the following orthogonal designs exist:*

$$\begin{bmatrix} a & -b & -c & 0 \\ b & a & 0 & c \\ c & 0 & a & -b \\ 0 & -c & b & a \end{bmatrix} \quad \begin{bmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ y & 0 & -x & 0 \\ 0 & y & 0 & -x \end{bmatrix} \quad \begin{bmatrix} x & x & y & y \\ x & -x & y & -y \\ y & y & -x & -x \\ y & -y & -x & x \end{bmatrix} \quad \begin{bmatrix} z & x & 0 & y \\ -x & z & y & 0 \\ 0 & y & -z & -x \\ y & 0 & x & -z \end{bmatrix}$$

(a) OD(4;1,1,1) (b) OD(4;1,1) (c) OD(4;2,2) (d) OD(4;1,1,1)

Figure 4.2. Orthogonal designs.

1. OD($n; u_1, u_2, \ldots, u_i + u_j, \ldots, u_t$) *on* $t-1$ *variables (i.e.,* $u_i + u_j$ *replaces* u_i, u_j, $i \neq j$*)*;
2. OD($n; u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t$) *on* $t-1$ *variables*;
3. OD($2n;\ u_1, u_2, \ldots, u_t$) *on* t *variables*;
4. OD($2n; 2u_1, 2u_2, \ldots, 2u_t$) *on* t *variables*;
5. OD($2n; u_1, u_1, u_2, \ldots, u_t$) *on* $t+1$ *variables*;
6. OD($2n; u_1, u_1, 2u_2, \ldots, 2u_t$) *on* $t+1$ *variables.*

The techniques of this lemma are exhibited in the following example:

Example 4.2. Let D_1 and D_2 be the designs of Figure 4.2(b) and (a), respectively. Applying Lemma 4.2 to these designs gives examples as follows: D_1 is an OD(4;1,1,1,1); letting $b = c$ as in case 1 of Lemma 4.2 gives the OD(4;1, 1,2) design in Figure 4.2(c); letting $d = 0$ as in case 2 gives the OD(4;1,1,1) design in Figure 4.2(a). D_2 is a (2;1,1) design; replacing variables by 2×2 matrices as in cases 3, 4, and 5 gives the designs OD(4;1,1), OD(4;2,2), OD(4;1,1,1), in Figure 4.2(b), (c), and (d), respectively.

Lemma 4.2 now lets us show

Lemma 4.3. *Suppose that for all choices of nonnegative integers* a, b, c *with* $a + b + c = n$, *an orthogonal design* OD($n; a, b, c$) *exists. Then for all choices of nonnegative integers* x, y, z *with* $x + y + z = 2n$, *an orthogonal design* OD($2n; x, y, z$) *exists.*

Proof. Notice first that we make the convention that an OD($n; a, b$) may also be considered as an OD($n; a, b, 0$), and so on.

Let x, y, z be nonnegative integers such that $x + y + z = 2n$, and assume that $0 \leq x \leq y \leq z$, so that $y \leq n$. Four cases arise:

1. Both x and y are even, so we may write $x = 2a, y = 2b$, and $a + b < n$. By hypothesis, an OD($n; a, b, c$) exists, where $c = n - a - b$. Hence, by case 6 of Lemma 4.2, an OD($2n; a, a, 2b, 2c$) exists and, by case 1, an OD($2n; 2a, 2b, 2c$) also exists. This is the design we want.

2. Next, let x be even and y odd, so we may take $x = 2a, y = 2a + l$. Now $a + y = 3a + l$, and $z = 2n - 4a - l$. Since $y \leq z$, we have $3a + l \leq n$. Thus, an $\mathrm{OD}(n; y, a, n - a - y)$ exists, and as before, this means that an $\mathrm{OD}(2n; y, y, 2a, 2n - 2a - 2y)$ also exists. Setting $x_1 = x_4$, we get an $\mathrm{OD}(2n; y, 2a, 2n - 2a - y)$. Since $2a = x$ and $2n - 2a - y = z$, the last design is the required one.
3. If x is odd and y is even, we can take $x = 2a + 1, y = 2b$ and $z = 2t + 1$. Since $x + y + z = 2n$, we have $a + b + t + 1 = n$. Now, by assumption, $a < t$, so $x + b = 2a + b + 1 < n$. Hence, we have the following orthogonal designs: $\mathrm{OD}(n; x, b, n - x - b)$, $\mathrm{OD}(2n; x, x, 2b, 2n - 2x - 2b)$, and $\mathrm{OD}(2n; x, 2b, 2n - x - 2b)$. Since $y = 2b$ and $z = 2n - x - y$, we have the required design.
4. Finally, if x and y are both odd, we let $y = x + 2b$, where $b \geq 0$. Since $x + b \leq n$, we have orthogonal designs

$$\mathrm{OD}(n; x, b, n - x - b), \qquad \mathrm{OD}(2n; x, x, 2b, 2n - 2x - 2b),$$

and finally, $\mathrm{OD}(2n; x, x + 2b, 2n - 2x - 2b)$, as required. □

Corollary 4.4. *If x, y, z are nonnegative integers such that $x + y + z = 2^m$, then an orthogonal design $\mathrm{OD}(2^m; x, y, z)$ exists.*

Proof. From the the array in Figure 4.1(a) and Lemma 4.2, the statement is true for $m = 2$. It then follows from Lemma 4.3 for all $m > 2$. □

Corollary 4.5. *If x, y, are nonnegative integers such that $x + y = 2^m$, then an orthogonal design $\mathrm{OD}(2^m; x, y)$ exists.*

Proof. Apply case 1 of Lemma 4.2 to the $\mathrm{OD}(2^m; x, y, z)$ obtained from the previous corollary. □

4.2. An Existence Theorem for Hadamard Designs

We need one further result from number theory.

Theorem 4.6. *Let x and y be positive integers such that $(x, y) = 1$. Then every integer $N \geq (x - 1)(y - 1)$ can be written as a linear combination $N = ax + by$, where a and b are nonnegative integers.*

Corollary 4.7. *Let z be an odd integer. Then there exist nonnegative integers a and b such that*

$$a(z + 1) + b(z - 3) = n = 2^t$$

for some t.

Proof. If $z \geq 9$, let

$$d = (z+1, z-3) = \begin{cases} 2 & \text{if } z \equiv 1 \pmod 4, \\ 4 & \text{if } z \equiv 3 \pmod 4. \end{cases}$$

Let

$$N = \left(\frac{z+1}{d} - 1\right)\left(\frac{z-3}{d} - 1\right),$$

and choose m so that $2^{m-1} < N \leq 2^m$. By Theorem 4.6 there exist nonnegative integers a and b such that

$$\frac{a(z+1)}{d} + \frac{b(z-3)}{d} = 2^m,$$

and thus

$$a(z+1) + b(z-3) = 2^{m+s},$$

where

$$s = \begin{cases} 1 & \text{if } z \equiv 1 \pmod 4, \\ 2 & \text{if } z \equiv 3 \pmod 4, \end{cases}$$

and $t = m + s$. It is easy to verify that this result also holds for odd $3 \leq z \leq 9$. □

Lemma 4.8. *Let p be a prime, $p \geq 11$. Then there exists a positive integer t such that an Hadamard matrix of size $2^s p$ exists for every $s > t$.*

Proof. Let $x = p+1$ and $y = p-3$. By Corollary 4.7 there exist nonnegative integers a and b such that $ax + by = 2^t = n$ for some t. By Corollary 4.4 there exists an OD$(n; a, b, n-a-b)$ orthogonal design D on the variables x_1, x_2, x_3.

The proof now divides into two cases.

Case 1 $p \equiv 3$ **(mod 4).** We replace each variable in D by a $p \times p$ $(1,-1)$ matrix: x_1 by J_p, x_2 by $J_p - 2I_p$, and x_3 by the back-circulant matrix N formed from the quadratic residues. This gives a $(1,-1)$ matrix E which is an Hadamard matrix of size $np = 2^t p$, and the Lemma follows for $p \equiv 3 \pmod 4$.

Case 2 $p \equiv 1$ **(mod 4).** There exists an OD$(2n; 2a, 2b, n-a-b, n-a-b)$ orthogonal design F on the variables x_1, x_2, x_3, x_4 by identity 4 of Lemma 4.2. We replace each variable in F by a $p \times p$ $(1,-1)$ matrix: x_1 by J_p, x_2 by $J_p - 2I_p$, x_3, and x_4, respectively, by the circulant matrices $X = Q + I$ and $Y =$

$Q - I$ formed from the quadratic residue matrix Q. This gives an $np \times np$ $(1, -1)$ matrix G which is an Hadamard matrix of size $2np = 2^{t+1}p$, and the lemma also follows for $p \equiv 1 \pmod 4$.

This completes the proof for all primes, except 2, 3, 5, and 7. □

Lemma 4.9. *There exist Hadamard matrices of sizes 2^t for all $t \geq 1$, and $2^t p$ for all $t \geq 2$ and $p = 3, 5, 7$.*

Proof. There exists an Hadamard matrix of size 2^t for $t \geq 1$.

By Sylvester's multiplication theorem, if there exist Hadamard matrices of sizes 12, 20, and 28, then there exist Hadamard matrices of sizes $2^t p$ for all $t \geq 2$ and $p = 3, 5, 7$.

Hadamard matrices of these orders are obtained by the Paley construction. □

Theorem 4.10. *Let q be any positive integer. Then there exists $t = t(q)$ such that an Hadamard matrix of size $2^s q$ exists for every $s \geq t$.*

Proof. We apply Lemma 4.8 and/or Lemma 4.9 to each prime factor of q. Since a Kronecker product of Hadamard matrices is an Hadamard matrix, the result follows. □

Theorem 4.11 (Seberry Wallis [121]). *Let q be any positive integer, then there exists an Hadamard matrix of order $2^s q$ for every $s \geq [2\log_2(q-3)]$.*

Proof. By the proof of Corollary 4.7, we can choose t so that

$$2^t \geq \left(\frac{z+1}{d} - 1\right)\left(\frac{z-3}{d} - 1\right),$$

where z is an odd prime and $d = (z+1, z-3)$.

If $z \equiv 1 \pmod 4$, then $d = 2$ and we must have

$$2^t \geq \frac{(z-1)(z-5)}{4}.$$

Since

$$(z-3)^2 > (z-1)(z-5),$$

it is sufficient to ensure that

$$2^{t+2} > (z-3)^2;$$

that is,

$$t + 2 > 2\log_2(z - 3).$$

Since t is an integer, we may choose

$$t = [2\log_2(z - 3)] - 1.$$

Similarly, if $z \equiv 3 \pmod 4$, then $d = 4$, and we may choose

$$t = [2\log_2(z - 5)] - 3.$$

As in the proof of Lemma 4.8, these choices of t ensure the existence of an Hadamard matrix of size $2^t z$.

If $z = pq$ where p and q are primes, $p \equiv 1 \pmod 4$, $q \equiv 1 \pmod 4$, then there exists an Hadamard matrix of size $2^r pq$, where

$$r = [2\log_2(p - 3)] + [2\log_2(q - 3)] < [2\log_2(pq - 3)].$$

Analogously, if $z = \prod_i p_i$ for p_i prime and $p_i \equiv 1 \pmod 4$, then

$$r = \sum_i 2\log_2(p_i - 3) < 2\log_2\left(\prod_i (p_i - 3)\right)$$

Since an integer z that is a product of primes congruent to 1 (mod 4) gives the greatest lower bound on the value of t for which we know an Hadamard matrix of size $2^t z$ exists, we have proved the theorem. □

We note that better bounds (i.e., smaller r) can be obtained if not all primes in the decomposition of z are congruent to 1 (mod 4). We use the equivalence of Hadamard matrices and Hadamard designs to obtain the following corollary:

Corollary 4.12. *Let λ be any positive integer; then there exists an $s \geq 0$ so that an SBIBD$(2^{s+2}\lambda - 1, 2^{s+1}\lambda - 1, 2^s\lambda - 1)$ exists.*

In fact, as was indicated in Theorem 3.13, the value of s in Theorem 4.11 is slightly smaller if the proof is applied carefully.

4.3. Orthogonal Designs in Order 24

In this section, we discuss the particular case of orthogonal designs of order 24. In so doing, we demonstrate how the power of s in Theorem 4.11 can be reduced in specific cases.

The following is an OD(12; 1, 2, 3, 6) on the variables A, B, C, D:

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
A & B & -B & C & B & B & C & -B & D & B & D & -C \\
-B & A & B & B & B & C & -B & D & C & D & -C & B \\
B & -B & A & B & C & B & D & C & -B & -C & B & D \\
\hline
-C & -B & -B & A & B & -B & -B & C & -D & C & D & -B \\
-B & -B & -C & -B & A & B & C & -D & -B & D & -B & C \\
-B & -C & -B & B & -B & A & -D & -B & C & -B & C & D \\
\hline
-C & B & -D & B & -C & D & A & B & -B & -C & -B & -B \\
B & -D & -C & -C & D & B & -B & A & B & -B & -B & -C \\
-D & -C & B & D & B & -C & B & -B & A & -B & -C & -B \\
\hline
-B & -D & C & -C & -D & B & C & B & B & A & B & -B \\
-D & C & -B & -D & B & -C & B & B & C & -B & A & B \\
C & -B & -D & B & -C & -D & B & C & B & B & -B & A
\end{array}\right]$$

Hence, there exists (equating variables) an OD(12; 4, 8).

Now, by identity 6 of Lemma 4.2, there are OD(24; 2, 4, 3, 3, 12), OD(24; 4, 4, 16), OD(24; 8, 8, 8), and OD(24; 1, 1, 4, 6, 12), giving

OD(24; 2, 4, 18);

OD(24; 3, a, 21 − a), $a = 3, 4, 5, 6, 7$;

OD(24; 4, a, 20 − a), $a = 4, 5, 6, 7, 8$;

OD(24; 8, 8, 8).

Robinson [72] has found OD(24; 1, 1, 1, 1, 1, 5, 5, 9) and OD(24; 1, 1, 1, 1, 1, 2, 8, 9) from which, by equating variables, all other OD(24; $x, y, 24 - x - y$) may be obtained.

Consider the following matrices, M_1 and M_2: (we use the convention that $\overline{x} = -x$):

$$M_1 = \begin{array}{c|c|c|c|c|c}
e & d\bar{h}f\bar{g} & gfhh & f\bar{g}\bar{h}h & \bar{g}\bar{f}\bar{h}\bar{h} & gfhh \\
\hline
\bar{d}h\bar{f}\bar{g} & \bar{e} & fghh & \bar{g}fh\bar{h} & \bar{g}fh\bar{h} & g\bar{f}\bar{h}h \\
\hline
\bar{g}\bar{f}\bar{h}h & \bar{f}\bar{g}\bar{h}h & g & dhef & \bar{h}hgg & hh\bar{f}f \\
\hline
\bar{f}ghh & g\bar{f}\bar{h}\bar{h} & \bar{d}\bar{h}\bar{e}f & \bar{g} & h\bar{h}ff & hhg\bar{g} \\
\hline
gfh\bar{h} & g\bar{f}\bar{h}\bar{h} & h\bar{h}\bar{g}g & \bar{h}h\bar{f}f & f & dhge \\
\hline
\bar{g}\bar{f}\bar{h}h & \bar{g}fhh & \bar{h}\bar{h}ff & \bar{h}\bar{h}\bar{g}\bar{g} & \bar{d}\bar{h}\bar{g}e & \bar{f}
\end{array}$$

	e	$d\bar{f}h\bar{f}$	$hhgg$	$h\bar{h}\bar{g}g$	$\bar{h}\bar{g}\bar{h}\bar{g}$	$hghg$
	$\bar{d}f\bar{h}\bar{f}$	$\bar{e}$	$hhgg$	$\bar{h}hg\bar{g}$	$h\bar{g}\bar{h}g$	$h\bar{g}\bar{h}g$
$M_2 =$	$\bar{h}\bar{h}\bar{g}g$	$\bar{h}\bar{h}\bar{g}g$	g	$dgeh$	$\bar{g}ghh$	$hh\bar{f}f$
	$\bar{h}hgg$	$h\bar{h}\bar{g}\bar{g}$	$\bar{d}\bar{g}\bar{e}h$	$\bar{g}$	$h\bar{h}ff$	$ggh\bar{h}$
	$hgh\bar{g}$	$\bar{h}ghg$	$g\bar{g}\bar{h}h$	$\bar{h}h\bar{f}f$	g	$dghe$
	$\bar{h}\bar{g}\bar{h}g$	$\bar{h}ghg$	$\bar{h}\bar{h}ff$	$\bar{g}\bar{g}\bar{h}\bar{h}$	$\bar{d}ghe$	$\bar{g}$

Let N_1 and N_2 be the matrices obtained from M_1 and M_2 by replacing the diagonal entries, y, of M_i by

$$\begin{matrix} a & b & c & y \\ \bar{b} & a & y & \bar{c} \\ \bar{c} & \bar{y} & a & b \\ \bar{y} & c & \bar{b} & a \end{matrix}$$

and the off-diagonal block entries p,q,r,s of M_i by

$$\begin{matrix} p & q & r & s \\ q & \bar{p} & s & \bar{r} \\ r & s & \bar{p} & q \\ s & \bar{r} & q & p. \end{matrix}$$

Then N_1 and N_2 give orthogonal designs of order 24 and types $(1,1,1,1,1,5,5,9)$ and $(1,1,1,1,1,2,8,9)$, respectively.

Hence, we have

Lemma 4.13 (P. Robinson [72]). *All three-tuples (x,y,z), $x+y+z=24$, are the types of orthogonal designs in order 24. That is, all $\mathrm{OD}(24;x,y,24-x-y)$ exist.*

Proceeding as in Theorem 4.10 we obtain

Theorem 4.14. *Let q be a positive integer. Then there exists a $t=t(q)$ so that there is an Hadamard matrix of order $2^s \cdot 3 \cdot q$ for all $s \geq t$.*

Remark. A few other results of the kind in this section are known for orders $4 \cdot p \cdot q$ and $3 < p \leq 11$. The importance of this result lies in the fact that the power s will be smaller than the power t obtained from Theorem 3.13 (see [81]).

5 SEQUENCES

A special orthogonal design, the OD$(4t;t,t,t,t)$, is especially useful in constructing Hadamard matrices. An OD$(12;3,3,3,3)$ was first found by Baumert-Hall [6] and an OD$(20;5,5,5,5)$ by Welch. These were given in Section 3. OD$(4t;t,t,t,t)$ are sometimes called *Baumert-Hall arrays*. This chapter concentrates on the powerful construction techniques for these OD$(4t;t,t,t,t)$ using disjoint orthogonal matrices and sequences with zero autocorrelation.

Since we are concerned with orthogonal designs, we will consider sequences of commuting variables. Let $X = \{\{a_{11},\ldots,a_{1n}\},\{a_{21},\ldots,a_{2n}\}\ldots\{a_{m1},\ldots,a_{mn}\}\}$ be m sequences of commuting variables of length n. The *nonperiodic autocorrelation function of the family of sequences* X (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j}(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \cdots + a_{m,i}a_{m,i+j}).$$

Early work of Golay [28, 29] was concerned with two $(1,-1)$ sequences with zero nonperiodic autocorrelation function, but Welti [123], Tseng [101], and Tseng and Liu [102] approached the subject from the point of view of two orthonormal vectors, each corresponding to one of two orthogonal waveforms. Later work, including Turyn's [108, 107], used four or more sequences.

Note that if the following collection of m matrices of order n is formed,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ 0 & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ 0 & & & a_{21} \end{bmatrix}, \ldots,$$

$$\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \\ & a_{m1} & & a_{m,n-1} \\ & & \ddots & \\ 0 & & & a_{m1} \end{bmatrix},$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

The *periodic autocorrelation function of the family of sequences* X (denoted P_X) is a function defined by

$$P_X(j) = \sum_{i=1}^{n}(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \cdots + a_{m,i}a_{m,i+j}),$$

where we assume the second subscript is actually chosen from the complete set of residues (mod n).

We can interpret the function P_X in the following way: Form the m circulant matrices that have first rows, respectively,

$$[a_{11}a_{12}\ldots a_{1n}],[a_{21}a_{22}\ldots a_{2n}],\ldots,[a_{m1}a_{m2}\ldots a_{mn}];$$

then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices. In these matrices, all a_{ij} are chosen from the set $\{0,1,-1\}$.

We say the *weight* of a set of sequences X is the number of nonzero entries in X. If X is as above with $N_X(j)=0$, $j=1,2,\ldots,n-1$, then we will call X *m-complementary sequences* of length n. If

$$X=\{A_1,A_2,\ldots,A_m\}$$

are m-complementary sequences of length n and weight $2k$ such that

$$Y=\left\{\frac{(A_1+A_2)}{2},\frac{(A_1-A_2)}{2},\ldots,\frac{(A_{2i-1}+A_{2i})}{2},\frac{(A_{2i-1}-A_{2i})}{2},\ldots\right\}$$

are also m-complementary sequences (of weight k), then X will be said to be *m-complementary disjointable sequences* of length n. X will be said to be *m-complementary disjoint sequences* of length n if all $\binom{m}{2}$ pairs of sequences are disjoint.

For example $\{1\ 1\ 0\ 1\}$, $\{0\ 0\ 1\ 0\ -1\}$, $\{0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ -1\}$, $\{0\ 0\ 0\ 0\ 0\ 0\ 1\ -1\}$ are disjoint as they have zero nonperiodic autocorrelation function and precisely one $a_{ij}\neq 0$ for each j.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r, then by xg_r we mean the sequence of integers of length r obtained from g_r by multiplying each member of g_r by x.

Proposition 5.1. *Let X be a family of m sequences of commuting variables. Then*

$$P_X(j)=N_X(j)+N_X(n-j),\qquad j=1,\ldots,n-1.$$

Corollary 5.2. *If $N_X(j)=0$ for all $j=1,\ldots,n-1$, then $P_X(j)=0$ for all $j=1,\ldots,n-1$.*

Note: $P_X(j)$ may equal 0 for all $j=1,\ldots,n-1$, even though the $N_X(j)$ do not.

If $X=\{\{a_1,\ldots,a_n\},\{b_1,\ldots,b_n\}\}$ are two sequences where $a_i,b_j\in\{1,-1\}$ and $N_X(j)=0$ for $j=1,\ldots,n-1$, then the sequences in X are called *Golay complementary sequences of length n*. For example, writing $-$ for minus 1, we have

$n = 2$	11 and 1−
$n = 10$	1−−1−1−−−1 and 1−−−−−−11−
$n = 26$	111−−111−1−−−−−1−11−−1−−−− and
	−−−11−−−1−11−1−1−11−−1−−−−.

We note that if X is as above, if A is the circulant matrix with first row $\{a_1,\ldots,a_n\}$, and, if B the circulant matrix with first row $\{b_1,\ldots,b_n\}$, then

$$AA^T + BB^T = \sum_{i=1}^{n}(a_i^2 + b_i^2)I_n = 2nI_n.$$

Consequently, such matrices may be used to obtain Hadamard matrices constructed from two circulants.

We would like to use Golay sequences to construct other orthogonal designs, but first we consider some of their properties.

Lemma 5.3. *Let $X = \{\{a_1,\ldots,a_n\},\{b_1,\ldots,b_n\}\}$ be Golay complementary sequences of length n. Suppose that k_1 of the a_i are positive and k_2 of the b_i are positive. Then*

$$n = (k_1 + k_2 - n)^2 + (k_1 - k_2)^2,$$

and n is even.

Proof. Since $P_X(j) = 0$ for all j, we may consider the two sequences as $2-\{n;k_1,k_2;\lambda\}$ supplementary difference sets with $\lambda = k_1 + k_2 - \frac{1}{2}n$. But the parameters (counting differences two ways) satisfy $\lambda(n-1) = k_1(k_1-1)+ k_2(k_2-1)$. On substituting λ in this equation we obtain the result of the enunciation. □

Geramita and Seberry [23, pp. 133–137], Andres [2] and James [38] have studied the smaller values of n, k_1, k_2 of the lemma, showing the only lengths ≤ 68 for which Golay sequences exist are 2, 4, 8, 10, 16, 20, 26, 32, 40, 52, and 64. Malcolm Griffin [30] has shown no Golay sequences can exist for lengths $n = 2\cdot 9^t$. The value $n = 18$, which was previously excluded by a complete search, is now theoretically excluded by Griffin's theorem and independently by a result of Kruskal [62] and C. H. Yang [133, 134]. Andres [2] and James [38] have found greatly improved computer algorithms for studying these sequences.

Recent theoretical work of Koukouvinos, Kounias, and Sotirakoglou [50] and Eliahou, Kervaire, and Saffari [20] shows that Golay sequences do not exist for $n = 2p$ where p has any prime factor $\equiv 3 \pmod 4$. This means the unresolved cases < 200 are $n = 74, 82, 106, 116, 122, 130, 136, 146, 148, 164, 170, 178, 194$.

Constraints can be found on the elements of a Golay sequence. One useful result (see Geramita and Seberry [23]) is

Lemma 5.4. *For Golay sequences* $X = \{\{x_i\},\{y_i\}\}$ *of length* n,

$$x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i,$$

where $e_i = \pm 1$. *That is,*

$$x_{n-i+1} x_i = -y_{n-i+1} y_i.$$

Example 5.1. The sequences of length 10 are

$$1--1-1---1 \text{ and}$$
$$1------11-.$$

Clearly, $e_1 = 1$, $e_2 = 1$, $e_3 = 1$, $e_4 = -1$, and $e_5 = -1$.

Proof (of Lemma 5.4*).* We use the fact that if x,y,z are $\pm 1, (x+y)z \equiv x+y \pmod 4$ and $x+y \equiv xy+1 \pmod 4$.

Let $i = 1$. Clearly, the result holds. We proceed by induction. Suppose that the result is true for every $i \le k-1$. Now consider $N(k) = N(n-k) = 0$, and we have

$$\begin{aligned}
0 &= x_1 x_{n+1-k} + x_2 x_{n+2-k} + \cdots + x_k x_n + y_1 y_{n+1-k} + y_2 y_{n+2-k} + \cdots + y_k y_n \\
&= x_1 e_k x_k + x_2 e_{k-1} x_{k-1} + \cdots + x_k e_1 x_1 + y_1 y_{n+1-k} - y_2 e_{k-1} y_{k-1} \\
&\quad - \cdots - y_k e_1 y_1 \\
&\equiv e_1 + e_2 + \cdots + e_k + y_1 y_{n+1-k} - e_{k-1} - \cdots - e_2 - y_k e_1 y_1 \pmod 4 \\
&\equiv e_1 + e_k + y_1 y_{n+1-k} - y_k e_1 y_1 \pmod 4 \\
&\equiv e_k + y_k y_{n+1-k} \pmod 4 \\
&\equiv 0 \pmod 4.
\end{aligned}$$

So $y_{n+1-k} = -e_k y_k$. □

5.1. Summary of Golay Properties

Two sequences $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ are called *Golay complementary sequences* of length n if all their entries are ± 1 and

$$\sum_{i=1}^{n-j} (x_i x_{i+j} + y_i y_{i+j}) = 0 \qquad \text{for every} \quad j \neq 0, \quad j = 1,\ldots,n-1,$$

that is, $N_X = 0$. These sequences have the following properties:

1. $\sum_{i=1}^{n}(x_i x_{i+j} + y_i y_{i+j}) = 0$ for every $j \neq 0$, $j = 1, \ldots, n-1$ (where the subscripts are reduced modulo n), i.e., $P_X = 0$.
2. n is even and the sum of two squares.
3. $x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i$, where $e_i = \pm 1$.
4.
$$\left[\sum_{i\in S} x_i \operatorname{Re}(\zeta^{2i+1})\right]^2 + \left[\sum_{i\in D} x_i \operatorname{Im}(\zeta^{2i+1})\right]^2 + \left[\sum_{i\in S} y_i \operatorname{Im}(\zeta^{2i+1})\right]^2 + \left[\sum_{i\in D} y_i \operatorname{Re}(\zeta^{2i+1})\right]^2 = \frac{1}{2}n,$$
   where $S = \{i : 0 \le i < n, e_i = 1\}$, $D = \{i : 0 \le i < n, e_i = -1\}$, and ζ is a $2n$th root of unity (Griffin [30]).
5. They exist for orders $2^a 10^b 26^c$, a, b, c nonnegative integers.
6. They do not exist for orders $2 \cdot 9^c$ (c a positive integer) (Griffin [30]), or for orders 34, 36, 50, 58, or 68.
7. They do not exist for orders $2 \cdot 49^c$ (c a positive integer) (Koukouvinos, Kounias, and Sotirakoglou [50]).
8. They do not exist for orders $2p$ where p has any prime factor $\equiv 3 \pmod 4$ (Eliahou, Kervaire, and Saffari [20]).

We now discuss other sequences with zero autocorrelation function.

5.2. Other Sequences with Zero Autocorrelation Function

Lemma 5.5. *Suppose that $X = \{X_1, X_2, \ldots, X_m\}$ is a set of $(0, 1, -1)$ sequences of length n for which $N_X = 0$ or $P_X = 0$. Further suppose that the weight of X_i is x_i and the sum of the elements of X_i is a_i. Then*

$$\sum_{i=1}^{m} a_i^2 = \sum_{i=1}^{m} x_i.$$

Proof. Form circulant matrices Y_i for each X_i. Then

$$Y_i J = a_i J \quad \text{and} \quad \sum_{i=1}^{m} Y_i Y_i^T = \sum_{i=1}^{m} x_i I.$$

Now considering

$$\sum_{i=1}^{m} Y_i Y_i^T J = \sum_{i=1}^{m} a_i^2 J = \sum_{i=1}^{m} x_i J,$$

we have the result. □

Example 5.2. Suppose that X_1, X_2, X_3, X_4 have elements from $+1$ and -1 and lengths 19, 19, 18, 18. The total weight of these sequences is 74. The sum of the squares of the four row sums must be 74, so we could have

$$\begin{array}{ll} 3^2 + 1^2 + 8^2 + 0^2 & 1^2 + 1^2 + 6^2 + 6^2 \\ 7^2 + 5^2 + 0^2 + 0^2 & \text{or} \\ 7^2 + 3^2 + 4^2 + 0^2 & 5^2 + 3^2 + 6^2 + 2^2 \end{array}$$

A row sum of 8 and length 18 would require that there are 13 elements $+1$ and 5 elements -1 considerably shortening any search.

Now a few simple observations are in order. For convenience, we put them together as a lemma—though more has been observed by Whitehead [124].

Lemma 5.6. *Let $X = \{A_1, A_2, \ldots, A_m\}$ be m-complementary sequences of length n. Then*

1. *$Y = \{A_1^*, A_2^*, \ldots, A_i^*, A_{i+1}, \ldots, A_m\}$ are m-complementary sequences of length n where A_i^* means "reverse the elements of A_i";*
2. *$W = \{A_1, A_2, \ldots, A_i, -A_{i+1}, \ldots, -A_m\}$ are m-complementary sequences of length n;*
3. *$Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \ldots, \{A_{2i-1}, A_{2i}\}, \{A_{2i-1}, -A_{2i}\}, \ldots\}$ are m-(or $m+1$- if m is odd, in which case we let A_{m+1} be n zeros) complementary sequences of length $2n$;*
4. *$U = \{\{A_1/A_2\}, \{A_1/-A_2\}, \ldots, \{A_{2i-1}/A_{2i}\}, \{A_{2i-1}/-A_{2i}\}, \ldots\}$, where A_j/A_k means that $a_{j1}, a_{k1}, a_{j2}, a_{k2}, \ldots a_{jn}, a_{kn}$, are m- (or $m+1$- if m is odd, in which case we let A_{m+1} be n zeros) complementary sequences of length $2n$.*
5. *$V = \{A_1^+, A_2^+, \ldots, A_m^+\}$, where $A_i^+ = \{a_{i1}, -a_{i2}, a_{i3}, -a_{i4}, \ldots\}$ are m-complementary sequences of length n.*

By a lengthy but straightforward calculation, it can be shown that

Theorem 5.7. *Suppose that $X = \{A_1, \ldots, A_{2m}\}$ are 2m-complementary sequences of length n and weight u and $Y = \{B_1, B_2\}$ are 2-complementary disjointable sequences of length t and weight $2k$. Then there are 2m-complementary sequences of length nt and weight ku.*

The same result is true if X are $2m$-complementary disjointable sequences of length n and weight $2u$ and Y are 2-complementary sequences of weight k.

Proof. Write X^* for the sequence whose elements are the reverse of those in the sequence X. Using an idea of R. J. Turyn, we consider

$$A_{2i-1} \times \frac{(B_1 + B_2)}{2} + A_{2i} \times \frac{(B_1 - B_2)}{2} \quad \text{and}$$
$$A_{2i-1} \times \frac{(B_1^* - B_2^*)}{2} - A_{2i} \times \frac{(B_1^* + B_2^*)}{2},$$

for $i = 1, \ldots, m$, which are the required sequences in the first case. While

$$\frac{(A_{2i-1} + A_{2i})}{2} \times B_1 + \frac{(A_{2i-1} - A_{2i})}{2} \times B_2^* \quad \text{and}$$
$$\frac{(A_{2i-1} + A_{2i})}{2} \times B_2 - \frac{(A_{2i-1} - A_{2i})}{2} \times B_1^*$$

for $i = 1, \ldots, m$, are the required sequences for the second case. (Note here that $\times$ is the normal Kronecker product.)

The proof now follows by an exceptionally tedious but straightforward verification. □

Corollary 5.8. *Since there are Golay sequences of lengths* 2, 10 *and* 26, *there are Golay sequences of length* $2^a 10^b 26^c$ *for* a, b, c *nonnegative integers.*

Corollary 5.9. *There are 2-complementary sequences of lengths* $2^a 6^b 10^c 14^d 26^e$ *of weights* $2^a 5^b 10^c 13^d 26^e$, *where* a, b, c, d, e *are nonnegative integers.*

Proof. Use the sequences of Tables 5 and 6 of Appendix H of [23]. □

5.3. T-Sequences and Base Sequences

The bulk of the remainder of this chapter will be devoted to obtaining T-sequences. We recall that T-sequences always yield T-matrices. If there are T-sequences of length t and Williamson matrices of order w there is an Hadamard matrix of order $4tw$.

Four sequences of elements $+1, -1$ of lengths $m+p, m+p, m, m$ where p is odd, and which have zero nonperiodic autocorrelation function, are called *base sequences*. In Table 5.1 base sequences are displayed for lengths $m + 1, m+1, m, m$ for $m+1 \in \{2, 3, \ldots, 30\}$. If X and Y are Golay sequences, $\{1, X\}, \{1, -X\}, \{Y\}, \{Y\}$ are base sequences of lengths $m+1, m+1, m, m$. So base sequences exist for all $m = 2^a 10^b 26^c$, a, b, c nonnegative integers, $p = 1$. The cases for $m = 17$, $p = 1$, were found by A. Sproul and J. Seberry; for

$m = 23$, $p = 1$ by R. Turyn; and for $m = 22, 24, 26, 27, 28$, $p = 1$ by Koukouvinos, Kounias, and Sotirakoglou [51]. These sequences are also discussed in Geramita and Seberry [23, pp. 129–148].

Base sequences are crucial to Yang's [138, 135, 136, 137] constructions for finding longer T-sequences of odd length.

Lemma 5.10. *Consider four* $(1,-1)$ *sequences* $A = \{X,U,Y,W\}$, *where*

$$\begin{aligned}
X &= \{x_1 = 1, x_2, x_3, \ldots, x_m, h_m x_m, \ldots, h_3 x_3, h_2 x_2, h_1 x_1 = -1\},\\
U &= \{u_1 = 1, u_2, u_3, \ldots, u_m, f_m u_m, \ldots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\},\\
Y &= \{y_1, y_2, \ldots, y_{m-1}, y_m, g_{m-1} y_{m-1}, \ldots, g_3 y_3, g_2 y_2, g_1 y_1\},\\
V &= \{v_1, v_2, \ldots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \ldots, e_3 v_3, e_2 v_2, e_1 v_1\}.
\end{aligned}$$

Then $N_A = 0$ *implies that* $h_i = f_i$ *for* $i \geq 2$ *and that* $g_j = e_j$ *for* $i \geq 1$. *Here*

$$8m - 2 = \left(\sum_{i=1}^{m} x_i + x_i h_i\right)^2 + \left(\sum_{i=1}^{m} u_i + u_i f_i\right)^2 + \left(y_m + \sum_{i=1}^{m-1} y_i + y_i g_i\right)^2 + \left(v_m + \sum_{i=1}^{m-1} v_i + v_i e_i\right)^2.$$

Corollary 5.11. *Consider four* $(1, -1)$ *sequences* $A = \{X,U,Y,V\}$, *where*

$$\begin{aligned}
X &= \{x_1 = 1, x_2, x_3, \ldots, x_m, -x_m, \ldots, -x_3, -x_2, -x_1 = -1\},\\
U &= \{u_1 = 1, u_2, u_3, \ldots, u_m, f_m u_m, \ldots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\},\\
Y &= \{y_1, y_2, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_3, y_2, y_1\},\\
V &= \{v_1, v_2, \ldots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \ldots, e_3 v_3, e_2 v_2, e_1 v_1\}.
\end{aligned}$$

Then $N_A = 0$ *implies that all* $e_i = +1$ *and that all* f_i *for* $i \geq 2 = -1$. *Here* $8m - 6$ *is the sum of two squares.*

Corollary 5.12. *Consider four* $(1,-1)$ *sequences* $A = \{X,U,Y,V\}$, *where*

$$\begin{aligned}
X &= \{x_1 = 1, x_2, x_3, \ldots, x_m, \; x_{m+1}, x_m, \ldots, x_3, x_2, x_1 = 1\},\\
U &= \{u_1 = 1, u_2, u_3, \ldots, u_m, u_{m+1}, f_m u_m, \ldots, f_3 u_3, f_2 u_2, -1\},\\
Y &= \{y_1, y_2, \ldots, y_m, -y_m, \ldots, -y_2, -y_1\},\\
V &= \{v_1, v_2, \ldots, v_m, e_m v_m, \ldots, e_2 v_2, e_1 v_1\}.
\end{aligned}$$

Then $N_A = 0$ *implies that* $e_i = -1$ *for all* i *and that* $f_i = +1$ *for all* i. *Here* $8m + 2$ *is the sum of two squares.*

TABLE 5.1 Base Sequences of Lengths $m + 1, m + 1, m, m$

Length	Sums of squares	Sequences
$m + 1 = 2$	$2^2 + 0^2 + 1^2 + 1^2$	++, +−, +, +
$m + 1 = 3$	$3^2 + 1^2 + 0^2 + 0^2$	+++, ++−, +−, +−
$m + 1 = 4$	$2^2 + 0^2 + 3^2 + 1^2$	++−+, ++−−, +++, +−+
$m + 1 = 5$	$3^2 + 3^2 + 0^2 + 0^2$	++−++, ++++−, ++−−, +−+−
$m + 1 = 5$	$3^2 + 1^2 + 2^2 + 2^2$	++++−, −+++−, ++−+, ++−+
$m + 1 = 6$	$2^2 + 0^2 + 3^2 + 3^2$	++−+−+, +++−−−, ++−++, ++−++
$m + 1 = 7$	$3^2 + 1^2 + 4^2 + 0^2$	++−+−++, ++−−−+−, −+++++, −−++−+
$m + 1 = 7$	$5^2 + 1^2 + 0^2 + 0^2$	+++−+++, ++−−−+−, ++−+−−, ++−+−−
$m + 1 = 8$	$2^2 + 0^2 + 5^2 + 1^2$	++++−−−+, ++−+−+−−, +++−+++, +−−+−−+
$m + 1 = 8$	$4^2 + 2^2 + 3^2 + 1^2$	−+++++−+, +++−−+−+, −++−+++, +−+++−−
$m + 1 = 9$	$5^2 + 3^2 + 0^2 + 0^2$	++++−++−+, −+++−++−+, +++−−−+−, +++−−−+−
$m + 1 = 10$	$4^2 + 2^2 + 3^2 + 3^2$	+++++−−+−+, −++++−−+−+, +++−+++−−, +−+++−++−
$m + 1 = 11$	$5^2 + 3^2 + 2^2 + 2^2$	++−−++++++−, −+−−++++++−, −+++−+−++−, −+++−+−++−
$m + 1 = 11$	$1^2 + 5^2 + 0^2 + 4^2$	++−++−−−−++, −++−−++++++, −−++−+−+−+, −++−+−++++
$m + 1 = 12$	$6^2 + 0^2 + 3^2 + 1^2$	−+++−+−+++++, −++−++−−−++−, −−+++++−++−, −++++−+−+−−
$m + 1 = 12$	$4^2 + 2^2 + 5^2 + 1^2$	++++++−−+−+−, +−−−−−++−+−+, ++++−−++−++, +−+−−++−−−+
$m + 1 = 13$	$7^2 + 1^2 + 0^2 + 0^2$	++++−+−+−++++, +++−−+−+−−++−, +++−++−−+−−−, +++−−+−++−−−
$m + 1 = 13$	$5^2 + 5^2 + 0^2 + 0^2$	+−−−+−+−−−+−−, ++−+−−+−+++++, ++−+−−+++−−−, ++−+++−−+−−−

TABLE 5.1 Base Sequences of Lengths $m+1, m+1, m, m$ *(continued)*

Length	Sums of squares	Sequences
$m+1=13$	$3^2+1^2+6^2+2^2$	+ + + + − + − − + + − − +,
		+ + + + − + − − + + − − −,
		+ + + + + − + − + − + +,
		+ + + − − + − + − − + +
$m+1=14$	$6^2+4^2+1^2+1^2$	+ + + + + + − − + + − + − +,
		+ − − − − − + + − − + − + −,
		+ + + + − − − + − − + − −,
		+ − + − − + − − − + + + −
$m+1=14$	$2^2+0^2+7^2+1^2$	− − − + + + − + + + + − + −,
		− − − + + − + − − + + − + +,
		+ − + + + − + + + − + + +,
		+ − + + − + − − − − + + +
$m+1=14$	$6^2+0^2+3^2+3^2$	+ + + − + + − + + + + + − −,
		+ + + + − − + − − − − + − +,
		+ + − − − + + + − + + − +,
		+ − − + + − + − + − + + +
$m+1=15$	$7^2+3^2+0^2+0^2$	+ + − + + + − + − + + + − + +,
		+ + + − + + − − − + + − + + −,
		+ + + + − − + − + + − − − −,
		+ − − − − + − + − + + + + −
$m+1=16$	$6^2+4^2+3^2+1^2$	+ + + + + − + + − − + + + − + −,
		+ − − − − + − − + + − − − + − +,
		+ − + + + + + − − + − + − − +,
		+ + + − + − + + − − − − − + +
$m+1=16$	$6^2+0^2+5^2+1^2$	+ − + − − + − − + + + + + + + +,
		+ + + − + − − + − + − − + + − −,
		+ − + − + − + + + − − + + + +,
		+ + − − − − + + + − + + − − +
$m+1=17$	$7^2+1^2+4^2+0^2$	+ + − − − − + − − − + − − + − − −,
		+ + + + − − + − − − + − − − + − +,
		+ − − − − + − + − + + − − − − +,
		+ + − + − − − − + + − − + − + +
$m+1=17$	$5^2+5^2+4^2+0^2$	+ − + + + − − + − + + + − + + + −,
		+ + + − + + − + + + + − − − + − +,
		+ − + + − + + + + − + + + − − −,
		+ − − − + + + + + − + − − + − −
$m+1=17$	$5^2+3^2+4^2+4^2$	+ + + + + − − + − + − + − − + + +,
		+ − − + + − + − + − + + − − − − −,
		+ + + + + − − + + − + − + + − −,
		+ + + + + − − + + − + − + + − −
$m+1=17$	$1^2+1^2+8^2+0^2$	+ − − − − + + + + − + − + − − − +,
		+ + + − + + − − − + − − − − + + −,
		+ + + + + − + + + + + − − + − +,
		+ − − + − + − + + − − + − − + +

TABLE 5.1 Base Sequences of Lengths $m+1, m+1, m, m$ *(continued)*

Length	Sums of squares	Sequences
$m+1=18$	$4^2+2^2+7^2+1^2$	+ + − − − − + − + + − − − + − + − −, + + + − − − − + − − + + − + − − − +, + − + − + − − + + − + + + + + + +, + + + + − − + + − − − + − − + − +
$m+1=18$	$4^2+2^2+5^2+5^2$	+ − + + − + − − + + + − − − + + + +, + − + + + − − + − − − − + + + + + −, + + + − + + + − + + + − + − − + −, + + + − + + + − + + + − + − − + −
$m+1=19$	$7^2+3^2+4^2+0^2$	+ + + − − + + − + − + + + + − + + − +, + + − + − − + + + − + − + − − − − − −, + + − − + − − + + − − − + + + + + +, + − − − − + + + − + − + − − + + − +
$m+1=19$	$3^2+1^2+8^2+0^2$	+ − + − − − + − + − + + − + − + + + +, + + + + + − + + − − − − − + + − + − −, + + + + + − − − + + + − + + + − + +, + − − + − + + − − − + + − − + + − +
$m+1=19$	$1^2+1^2+6^2+6^2$	+ + + + − − − − + − + − + − + − − + +, + − + − − + − − + + + − + + + + − − −, + − − + + + − + + − + + + + + − + −, + + + + + − + + − + + − − + + + − −
$m+1=20$	$2^2+0^2+7^2+5^2$	+ + − − − − + − − − + + − + + + + + − +, + + − + − + − + + + − − + − − + − + − −, + − + + − + + − − + + + − + − + + + +, + + − − − − − + + + − − + − − − − − +
$m+1=21$	$7^2+5^2+2^2+2^2$	+ + − − + + + + + + − − + + + − + − + + −, − + − − + + + + + + − − + + + − + − + + −, + − − + + + + + + − + − − − + − + − − +, + − − + + + + + + − + − − − + − + − − +
$m+1=21$	$3^2+1^2+6^2+6^2$	+ + − − + + + + + + − + − − + − + − − − +, − + − − + + + + + + − + − − + − + − − − +, + − − + + + + + + − − + + − + − + + + −, + − − + + + + + + − − + + − + − + + + −
$m+1=22$	$6^2+0^2+7^2+1^2$	+ + − + − + − − − − − − − + − − + + − + − −, + + − − − − + − + − + + − − − + − + + + − +, + − + + − + + + + + + − − − + + − − + + +, + + − + − + + + + − − + − − + + − − − − +
$m+1=23$	$3^2+3^2+6^2+6^2$	+ − + − − + − − − − + − + + + + + − − + + + +, − + + − − + + + + − + + + + − − − − − + + − +, + + − + − − + − + + + − + + + − + − + + + −, − + − − + + + − + + + − + + + − − + − + + +

TABLE 5.1 Base Sequences of Lengths $m+1, m+1, m, m$ *(continued)*

Length	Sums of squares	Sequences
$m+1=24$	$8^2+2^2+5^2+1^2$	+ − − + − − + − + + + − − − − − + − − − + − − −, + − − − + − + − − − − + + + + + + − − + − − − +, + + + − − − + + − − + − − − − + − − − − + − +, + + − − + − + + − + − + + + − + − − + − − − +
$m+1=25$	$7^2+7^2+0^2+0^2$	− − − − + + + − + + − + + + + + + + + − + − + − +, + − − + + + + + + + + − − − − + + − + − + + + − +, − + − + − + + − + − − + + − − + + − + + + − − −, + + − + − − + − + + − − − − + + + − − + + − − +
$m+1=26$	$8^2+6^2+1^2+1^2$	+ + + + + + + + − − + + + − − + − − + + − + − + − +, + − − − − − − − + + − − − + + − + + − − + − + − + −, + + + + + + − − − − + + − − + + − + − − + − + − −, + − + − + − − + − + + − − + + − − − − − + + + + + −
$m+1=27$	$7^2+5^2+4^2+4^2$	+ + + + − − + + + − + − − + − + − + − − + + − + + + +, − + + + − − + + + − + − − + − + − + − − + + − + + + +, − − − + + − − − + − + + + + + − + − − + + − + + + +, − − − + + − − − + − + + + + + − + − − + + − + + + +
$m+1=28$	$4^2+2^2+3^2+9^2$	− + + − + + − − − + + + + − − − − + − + − + + + − + + +, + + + − + − + + − + − + − − − + − − − + + − − + − + + +, + + − + − + + + + + − − + − + + − − + + + − − + − − −, + + − + + − + + + − + + − + − − + + + + + + + + − − −
$m+1=29$	$3^2+1^2+2^2+10^2$	+ − − + + + + + − + + + − − + − + − − + − − + − + + − − +, + + − − + − + − − − + − − + + + + + − − − + + + + − − + −, + + − + − + + + + + − − + + − − + + + − + − − − − − − +, + + + + + + − + + + − − − + − + + + + − + + − + − + − +
$m+1=30$	$8^2+6^2+3^2+3^2$	+ + + + + − + + + + − + − − + + − − + + + + − + − − − + − +, + − − − − + − − − − + − + + − − + + − − − − + − + + + − + −, + + + − + − − − + − + + − − − + − − + + + + − + + + + − −, + − + + + + − + + + + − − + − − − + + − + − − − + − + + −

TABLE 5.1 Base Sequences of Lengths $m+1, m+1, m, m$ (continued)

Length	Sums of squares	Sequences
$m+1=31$	$5^2+4^2+4^2+2^2$	+ − + − + − + − + − − + − − + − − + + + + + + − + + − − − + +, + − − − − − − − − − + + + − − − + + − + − + − − − + + − + + −, − − + − − + + + + + − + + − + − + + + + − + + − − + + + − +, − − − + − + + − + − − + + + + − − + + + + + − − − + − − − +

Definition 5.1 (Turyn Sequences). Four $(1,-1)$ sequences $A=(X,U,Y,V)$, where

$$X=\{x_1=1,x_2,x_3,\ldots,x_m,-x_m,\ldots,-x_3,-x_2,-x_1=-1\},$$

$$U=\{u_1=1,u_2,u_3,\ldots,u_m,-u_m,\ldots,-u_3,-u_2,1\},$$

$$Y=\{y_1,y_2,\ldots,y_{m-1},y_m,y_{m-1},\ldots,y_3,y_2,y_1\},$$

$$V=\{v_1,v_2,\ldots,v_{m-1},v_m,v_{m-1},\ldots,v_3,v_2,v_1\},$$

which have $N_A=0$ and $8m-6$ is the sum of two squares, or where

$$X=\{x_1=1,x_2,x_3,\ldots,x_m,x_{m+1},x_m,\ldots,x_3,x_2,x_1=1\},$$

$$U=\{u_1=1,u_2,u_3,\ldots,u_m,u_{m+1},u_m,\ldots,u_3,u_2,-1\},$$

$$Y=\{y_1,y_2,\ldots,y_m,-y_m,\ldots,-y_2,-y_1\},$$

$$V=\{v_1,v_2,\ldots,v_m,-v_m,\ldots,-v_2,-v_1\},$$

which have $N_A=0$ and $8m+2$ is the sum of two squares will be called *Turyn sequences* of length $n+1,n+1,n,n$ (they have weights $n+1,n+1,n,n$ also), where $n=2m-1$ in the first case and $n=2m$ in the second case.

Known Turyn sequences are given in Table 5.2. Note that in that table n represents the length of the shorter sequences.

Geramita and Seberry [23, pp. 142–143] quote Robinson and Seberry (Wallis) [68] results giving such sequences where the longer sequence is of length 2, 3, 4, 5, 6, 7, 8, 13, 15 (though the result for 5 has a typographical error and the last sequence should be $1-1-$), that they cannot exist for 11, 12, 17, or 18. A complete machine search showed they do not exist for (longer) lengths 9, 10, 14, or 16. Koukouvinos, Kounias, and Sotirakoglou [51] developed an algorithm and proved through an exhaustive search that Turyn sequences do not exist for (longer) lengths 19,...,28 (Genet Edmondson [19] has now estab-

TABLE 5.2 Turyn Sequences of Lengths $n+1, n+1, n, n$

Length	Sequences
$n=1$	{{1 −1}, {1 1}, {1}, {1}}
$n=2$	{{1 1 1}, {1 1 −1}, {1 −1}, {1 −1}}
$n=3$	{{1 1 −1 −1}, {1 1 −1 1}, {1 1 1}, {1 −1 1}}
$n=4$	{{1 1 −1 1 1}, {1 1 1 1 −1}, {1 1 −1 −1}, {1 −1 1 −1}}
$n=5$	{{1 1 1 −1 −1 −1}, {1 1 −1 1 −1 1}, {1 1 −1 1 1}, {1 1 −1 1 1}}
$n=6$	{{1 1 1 −1 1 1 1}, {1 1 −1 −1 −1 1 −1}, {1 1 −1 1 −1 −1}, {1 1 −1 1 −1 −1}}
$n=7$	{{1 1 −1 1 −1 1 −1 −1}, {1 1 1 1 −1 −1 −1 1}, {1 1 1 −1 1 1 1}, {1 −1 −1 1 −1 −1 1}}
$n=12$	{{1 1 1 1 −1 1 −1 1 −1 1 1 1 1}, {1 1 1 −1 −1 1 −1 1 −1 −1 1 1 −1}, {1 1 1 −1 1 1 −1 −1 1 −1 −1 −1}, {1 1 1 −1 −1 1 −1 1 1 −1 −1 −1}}
$n=14$	{{1 1 −1 1 1 1 −1 1 −1 1 1 1 −1 1 1}, {1 1 1 −1 1 1 −1 −1 −1 1 1 −1 1 1 −1}, {1 1 1 1 −1 −1 1 −1 1 1 −1 −1 −1 −1}, {1 −1 −1 −1 −1 1 −1 1 −1 1 1 1 1 −1}}

lished that they do not exist for all lengths less than 42 aside fron those listed here). The first unsettled case is $m+1=43$.

A sequence $X=\{x_1,\ldots,x_n\}$ will be called *skew* if n is even and $x_i=-x_{n-i+1}$, and *symmetric* if n is odd and $x_i=x_{n-i+1}$.

Theorem 5.13 (Turyn). *Suppose that $A=\{X,U,Y,V\}$ are Turyn sequences of lengths $m+1, m+1, m, m$. Then there are T-sequences of lengths $2m+1$ and $4m+3$.*

Proof. We use the notation A/B as before to denote the *interleaving* of two sequences $A=\{a_1,\ldots,a_m\}$ and $B=\{b_1,\ldots,b_{m-1}\}$:

$$\frac{A}{B}=\{a_1,b_1,a_2,b_2,\ldots,b_{m-1},a_m\}.$$

Let 0_t be a sequence of zeros of length t. Then

$$T_1=\{\{\tfrac{1}{2}(X+U),0_m\},\{\tfrac{1}{2}(X-U),0_m\},\{0_{m+1},\tfrac{1}{2}(Y+V)\},\{0_{m+1},\tfrac{1}{2}(Y-V)\}\}$$

and

$$T_2=\left\{\{1,0_{4m+2}\},\left\{0,\frac{X}{Y},0_{2m+1}\right\},\left\{0,0_{2m+1},\frac{U}{0_m}\right\},\left\{0,0_{2m+1},\frac{0_{m+1}}{V}\right\}\right\}$$

are T-sequences of lengths $2m+1$ and $4m+3$, respectively. □

Theorem 5.14. *If X and Y are Golay sequences of length r, then writing 0_r for the vector of r zeros, we have that* $T = \{\{1, 0_r\}, \{0, \frac{1}{2}(X+Y)\}, \{0, \frac{1}{2}(X-Y)\}, \{0_{r+1}\}\}$ *are T-sequences of length $r+1$.*

Corollary 5.15 (Turyn). *There exist T-sequences of lengths* $1 + 2^a 10^b 26^c$, *where a, b, c are nonnegative integers.*

Combining the two theorems we find

Corollary 5.16. *There exist T-sequences of lengths* $3, 5, 7, \ldots, 33, 41, 51, 53, 59, 65, 81, 101$.

A desire to fill the gaps in the list in Corollary 5.7 leads to the following idea:

Lemma 5.17. *Suppose that $X = \{A, B, C, D\}$ are 4-complementary sequences of length $m+1, m+1, m, m$, respectively, and weight k. Then*

$$Y = \{\{A, C\}, \{A, -C\}, \{B, D\}, \{B, -D\}\}$$

are 4-complementary sequences of length $2m+1$ and weight $2k$. Further, if $\frac{1}{2}(A+B)$ and $\frac{1}{2}(C+D)$ are also $(0, 1, -1)$ sequences, then, with 0_t the sequence of t zeros,

$$Z = \{\{\tfrac{1}{2}(A+B), 0_m\}, \{\tfrac{1}{2}(A-B), 0_m\}, \{0_{m+1}, \tfrac{1}{2}(C+D)\}, \{0_{m+1}, \tfrac{1}{2}(C-D)\}\}$$

are 4-complementary sequences of length $2m+1$ and weight k. If A, B, C, D are $(1, -1)$ sequences, then Z consists of T-sequences of length $2m+1$.

Lemma 5.18. *If there are Turyn sequences of length $m+1, m+1, m, m$, there are base sequences of lengths $2m+2, 2m+2, 2m+1, 2m+1$.*

Proof. Let X, U, Y, V be the Turyn sequences as in Table 5.2. Then

$$E = \left\{1, \frac{X}{Y}\right\}, \quad F = \left\{-1, \frac{X}{Y}\right\}, \quad G = \left\{\frac{U}{V}\right\}, \quad H = \left\{\frac{U}{-V}\right\}$$

are 4-complementary base sequences of lengths $2m+2, 2m+2, 2m+1, 2m+1$, respectively. □

Corollary 5.19. *There are base sequences of lengths $m+1, m+1, m, m$ for m equal to*

1. t, $2t+1$, *where there are Turyn sequences of length* $t+1$, $t+1$, t, t;
2. $9, 11, 13, 25, 29$;
3. g, *where there are Golay sequences of length* g;

4. 17 *(Seberry–Sproul)*, 23 *(Turyn)*, 22,24,26,27,28 *(Koukouvinos, Kounias, Sotirakoglou) given in Table* 5.1 *and Table* 5.3.

Corollary 5.20. *There are base sequences of lengths* $m+1, m+1, m, m$ *for* $m \in \{1,2,\ldots,29\} \cup G$, *where* $G = \{g : g = 2^a \cdot 10^b \cdot 26^c,\ a,b,c$ *non-negative integers*$\}$.

Now Cooper-(Seberry)Wallis-Turyn have shown how 4 disjoint complementary sequences of length t and zero nonperiodic (or periodic) autocorrelation function can be used to form OD$(4t;t,t,t,t)$ (formerly called *Baumert-Hall arrays*) [12]. First, the sequences (variously called *T-sequences* or *Turyn sequences*, but the latter has two different usages) are turned into T-matrices and then the Cooper-(Seberry)Wallis construction can be applied (see Section 3). Thus, it becomes important to know for which lengths (and decomposition into squares) T-sequences exist. First,

Lemma 5.21. *If there are base sequences of length* $m+1, m+1, m, m$, *there are*

1. 4 *(disjoint) T-sequences of length* $2m+1$,
2. 4*-complementary sequences of length* $2m+1$.

Proof. Let X, U, Y, V be the base sequences of lengths $m+1, m+1, m, m$, then

$$\{\tfrac{1}{2}(X+U), 0_m\}, \{\tfrac{1}{2}(X-U), 0_m\}, \{0_{m+1}, \tfrac{1}{2}(Y+V)\}, \{0_{m+1}, \tfrac{1}{2}(Y-V)\}$$

are the T-sequences of length $2m+1$ and

$$\{X,Y\}, \{X,-Y\}, \{U,V\}, \{U,V\}$$

are 4-complementary sequences of length $2m+1$. □

Corollary 5.22. *There are T-sequences of lengths t for the following* $t < 106$:

$$1,3,\ldots,59,65,81,101,105.$$

5.4. On Yang's Theorems on T-Sequences

In an a series of papers in 1982 and 1983, Yang [135, 136, 137] found that base sequences can be multiplied by 3, 7, 13, and $2g+1$, where $g = 2^a 10^b 26^c$, $a,b,c \geq 0$: These are instances of what are termed *Yang numbers*. If y is a Yang number and there are base sequences of lengths $m+p, m+p, m, m$, then there are (4-complementary) T-sequences of length $y(2m+p)$. This is of most interest when $2m+p$ is odd. (A new construction for the Yang number 57 is given in [58].)

Yang numbers currently exist for $y \in \{3,5,\ldots,33,37,39,41,45,49,51,53,57,59,65,81,\ldots,$ and $2g+1>81,\ g \in G\}$, where

$$G = \{g : g = 2^a 10^b 26^c,\ a,b,c \text{ nonnegative integers}\}.$$

Base sequences currently exist for $p = 1$ and $m \in \{1,2,\ldots,29\} \cup G$. We reprove and restate Yang's theorems from [138] to illustrate why they work.

Theorem 5.23 (Yang). *Let A,B,C,D be base sequences of lengths $m+p, m+p, m, m$, and let $F=(f_k)$ and $G=(g_k)$ be Golay sequences of length s. Then the following Q,R,S,T become 4-complementary sequences (i.e., the sum of nonperiodic autocorrelation functions is 0), using X^* to denote the reverse of X:*

$$Q = (Af_s, Cg_1;\ 0,0;\ Af_{s-1}, Cg_2; 0,0; \ldots; Af_1, Cg_s;\ 0,0;\ -B^*,0);$$

$$R = (Bf_s, Dg_s;\ 0,0;\ Bf_{s-1}, Dg_{s-1};\ 0,0;\ldots;\ Bf_1, Dg_1;\ 0,0;\ A^*,0);$$

$$S = (0,0;\ Ag_s, -Cf_1;\ 0,0;\ Ag_{s-1}, -Cf_2;\ldots;\ 0,0;\ Ag_1, -Cf_s;\ 0,-D^*);$$

$$T = (0,0;\ Bg_1, -Df_1;\ 0,0;\ Bg_2, -Df_2;\ldots;\ 0,0;\ Bg_s, -Df_s;\ 0, C^*).$$

Furthermore, if we define sequences

$$X = (Q+R)/2, \qquad Y = (Q-R)/2, \qquad V = (S+T)/2, \qquad W = (S-T)/2,$$

then these sequences become T-sequences of length $t(2s+1)$, $t = 2m+p$.

Note: The interesting case for Yang's theorem is for base sequences of lengths $m+p, m+p, m, m$, where p is odd for then Yang's theorem produces T-sequences of odd length, for example, $3(2m+p)$.

Restatement 5.24 (Yang). *Suppose that E,F,G,H are base sequences of lengths $m+p, m+p, m, m$. Define $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, and $D = \frac{1}{2}(G-F)$ to be suitable sequences. Then the following sequences are disjoint T-sequences of length $3(2m+p)$:*

$$X = A, C;\ 0, 0';\ B^*, 0';$$

$$Y = B, D;\ 0, 0';\ -A^*, 0';$$

$$Z = 0, 0';\ A, -C;\ 0, D^*;$$

$$W = 0, 0';\ B, -D;\ 0, -C^*;$$

and

$$X = B^*,0';\ A,C;\ 0,0';$$
$$Y = -A^*,0';\ B,D;\ 0,0';$$
$$Z = 0,D^*;\ 0,0';\ A,-C;$$
$$W = 0,-C^*;\ 0,0';\ B,-D.$$

In these sequences 0 *and* 0′ *are sequences of zeros of lengths* $m+p$ *and* m, *respectively.*

The next two theorems deal with multiplication by 7 and 13. They can be used recursively, but as the sequences produced are of equal lengths, the next recursive use of the theorems gives sequences of (equal) even length.

Theorem 5.25 (Yang [137]). *Let* (E,F,G,H) *be the base sequences of length* $m+p, m+p, m, m$. *Let* $t = 2m+p$ *and define the suitable sequences* $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, *and* $D = \frac{1}{2}(G-H)$ *of lengths* $m+p, m+p, m$ *and* m. *Then the following* X,Y,Z,W *are 4-disjoint T-sequences of length* $7t$ *(where* $\overline{X}$ *means negate all the elements of the sequence and* X^* *means reverse all the elements of the sequence):*

$$X = (\overline{A},C;\ 0,0;\ A,D;\ 0,0;\ A,C;\ 0,0;\ \overline{B}^*,0);$$
$$Y = (\overline{B},D;\ 0,0;\ B,\overline{C};\ 0,0;\ B,D;\ 0,0;\ A^*,0);$$
$$Z = (0,0;\ A,\overline{C};\ 0,0;\ \overline{B},\overline{C};\ 0,0;\ A,C;\ 0,\overline{D}^*);$$
$$W = (0,0;\ B,\overline{D};\ 0,0;\ A,\overline{D};\ 0,0;\ B,D;\ 0,C^*).$$

Theorem 5.26 (Yang [137]). *Let* (E,F,G,H) *be the base sequences of length* $m+p, m+p, m, m$. *Let* $t = 2m+p$, *and define the suitable sequences* $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, *and* $D = \frac{1}{2}(G-H)$ *of lengths* $m+p, m+p, m$, *and* m. *Then the following* X,Y,Z,W *are 4-disjoint T-sequences of length* $13t$:

$$Q = (A,D^*;\ \overline{A},\overline{C};\ \overline{A},D^*;\ \overline{A},C;\ \overline{A},D^*;\ A,\overline{C};\ 0,C;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0);$$
$$R = (\overline{B},C^*;\ B,D;\ B,C^*;\ B,\overline{D};\ B,C^*;\ \overline{B},D;\ 0,\overline{D};\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0);$$
$$S = (0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ \overline{A},0;\ A,C;\ B^*,\overline{C};\ \overline{A},\overline{C};\ B^*,\overline{C};\ A,\overline{C};\ B^*,C);$$
$$T = (0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ B,0;\ \overline{B},\overline{D};\ A^*,D;\ B,D;\ A^*,D;\ \overline{B},D;\ A^*,\overline{D}).$$

Yang [137] has also shown how to multiply by 11. The sequences obtained are not disjoint and so cannot be used in another iteration but still are vital in that they give complementary sequences of length $11(2m+p)$, and hence Hadamard matrices of order $44(2m+p)$. Using the Yang numbers $y = 3,5,7,9,13,17,21,33,41,53,61,65,81$ with base sequences gives T-sequences, so

Corollary 5.27. *Yang numbers and base sequences of lengths $m+1, m+1, m, m$ can be used to give T-sequences of lengths $t = y(2m+1)$ for the following $t < 200$:*

$$1,3,\ldots,41,45,\ldots,59,61,63,65,69,75,77,81,85,91,$$

$$93,95,99,101,105,111,115,\ldots,125,133,135,141,\ldots,$$

$$147,153,155,159,161,165,169,171,175,177,183,187,189,195.$$

The gaps in these sets can sometimes be filled by T-matrices. Thus, using Table 5.3 and Corollary 5.22 and noting that T-sequences give T-matrices, we have

Lemma 5.28. *T-matrices exist for the following $t < 196$:*

$$1,3,\ldots,71,75,77,81,85,87,91,93,95,99,101,105,111,115,\ldots,$$

$$125,129,\ldots,135,141,\ldots,147,153,155,159,\ldots,$$

$$165,169,171,175,177,187,189,195.$$

These are given in more detail in Cohen, Rubie, Koukouvinos, Kounias, Seberry, and Yamada [10], Koukouvinos, Kounias, and Seberry [56], and Koukouvinos, Kounias and Sotirakoglou [51]. Further results, including multiplication and construction theorems, are given in recent work of Koukouvinos, Kounias, Seberry, C. H. Yang, and J. Yang [57, 58].

5.5. Koukouvinos and Kounias

We call κ a *Koukouvinos-Kounias number*, or *KK number*, if

$$\kappa = g_1 + g_2,$$

where g_1 and g_2 are both the lengths of Golay sequences. Then we have

Lemma 5.29. *Let κ be a KK number and y be a Yang number. Then there are T-sequences of length t and* OD$(4t; t,t,t,t)$ *for $t = y\kappa$.*

TABLE 5.3 *T***-Matrices Used**

Order	Sum of squares	T_i	Sets
31	$3^2+3^2+3^2+2^2$	T_1	{1, 5, −8, −9, 11, −14, 24, 25, 27}
		T_2	{2, 6, 10, −12, 19, −21, 26, −29, 30}
		T_3	{4, 7, −16, 17, −18, 20, 22, 23, −28}
		T_4	{3, 13, 15, −31}
39	$6^2+1^2+1^2+1^2$	T_1	{17, 20, −21, 23, 24, 26, 35, 38}
		T_2	{14, 15, −16, −18, 19, 22, 25, −34, −36, −37, 39}
		T_3	{−4, −7, −8, 10, 11, 13, 28, −29, −31, 32, 33}
		T_4	{1, 2, −3, −5, 6, −9, −12, 27, 30}
43	$4^2+3^2+3^2+3^2$	T_1	{1, 4, −5, 6, 7, 8, 9, −13, −14, 15, 16, −17, −18, 21}
		T_2	{−2, 3, 10, 11, −12, 19, 20}
		T_3	{−22, −23, 24, 26, 29, 31, 34, 36, −39, −41, 42}
		T_4	{−25, −27, 28, 30, 32, 33, −35, 37, −38, 40, 43}
49	$4^2+4^2+4^2+1^2$	T_1	{4, 6, −18, 19, 21, −32, 34, 44, 45, −46}
		T_2	{−8, 9, 10, 12, 14, 25, 26, 28, −36, 37, 38, −40, −42, −48}
		T_3	{11, 13, 22, −23, −24, 27, 39, −41, 47, 49}
		T_4	{−1, 2, 3, 5, 7, 15, −16, −17, −20, 29, −30, −31, 33, 35, −43}
49	$5^2+4^2+2^2+2^2$	T_1	{1, −2, −3, 5, 7, −15, 16, 17, 20, −29, 30, 31, 33, 35, −43}
		T_2	{11, −13, −22, 23, 24, −27, 39, 41, 47, 49}
		T_3	{−4, 6, 18, −19, −21, 32, 34, 44, 45, −46}
		T_4	{8, −9, −10, 12, 14, 25, 26, 28, 36, −37, −38, −40, −42, 48}
55	$5^2+5^2+2^2+1^2$	T_1	{1, 2, −5, 7, 8, −9, 10, 11, −23, −24, 27, 29, 30, −31, 32, 33, −45, −47, 48}
		T_2	{−14, 15, 17, −36, 37, 39, 51, 52, −53, 54, 55}
		T_3	{12, 13, −16, 18, 19, −20, 21, 22, 34, 35, −38, −40, −41, 42, −43, −44}
		T_4	{−3, 4, 6, 25, −26, −28, −46, 49, 50}
57	$4^2+4^2+4^2+3^2$	T_1	{−24, −25, 29, 30, −31, 32, 33, −35, 36, 37, 38, 53}
		T_2	{20, 21, 22, −23, −26, 27, −28, −34, 49, 50, 51, −52, 54, 55, −56, 57}
		T_3	{5, 6, −10, 11, −12, 13, 14, −16, 17, 18, 19, 40, −41, 42, 45, −46, −47, −48}
		T_4	{1, 2, 3, −4, −7, 8, −9, 15, −39, 43, 44}

TABLE 5.3 ***T*-Matrices Used** *(continued)*

Order	Sum of squares	T_i	Sets
61	$6^2 + 5^2$	T_1	{2, 7, 10, 17, 18, −26, 29, −30, 31, −32, 35, 40, −44, −51, 55, 61}
		T_2	{3, 4, −8, −11, −12, 13, 14, 15, 16, 19, 22, −25, 27, −28, 36, −37, −38, 41, −42, −47, 49, 52, 56, −57, 60}
		T_3	{−1, 5, 6, −9, −20, 21, −23, −24, −33, −34, 39, 43, 45, 46, 48, −50, −53, 54, −58, 59}
		T_4	{ϕ}
67	$8^2 + 1^2 + 1^2 + 1^2$	T_1	{−1, 5, 9, 13, 14, 15, 18, 25, 27, 29, −31, 32, −39, 43, 50, −67}
		T_2	{2, −8, −12, 16, 17, 23, −40, 41, 42, −45, −46, −47, −53, 54, −56, 65, 66}
		T_3	{−6, 7, 11, 19, 20, −21, 24, −26, −28, −37, 38, 44, −49, 57, −58, −59, 61}
		T_4	{−3, −4, 10, 22, 30, −33, 34, −35, 36, 48, 51, −52, −55, −60, −62, 63, 64}
71	$6^2 + 5^2 + 3^2 + 1^2$	T_1	{1, −2, −3, 4, 5, 6, −7, 8, 9, 10, −11, −12, −13, −14, 15, 16, −17, 18, 19, −20, 21, 22, 23, 24}
		T_2	{25, 26, 27, 28, −29, 30, 31, −32, 33, 34, 35, −36, 37, −38, 39, −40, 41, −42, −43, −44, −45, 46, 47}
		T_3	{48, 49, 50, 51, −52, −56, 57, 58, 60, −64, 65, −66, −71}
		T_4	{53, 54, −55, 59, −61, 62, −63, −67, −68, 69, 70}
85	$7^2 + 6^2$	T_1	{1, 2, 4, −5, −11, −12, 14, −15, 21, 22, 24, −25, 31, 32, −34, 35, −41, −42, −44, 45, 51, 52, −54, 55, 61, 62, 64, −65, 71, 72, −74, 75, −81}
		T_2	{3, −13, 23, 33, −43, 53, 63, 73, 82, 83}
		T_3	{−6, −7, −9, 10, 16, 17, −19, 20, −26, −27, −29, 30, −36, −37, 39, −40, −46, −47, −49, 50, 56, 57, −59, 60, 66, 67, 69, −70, 76, 77, −79, 80}
		T_4	{8, 18, 28, −38, −48, −58, 68, −78, −84, 85}
87	$7^2 + 6^2 + 1^2 + 1^2$	T_1	{−2, −3, 5, 6, 10, 11, 13, −14, 15, 16, −17, 20, 21, 24, −25, 28, 29, −62, −65, 66, 67, 70, −73}
		T_2	{30, −33, −36, −37, 38, 41, −47, 48, 51, 52, 55, −56, 74, 75, −78, 79, 82, 83, −86, 87}
		T_3	{1, −4, −7, −8, 9, 12, 18, −19, −22, −23, −26, 27, 59, −60, 61, 63, 64, 68, 69, −71, −72}
		T_4	{−31, −32, 34, 35, 39, 40, 42, −43, 44, −45, 46, −49, −50, −53, 54, −57, −58, −76, 77, 80, 81, 84, −85}

TABLE 5.3 ***T*-Matrices Used** *(continued)*

Order	Sum of squares	T_i	Sets
91	$5^2+5^2+5^2+4^2$	T_1	$\{-1, -2, 3, 5, -6, -8, 10, 11, 13, 27, 28, -29, -31, 32, 35, 38, 53, 54, -55, -57, 58, -60, 62, 63, 65, -79, -82\}$
		T_2	$\{-4, -7, 9, 12, 30, 33, 34, -36, -37, -39, 56, 59, 61, 64, 80, -81, -83, 84, 85\}$
		T_3	$\{17, 20, , -22, -25, 40, 41, -42, -44, 45, -48, -51, 69, 72, 74, 77, 86, 88, 89, -91\}$
		T_4	$\{-14, -15, 16, 18, -19, -21, 23, 24, 26, 43, 46, -47, 49, 50, 52, -66, -67, 68, 70, -71, 73, -75, -76, -78, 87, 90\}$
93	$6^2+5^2+4^2+4^2$	T_1	$\{2, 3, 4, 5, -6, 7, 8, -9, -10, 11, 12, 13, -14, 15, -16, 17, 19, 21, 23, -25, -27, -29, 31, -78\}$
		T_2	$\{1, 18, -20, -22, 24, -26, -28, 30, -63, 64, -65, 66, 67, 68, -69, -70, 71, 72, -73, 74, 75, 76, 77\}$
		T_3	$\{33, 34, 35, 36, -37, 38, 39, -40, -41, 42, 43, 44, -45, 46, -47, -48, -50, -52, -54, 56, 58, 60, -62, -80, 82, 84, -86, 88, 90, -92\}$
		T_4	$\{32, -49, 51, 53, -55, 57, 59, -61, 79, -81, -83, -85, 87, 89, 91, 93\}$

This gives T-sequences of lengths

$$2\cdot 101, 2\cdot 109, 2\cdot 113, 8\cdot 127, 2\cdot 129, 2\cdot 131, 8\cdot 151,$$
$$8\cdot 157, 16\cdot 163, 2\cdot 173, 4\cdot 179, 4\cdot 185, 4\cdot 193, 2\cdot 201.$$

6 AMICABLE HADAMARD MATRICES AND AOD

Two matrices $M = I + U$ and N will be called [complex] *amicable Hadamard matrices* if M is a (complex) skew Hadamard matrix and N a [complex] Hadamard matrix satisfying

$$N^T = N, \qquad MN^T = NM^T \qquad \text{if real,}$$
$$N^* = N, \qquad MN^* = NM^* \qquad \text{if complex.}$$

Amicable Hadamard matrices are useful in constructing skew Hadamard matrices: They are algebraically powerful and elegant. We will only use constructions with real matrices to construct (real) amicable Hadamard matrices. It is obvious, however, that if complex matrices are used, then complex amicable Hadamard matrices can be obtained.

We note that the truth of the conjecture implicit in Seberry [77] and Seberry-Yamada [86], that "amicable Hadamard matrices exist for every order 2 and $4n$, $n \geq 1$," would imply the two conjectures that "skew Hadamard matrices exist for every order 2 and $4n$, $n \geq 1$" (which appears to be hard to prove) and that "symmetric Hadamard matrices exist for every order 2 and $4n$, $n \geq 1$" (which appears to be the easier to prove).

6.1. Other Amicable Matrices

M and N of order n are said to be *amicable orthogonal designs* of type $\mathrm{AOD}(n;(m_1,\ldots,m_p);\ (n_1,\ldots,n_q))$ if M is an $\mathrm{OD}(n;m_1,\ldots,m_p)$, N is an *orthogonal design* $\mathrm{OD}(n;n_1,\ldots,n_q)$, and $MN^T = NM^T$. If M comprises the variables $x_1,\ldots,x_p$ and N comprises the variables $y_1,\ldots,y_q$, then

$$MM^T = \sum_{i=1}^{p} m_i x_i^2 I_n, \qquad NN^T = \sum_{j=1}^{q} n_j y_j^2 I_n$$

and

$$ZZ^T = (m_1x_1^2 + \cdots + m_p x_p^2)(n_1 y_1^2 + \cdots + n_q y_q^2) I_n,$$

where $Z = MN^T$. Wolfe and Shapiro (see [23]) have studied and solved the algebraic necessary conditions for amicable orthogonal designs, but the sufficiency conditions are largely unresolved (see [71, 23, 79] for partial results).

Amicable orthogonal designs $\mathrm{AOD}(n;(1,n-1);(n))$ give amicable Hadamard matrices (they are not the same since the orthogonal designs have no symmetry or skew symmetry conditions). *Normalized amicable Hadamard matrices* of order h can be written in the form

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ - & & & \\ \vdots & & I+S & \\ - & & & \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & P+R & \\ 1 & & & \end{bmatrix},$$

where

$$S^T = -S, \qquad P^T = P, \qquad R^T = R, \qquad PR^T + RP^T = 0,$$
$$RR^T = I, \qquad SJ = PJ = 0, \qquad RJ = -J, \qquad SP^T = PS^T,$$
$$SR^T = RS^T, \qquad SS^T = PP^T = (h-1)I - J.$$

Amicable orthogonal designs, amicable Hadamard matrices, and skew Hadamard matrices have proved difficult to find. The Kronecker product of skew Hadamard matrices is not a skew Hadamard matrix. However, if h_1 and h_2 are the orders of amicable Hadamard matrices, then there are amicable Hadamard matrices of order h_1h_2; further, if g is the order of a skew Hadamard matrix, there are skew Hadamard matrices of orders h_1g and h_2g [114]. We list the orders for which amicable matrices are known, but we do not prove these results here. The recent result of Seberry and Yamada [86], which is class AIII, indicate that powerful results may remain to be discovered.

6.2. Summary and Tables of Amicable Hadamard Matrices

AI	2^t	t a nonnegative integer; J. Wallis [110]
AII	$p^r + 1$	p^r (prime power) $\equiv 3 \pmod 4$; J. Wallis [110]
AIII	$(p-1)^u + 1$	p the order of normalized amicable Hadamard matrices, there are normalized amicable Hadamard matrices of order $(p-1)^u + 1$, $u > 0$ an odd integer; Seberry and Yamada [86]
AIV	$2(q+1)$	$2q+1$ is a prime power, q (prime) $\equiv 1 \pmod 4$; J. Wallis [114, p. 304]
AV	$(\lvert t\rvert + 1)(q+1)$	q (prime power) $\equiv 5 \pmod 8 = s^2 + 4t^2$, $s \equiv 1 \pmod 4$, and $\lvert t\rvert + 1$ is the order of amicable orthogonal designs of type $\mathrm{AOD}(1 + \lvert t\rvert; (1, \lvert t\rvert); (\frac{1}{2}(\lvert t\rvert + 1), \frac{1}{2}(\lvert t\rvert + 1))$; [23, §5.7]
	$2^r(q+1)$	q (prime power) $\equiv 5 \pmod 8 = s^2 + 4(2^r - 1)^2$, $s \equiv 1 \pmod 4$, r some integer; [23, §5.7]
	$2(q+1)$	$q \equiv 5 \pmod 8$; J. Wallis [116]
AVI	S	S is a product of the above orders; J. Wallis [110]

Constructions for amicable orthogonal designs can be found in [23], [70], [69], [77], [79], [86], [96], [110], [116], [114], [119]. A summary of the orders for which skew Hadamard matrices are known can be found at the end of Section 7. Amicable Hadamard matrices appear in Table 6.1. In this table, a "." means "unknown" and a blank means "2."

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist

q	t	q	t	q	t	q	t	q	t
1		23	4	45		67	5	89	4
3		25	3	47	4	69	4	91	3
5		27		49	4	71		93	3
7		29	4	51	4	73	7	95	
9	3	31	3	53		75	3	97	9
11		33		55	3	77		99	4
13	3	35		57		79	3	101	.
15		37	.	59	.	81	3	103	3
17		39	3	61	3	83		105	
19	3	41		63		85	4	107	.
21		43	3	65	4	87		109	9

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist *(continued)*

q	t	q	t	q	t	q	t	q	t
111		201	3	291		381		471	3
113	8	203		293		383		473	5
115		205	4	295	5	385	3	475	4
117		207		297		387	5	477	
119	4	209	4	299	4	389	.	479	.
121	3	211	4	301	5	391	5	481	3
123		213	4	303	3	393		483	
125		215		305	5	395		485	4
127	.	217	5	307	.	397	5	487	5
129	3	219	7	309	4	399	3	489	3
131		221		311	.	401	.	491	.
133	3	223	3	313	3	403	6	493	3
135	4	225	4	315	4	405		495	
137		227		317	6	407		497	
139	4	229	3	319	3	409	3	499	3
141		231	3	321		411	4	501	
143		233	4	323		413	4	503	
145	5	235	3	325	5	415	3	505	.
147		237		327		417		507	
149	4	239	4	329	6	419	4	509	.
151	5	241	.	331	3	421	7	511	5
153	3	243		333	3	423	4	513	4
155		245	4	335	7	425		515	5
157	5	247	6	337	.	427	4	517	6
159	4	249	4	339	3	429	4	519	4
161		251	6	341	5	431		521	
163	3	253	6	343	6	433	3	523	7
165		255		345	4	435	4	525	
167	4	257	4	347	.	437		527	4
169	5	259	5	349	3	439	3	529	3
171		261	3	351	4	441	3	531	7
173		263		353	4	443	6	533	
175	3	265	4	355	4	445	3	535	
177	.	267	4	357		447		537	5
179	8	269	8	359	4	449	.	539	4
181	3	271	7	361	3	451	3	541	3
183	4	273		363		453		543	5
185		275	5	365		455	5	545	
187	4	277	5	367		457	.	547	
189	3	279	4	369	4	459	3	549	3
191	.	281		371		461	.	551	
193	3	283	.	373	7	463	7	553	3
195	3	285	4	375		465	3	555	4
197		287	4	377	7	467		557	.
199	3	289	3	379	.	469	7	559	5

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist *(continued)*

q	t	q	t	q	t	q	t	q	t
561		649	7	737	7	825		913	4
563		651	5	739	.	827		915	
565	3	653		741		829	.	917	4
567		655	4	743		831		919	3
569	4	657	5	745	6	833		921	6
571	3	659	.	747	4	835	3	923	
573	3	661	.	749	.	837		925	4
575	4	663	3	751	3	839	.	927	4
577	.	665		753		841	8	929	.
579	5	667	8	755		843		931	5
581	4	669	3	757	5	845	7	933	.
583	3	671		759	4	847	4	935	
585		673	7	761	.	849	3	937	5
587		675		763	11	851	.	939	4
589	6	677		765	3	853	3	941	.
591	4	679	3	767		855	4	943	6
593		681	4	769	3	857	4	945	
595	3	683		771	5	859	3	947	6
597	4	685	3	773	.	861	4	949	3
599	.	687		775	3	863	4	951	
601	5	689	5	777	4	865	4	953	.
603		691	3	779	5	867		955	3
605	4	693	4	781	3	869	4	957	5
607	5	695	4	783	3	871	3	959	4
609	3	697	4	785	7	873		961	3
611	6	699	3	787	5	875		963	.
613	3	701		789	3	877	.	965	4
615		703	3	791		879	4	967	.
617		705		793	3	881	6	969	4
619	3	707	6	795	3	883	.	971	6
621	3	709	.	797		885		973	6
623	4	711		799	6	887		975	5
625	3	713		801		889	5	977	
627	4	715	4	803	9	891	3	979	5
629	.	717	4	805	4	893		981	
631	.	719	4	807	4	895	3	983	
633		721	5	809	.	897	5	985	4
635		723	3	811	5	899	7	987	
637	5	725	6	813		901	3	989	4
639	4	727	.	815		903	4	991	3
641	6	729	4	817	6	905	4	993	4
643	.	731	5	819	3	907	5	995	4
645		733	.	821	6	909	4	997	.
647	.	735		823	.	911		999	5

7 CONSTRUCTIONS FOR SKEW HADAMARD MATRICES

Some of the most powerful methods for constructing Hadamard matrices depend on the existence of skew Hadamard matrices. Skew Hadamard matrices are known to be equivalent to doubly regular tournaments. The analogue of a skew Hadamard matrix in orders $\equiv 2 \pmod 4$ is a symmetric conference matrix, but very few symmetric conference matrices are known whose orders are not of the form prime power plus one or those derived from skew Hadamard matrices.

The properties of these matrices were noticed as long ago as 1933 and 1944 by Paley and Williamson, but it has only been recently when the talents of Szekeres, Seberry, and Whiteman (among others) were directed toward their study that significant understanding of their nature was achieved.

N. Ito has determined that for general skew Hadamard matrices, there is a unique matrix of each order less than 16, two of order 16, and 16 of order 24. Kimura has found 49 of order 28 [45] and 6 of order 32 [46].

For completeness, we will restate results given earlier that are corollaries of the stronger theorems on amicable Hadamard matrices. The smallest known skew Hadamard matrices are listed. The first rows of circulant matrices of small order that give skew Hadamard matrices are listed.

Jennifer Wallis [111] used a computer to obtain skew Hadamard matrices using the Williamson matrix

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}.$$

Those of order < 92 only took at most a few minutes to find, but the matrix of order 92 took many hours on an ICL 1904A. Subsequently, Szekeres and Hunt [35], using a bigger computer, developed indexing techniques that allowed the matrix of order 100 to be found in about one hour. Szekeres [100] has now extended these results and corrected minor errors. The number of inequivalent Hadamard matrices of this type depends on the decomposition into squares, but for order 12, he found one; for 20, one; for 28, three; for 36, one; for 44, three; for 52, six; for 60, eleven; for 68, two; for 76, eight; for 84, ten; for 92, six; for 100, nine; for 108, twelve; for 116, five; and for 124, three.

The following first rows for A, B, C, D generate the required matrices: The results for 21,25 were found by Hunt; for 27,29,31 by Szekeres; and the remainder by (Seberry) Wallis:

3	1 -1 1
	1 -1 -1
	1 -1 -1
	1 1 1
5	1 -1 -1 1 1
	1 -1 -1 -1 -1
	1 -1 -1 -1 -1
	1 -1 1 1 -1
7	1 -1 -1 -1 1 1 1
	1 -1 -1 -1 -1 -1 -1
	1 -1 -1 1 1 -1 -1
	1 -1 1 -1 -1 1 -1
9	1 -1 -1 -1 1 -1 1 1 1
	1 -1 -1 -1 1 1 -1 -1 -1
	1 1 -1 1 -1 -1 1 -1 1
	1 1 1 -1 1 1 -1 1 1
11	1 -1 -1 -1 -1 1 -1 1 1 1 1
	1 -1 -1 -1 -1 1 1 -1 -1 -1 -1
	1 1 -1 1 -1 1 1 -1 1 -1 1
	1 1 -1 1 1 -1 -1 1 1 -1 1
13	1 -1 -1 -1 -1 1 -1 1 -1 1 1 1 1
	1 -1 1 1 1 -1 1 1 -1 1 1 1 -1
	1 1 -1 -1 1 -1 1 1 -1 1 -1 -1 1
	1 1 1 1 -1 -1 1 1 -1 -1 1 1 1
15	1 -1 -1 -1 -1 -1 1 -1 1 -1 1 1 1 1 1
	1 -1 1 -1 -1 1 1 -1 -1 1 1 -1 -1 1 -1
	1 1 1 -1 -1 1 -1 1 1 -1 1 -1 -1 1 1
	1 1 -1 1 1 1 1 -1 -1 1 1 1 1 -1 1
17	1 -1 -1 -1 -1 -1 1 -1 -1 1 1 -1 1 1 1 1 1
	1 1 -1 -1 -1 1 -1 -1 1 1 -1 -1 1 -1 -1 -1 1
	1 1 -1 -1 -1 1 -1 1 -1 -1 1 -1 1 -1 -1 -1 1
	1 -1 -1 1 -1 1 -1 -1 -1 -1 -1 -1 1 -1 1 -1 -1
19	1 -1 1 -1 -1 -1 -1 -1 1 1 -1 -1 1 1 1 1 1 -1 1
	1 1 -1 1 1 1 1 1 -1 -1 -1 -1 1 1 1 1 1 -1 1
	1 -1 1 -1 -1 -1 -1 1 1 -1 -1 1 1 -1 -1 -1 -1 1 -1
	1 -1 -1 1 -1 1 -1 1 1 -1 -1 1 1 -1 1 -1 1 -1 -1
21	1 -1 -1 -1 -1 -1 1 -1 -1 1 -1 1 -1 1 1 -1 1 1 1 1 1
	1 1 1 -1 -1 1 -1 1 1 1 -1 -1 1 1 1 -1 1 -1 -1 1 1
	1 1 -1 1 -1 -1 -1 -1 -1 1 -1 -1 1 -1 -1 -1 -1 -1 1 -1 1
	1 -1 1 -1 -1 1 1 -1 -1 -1 1 1 -1 -1 -1 1 1 -1 -1 1 -1
23	1 -1 -1 -1 -1 -1 -1 -1 1 1 -1 1 -1 1 -1 -1 1 1 1 1 1 1 1
	1 1 -1 -1 1 -1 -1 1 1 1 1 -1 -1 1 1 1 1 -1 -1 1 -1 -1 1
	1 1 -1 -1 -1 1 -1 1 -1 1 -1 1 1 -1 1 -1 1 -1 1 -1 -1 -1 1
	1 -1 -1 -1 -1 1 -1 -1 1 -1 -1 1 1 -1 -1 1 -1 -1 1 -1 -1 -1 -1

(continued)

25	1 1 −1 −1 −1 −1 −1 1 −1 1 −1 −1 −1 1 1 1 −1 1 −1 1 1 1 1 1 −1
	1 1 −1 1 1 1 1 −1 1 −1 1 1 −1 −1 1 1 −1 1 −1 1 1 1 1 −1 1
	1 −1 −1 −1 1 −1 −1 1 1 1 1 −1 −1 −1 −1 1 1 1 1 −1 −1 1 −1 −1 −1
	1 −1 −1 1 −1 1 1 1 −1 −1 1 −1 −1 −1 −1 1 −1 −1 1 1 1 −1 1 −1 −1
27	1 1 −1 −1 −1 1 −1 1 −1 −1 1 1 −1 1 −1 1 −1 −1 1 1 −1 1 −1 1 1 1 −1
	1 1 1 1 −1 −1 −1 1 1 −1 1 1 −1 1 1 −1 1 1 −1 1 1 −1 −1 −1 1 1 1
	1 1 1 1 −1 1 1 1 −1 1 1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 1 −1 1 1 1
	1 1 −1 −1 −1 1 1 1 1 −1 1 −1 1 −1 −1 1 −1 1 −1 1 1 1 1 −1 −1 −1 1
29	1 1 −1 −1 1 −1 −1 −1 −1 1 −1 1 −1 1 1 −1 −1 1 −1 1 −1 1 1 1 1 −1 1 1 −1
	1 1 −1 −1 −1 1 1 1 1 −1 1 1 −1 −1 1 1 −1 −1 1 1 −1 1 1 1 1 −1 −1 −1 1
	1 1 −1 1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 −1 −1 1 −1 1 1 −1 −1 −1 −1 1 −1 1
	1 1 1 1 1 1 1 −1 −1 −1 1 −1 1 −1 1 1 −1 1 −1 1 −1 −1 −1 1 1 1 1 1 1
31	1 1 −1 −1 1 1 −1 −1 −1 −1 1 −1 −1 −1 1 −1 1 −1 1 1 1 −1 1 1 1 1 −1 −1 1 1 −1
	1 −1 1 1 −1 1 −1 −1 −1 1 −1 1 1 1 −1 −1 −1 −1 1 1 1 −1 1 −1 −1 −1 1 −1 1 1 −1
	1 1 1 −1 −1 −1 1 −1 −1 1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 1 −1 −1 1 −1 −1 −1 1 1
	1 −1 1 1 1 1 1 1 1 1 −1 −1 1 −1 1 −1 −1 1 −1 1 −1 −1 1 1 1 1 1 1 1 1 −1

7.1. The Goethals-Seidel Type

Goethals and Seidel modified the Williamson matrix so that the matrix entries did not have to be circulant and symmetric. Their matrix, which has been valuable in constructing many new Hadamard matrices, was orginally given to construct a skew Hadamard matrix of order 36 [27].

Theorem 7.1 (Goethals and Seidel [27]). *If A,B,C,D are square circulant matrices of order m, and $R = (r_{ij})$ is defined by $r_{i,m-i} = 1$, $i = 1,\dots,m$, then if A is skew type, and if*

$$AA^T + BB^T + CC^T + DD^T = 4mI, \tag{6}$$

then the array 7 in Section 3 is skew Hadamard of order $4m$.

This construction gave the first skew Hadamard matrices of orders 36 and 52.

Recently, Djokovic [17, 16] has carried out a computer search for circulant matrices that can be used in the Goethals-Seidel array and found matrices to give skew Hadamard matrices of order $4n$, $n = 37, 43, 49, 67, 113, 127, 157, 163,$ 181, and 241.

The following two pairs of four sets are 4-(37; 18, 18, 16, 13; 28) and 4-(37, 18, 15, 15, 15; 26) supplementary difference sets, respectively, found by Djokovic [17], which may be used to construct circulant $(1, -1)$ matrices that give, using the Goethals-Seidel array, skew Hadamard matrices of order $4 \cdot 37 = 148$:

1, 3, 4, 10, 14, 17, 18, 21, 22, 24, 25, 26, 28, 29, 30, 31, 32, 35
1, 6, 8, 9, 10, 11, 12, 14, 16, 17, 22, 23, 26, 27, 29, 31, 35, 36
0, 5, 6, 7, 8, 11, 13, 18, 19, 23, 24, 27, 32, 33, 34, 36
0, 2, 5, 11, 13, 15, 17, 19, 20, 22, 27, 35, 36

1, 7, 9, 10, 12, 14, 16, 17, 18, 22, 24, 26, 29, 31, 32, 33, 34, 35
1, 5, 6, 7, 8, 10, 13, 18, 19, 23, 24, 26, 32, 33, 34
2, 5, 11, 13, 14, 15, 18, 19, 20, 24, 27, 29, 31, 32, 36
2, 5, 6, 8, 9, 12, 13, 14, 15, 16, 19, 20, 23, 29, 31

The following four sets, also found by Djokovic [17], give 4-(43; 21, 21, 21, 15; 35) supplementary difference sets and may be used similarly to form a skew Hadamard matrix of order $4 \cdot 43 = 172$:

2, 3, 5, 7, 8, 12, 18, 19, 20, 22, 26, 27, 28, 29, 30, 32, 33, 34, 36, 39, 42 (twice)
1, 3, 4, 5, 6, 10, 11, 12, 16, 19, 20, 21, 23, 24, 31, 33, 35, 36, 38, 40, 41
0, 6, 7, 10, 18, 23, 24, 26, 28, 29, 30, 31, 34, 38, 40.

7.2. An Adaption of Wallis-Whiteman

We note the following adapation of the Goethals-Seidel matrix that does not require the matrix entries to be circulant at all:

Theorem 7.2 (J. Wallis-Whiteman [113]). *Suppose that X, Y, and W are type one incidence matrices and that Z is a type two incidence matrix of* 4-$\{v; k_1, k_2, k_3, k_4; \sum_{i=1}^{4} k_i - v\}$ *supplementary difference sets. If*

$$A = 2X - J, \qquad B = 2Y - J, \qquad C = 2Z - J, \qquad D = 2W - J,$$

then

$$H = \begin{bmatrix} A & B & C & D \\ -B^T & A^T & -D & C \\ -C & D^T & A & -B^T \\ -D^T & -C & B & A^T \end{bmatrix} \tag{7}$$

is an Hadamard matrix of order $4v$.

Further, if A is skew-type ($C^T = C$ as Z is of type two) then H is skew Hadamard.

This matrix can be used when the sets are from any finite abelian group. We now show how Theorem 7.2 may be further modified to obtain useful results.

Theorem 7.3 (J. Wallis-Whiteman [113]). *Suppose that X, Y, and W are type one incidence matrices and that Z is a type two incidence matrix of* 4-$\{2m + 1; m; 2(m - 1)\}$ *supplementary difference sets. If*

$$A = 2X - J, \qquad B = 2Y - J, \qquad C = 2Z - J, \qquad D = 2W - J,$$

and e is the $1 \times (2m + 1)$ *matrix with every entry 1, then*

$$H = \begin{bmatrix} -1 & -1 & -1 & -1 & e & e & e & e \\ 1 & -1 & 1 & -1 & -e & e & -e & e \\ 1 & -1 & -1 & 1 & -e & e & e & -e \\ 1 & 1 & -1 & -1 & -e & -e & e & e \\ e^T & e^T & e^T & e^T & A & B & C & D \\ -e^T & e^T & -e^T & e^T & -B^T & A^T & -D & C \\ -e^T & e^T & e^T & -e^T & -C & D^T & A & -B^T \\ -e^T & -e^T & e^T & e^T & -D^T & -C & B & A^T \end{bmatrix}$$

is an Hadamard matrix of order $8(m + 1)$. *Further, if A is skew type, H is skew Hadamard.*

Delsarte, Goethals, and Seidel's [15] important result states that if there exists a $W(n, n-1)$ for $n \equiv 0 \pmod 4$, then there exists a skew symmetric $W(n, n-1)$. This is used in the next result which uses orthogonal designs and is due to Seberry. The results for skew Hadamard matrices are far less complete than for Hadamard matrices.

Theorem 7.4 (Seberry [77]). *Let* $q \equiv 5 \pmod 8$ *be a prime power and* $p = \frac{1}{2}(q+1)$ *be a prime. Then there is a skew Hadamard matrix of order* $2^t p$, *where* $t \leq [2\log_2(p-2)]$.

7.3. Summary and Tables of Skew Hadamard Orders

Skew Hadamard matrices are known for the following orders (the reader should consult [114, pp. 451], [77] and Geramita and Seberry [23]):

SI	$2^t \Pi k_i$	t, r_i, all nonnegative positive integers $k_i - 1 \equiv 3 \pmod 4$ a prime power [66]
SII	$(p-1)^u + 1$	p the order of a skew Hadamard matrix, $u > 0$ an odd integer [105]
SIII	$2(q+1)$	$q \equiv 5 \pmod 8$ a prime power [98]
SIV	$2(q+1)$	$q = p^t$ is a prime power with $p \equiv 5 \pmod 8$ and $t \equiv 2 \pmod 4$ [99, 125]
SV	$4m$	$m \in \{$odd integers between 3 and 31 inclusive$\}$ [35, 100]; $m \in \{37, 39, 43, 49, 65, 67, 93, 113, 121, 127, 129, 133, 157, 163, 181, 217, 219, 241, 267\}$ [17, 16]
SVI	$mn(n-1)$	n the order of amicable orthogonal designs of types $((1, n-1); (n))$ and nm the order of an orthogonal design of type $(1, m, mn-m-1)$ [77]
SVII	$4(q+1)$	$q \equiv 9 \pmod{16}$ a prime power [113]
SVIII	$(\lvert t\rvert+1)(q+1)$	$q = s^2 + 4t^2 \equiv 5 \pmod 8$ a prime power, and $\lvert t\rvert + 1$ the order of a skew Hadamard matrix [117]
SIX	$4(q^2+q+1)$	q a prime power and $q^2 + q + 1 \equiv 3, 5$, or $7 \pmod 8$ a prime power or $2(q^2+q+1)+1$ a prime power [94]
SX	$2^t q$	$q = s^2 + 4r^2 \equiv 5 \pmod 8$ a prime power, and an orthogonal design $\mathrm{OD}(2^t; 1, a, b, c, c + \lvert r\rvert)$ exists where $1 + a + b + 2c + \lvert r\rvert = 2^t$ and $a(q+1) + b(q-4) = 2^t$ [77]
SXI	hm	h the order of a skew Hadamard matrix; m the order of amicable Hadamard matrices [121]

Spence [95] has found a new construction for SIV and Whiteman [125] a new construction for SI when $k_i - 1 \equiv 3 \pmod 8$. These are of considerable interest because of the structure involved and have use in the construction of orthogonal designs.

In Table 7.1, the lowest power of two for which a skew Hadamard matrix is known is indicated. For example, the entry (193,3) means a skew Hadamard matrix of order $2^3 \cdot 193$ is known, the entry (59,.) means a skew Hadamard matrix of order $2^t \cdot 59$ is not yet known for any t. Also, a blank represents 2.

8 M-STRUCTURES

Named after Mieko Yamada and Masahiko Miyamoto, M-structures have proved to be very powerful in attacking the question "if there is an Hadamard matrix of order $4t$, is there an Hadamard matrix of order $8t+4$?" M-structures provide another variety of "plug in" matrices that have yet to be fully exploited.

Table A.1 gives the present knowledge of Williamson matrices. The theorems were applied to get the table.

Definition 8.1. An orthogonal matrix of order $4t$ can be divided into 16 $t \times t$ blocks M_{ij}. This partitioned matrix is said to be an *M-structure*. If the orthogonal matrix can be partitioned into 64 $s \times s$ blocks M_{ij}, it will be called a 64 *block M-structure*.

An Hadamard matrix made from (symmetric) Williamson matrices W_1, W_2, W_3, W_4 is an M-structure with

$$W_1 = M_{11} = M_{22} = M_{33} = M_{44},$$
$$W_2 = M_{12} = -M_{21} = M_{34} = -M_{43},$$
$$W_3 = M_{13} = -M_{31} = -M_{24} = M_{42},$$
$$W_4 = M_{14} = -M_{41} = M_{23} = -M_{32}.$$

An Hadamard matrix made from four (4) circulant (or type 1) matrices A_1, A_2, A_3, A_4 of order n [where R is the matrix that makes all of the A_iR back circulant (or type 2)] is an M-structure with

$$A_1 = M_{11} = M_{22} = M_{33} = M_{44},$$
$$A_2 = M_{12}R = -M_{21}R = RM_{34}^T = -RM_{43}^T,$$
$$A_3 = M_{13}R = -M_{31}R = -RM_{24}^T = RM_{42}^T,$$
$$A_4 = M_{14}R = -M_{41}R = RM_{23}^T = -RM_{32}^T.$$

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist

q	t	q	t	q	t	q	t	q	t
1		89	4	177	.	265	4	353	4
3		91		179	8	267		355	
5		93		181		269	8	357	
7		95		183		271		359	4
9		97	9	185		273		361	3
11		99		187		275	4	363	
13		101	10	189		277	5	365	
15		103	3	191	.	279		367	
17		105		193	3	281		369	4
19		107	.	195		283	.	371	
21		109	9	197		285	3	373	7
23		111		199		287	4	375	
25		113		201	3	289	3	377	6
27		115		203		291		379	
29		117		205	3	293		381	
31		119	4	207		295	5	383	
33		121		209	4	297		385	3
35		123		211		299	4	387	
37		125		213	4	301	3	389	15
39		127		215		303	3	391	4
41		129		217		305	4	393	
43		131		219		307		395	
45		133		221		309	3	397	5
47	4	135		223	3	311	.	399	
49		137		225	4	313		401	.
51		139		227		315		403	5
53		141		229	3	317	6	405	
55		143		231		319	3	407	
57		145	5	233	4	321		409	3
59	.	147		235	3	323		411	
61		149	4	237		325	5	413	4
63		151	5	239	4	327		415	
65		153	3	241		329	6	417	
67		155		243		331	3	419	4
69	3	157		245	4	333		421	
71		159		247	6	335	7	423	4
73		161		249	4	337	18	425	
75		163		251	6	339		427	
77		165		253	4	341	4	429	3
79		167	4	255		343	6	431	
81	3	169	5	257	4	345	4	433	3
83		171		259	5	347	.	435	4
85		173		261	3	349	3	437	
87		175		263		351		439	

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist *(continued)*

q	t	q	t	q	t	q	t	q	t
441	3	529	3	617		705		793	3
443	6	531		619		707	4	795	3
445	3	533		621	3	709	.	797	
447		535		623	4	711		799	
449	.	537	5	625	3	713		801	
451	3	539	4	627	4	715		803	9
453		541	3	629	.	717	4	805	4
455	4	543	5	631	.	719	4	807	
457	.	545		633		721	5	809	.
459	3	547		635		723	3	811	
461	17	549	3	637	4	725	6	813	
463	7	551		639		727		815	
465	3	553	3	641	6	729	4	817	5
467		555		643	.	731	5	819	
469	3	557	.	645		733	.	821	6
471		559		647	.	735		823	3
473	5	561		649	7	737	7	825	
475	4	563		651		739	.	827	
477		565	3	653	.	741		829	.
479	.	567		655	4	743		831	
481	3	569	4	657	5	745	6	833	
483		571	3	659	.	747		835	
485	4	573	3	661	.	749	.	837	
487	5	575	4	663		751	3	839	.
489	3	577	.	665		753		841	8
491	.	579	5	667	6	755		843	
493	3	581	4	669	3	757		845	6
495		583	3	671		759	4	847	
497		585		673	7	761	.	849	3
499		587		675		763	11	851	.
501		589	5	677		765	4	853	3
503		591		679	3	767		855	
505	.	593		681	4	769	3	857	4
507		595	3	683		771		859	3
509	.	597	4	685		773	.	861	4
511		599	.	687		775		863	4
513	4	601	5	689	4	777	4	865	4
515	5	603		691		779	4	867	
517	6	605	4	693	4	781	3	869	4
519	4	607		695	4	783		871	
521		609	3	697	4	785	7	873	
523	7	611	6	699	3	787	5	875	
525		613	3	701		789	3	877	
527	4	615		703	3	791		879	

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist *(continued)*

q	t	q	t	q	t	q	t	q	t
881	6	905	4	929	.	953	.	977	
883	.	907	5	931		955	3	979	5
885		909	4	933	.	957	4	981	
887		911		935		959	4	983	
889	5	913	4	937	5	961	3	985	3
891	3	915		939		963		987	
893		917	4	941	6	965	4	989	4
895		919	3	943	4	967		991	3
897	5	921	4	945		969	4	993	
899	6	923		947	6	971	6	995	4
901	3	925	3	949	3	973	4	997	.
903	4	927	4	951		975		999	

8.1. Multiplication Theorems Using M-Structures

In this section, the reader wishing more details of constructions is referred to Seberry and Yamada [87]. As shown in Section 3, the power of M-structures comprising wholly circulant or type one blocks permits them to be multiplied by the order of T-matrices.

Theorem 8.1. *Suppose that there is an M-structure orthogonal matrix of order* $4m$ *with each block circulant or type one. Then there is an M-structure orthogonal matrix of order* $4mt$ *where* t *is the order of T-matrices.*

Further,

Theorem 8.2. *Let* $N = (N_{ij})$, $i,j = 1,2,3,4$, *be an Hadamard matrix of order* $4n$ *of M-structure. Further, let* T_{ij}, $i,j = 1,2,3,4$, *be* 16 $(0,+1,-1)$ *type* 1 *or circulant matrices of order* t *that satisfy*

$$\begin{aligned}
&1.\ T_{ij} * T_{ik} = 0,\ T_{ji} * T_{ki} = 0,\ j \neq k\ (* \text{ is the Hadamard product});\\
&2.\ \textstyle\sum_{k=1}^{4} T_{ik} \text{ is a } (1,-1) \text{ matrix};\\
&3.\ \textstyle\sum_{k=1}^{4} T_{ik}T_{ik}^T = tI_t = \sum_{k=1}^{4} T_{ki}T_{ki}^T;\\
&4.\ \textstyle\sum_{k=1}^{4} T_{ik}T_{jk}^T = 0 = \sum_{k=1}^{4} T_{ki}T_{kj}^T,\ i \neq j.
\end{aligned} \tag{8}$$

Then there is an M-structure Hadamard matrix of order $4nt$.

Corollary 8.3. *If there exists an Hadamard matrix of order* $4h$ *and an orthogonal design* $\mathrm{OD}(4u; u_1, u_2, u_3, u_4)$, *then an* $\mathrm{OD}(8hu; 2hu_1, 2hu_2, 2hu_3, 2hu_4)$ *exists. In particular, the* u_i*'s can be equal.*

This gives the theorem of Agayan and Sarukhanyan [1] as a corollary by setting all variables equal to one:

Corollary 8.4. *If there exist Hadamard matrices of orders* $4h$ *and* $4u$, *then there exists an Hadamard matrix of order* $8hu$.

We now give as a corollary a result motivated by (and a little stronger than) that of Agayan and Sarukhanyan [1]:

Corollary 8.5. *Suppose that there are Williamson or Williamson-type matrices of orders* u *and* v. *Then there are Williamson-type matrices of order* $2uv$. *If the matrices of orders* u *and* v *are symmetric, the matrices of order* $2uv$ *are also symmetric. If the matrices of orders* u *and* v *are circulant and/or type one, the matrices of order* $2uv$ *are type* 1.

Proof. Suppose A, B, C, D are (symmetric) Williamson or Williamson type matrices of order u, then they are pairwise amicable. Define

$$E = \tfrac{1}{2}(A+B), \qquad F = \tfrac{1}{2}(A-B), \qquad G = \tfrac{1}{2}(C+D), \qquad H = \tfrac{1}{2}(C-D),$$

then E, F, G, H are pairwise amicable (and symmetric) and satisfy

$$EE^T + FF^T + GG^T + HH^T = 2uI_u.$$

Now define

$$T_1 = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \qquad T_2 = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \qquad T_3 = \begin{bmatrix} 0 & G \\ G & 0 \end{bmatrix},$$

$$\text{and} \qquad T_4 = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix},$$

so that

$$T_1 = T_{11} = T_{22} = T_{33} = T_{44},$$
$$T_2 = T_{12} = -T_{21} = T_{34} = -T_{43},$$
$$T_3 = T_{13} = -T_{31} = -T_{24} = T_{42},$$
$$T_4 = T_{14} = -T_{41} = T_{23} = -T_{32},$$

in the theorem. Note that T_1, T_2, T_3, T_4 are pairwise amicable. If A, B, C, D were circulant (or type 1) they would be type 1 of order $2u$.

Let X, Y, Z, W be the Williamson or Williamson-type (symmetric) matrices of order v. Then X, Y, Z, W are pairwise amicable and

$$XX^T + YY^T + ZZ^T + WW^T = 4vI_v.$$

Then

$$L = T_1 \times X + T_2 \times Y + T_3 \times Z + T_4 \times W,$$

$$M = -T_1 \times Y + T_2 \times X + T_3 \times W - T_4 \times Z,$$

$$N = -T_1 \times Z - T_2 \times W + T_3 \times X + T_4 \times Y,$$

$$P = -T_1 \times W + T_2 \times Z - T_3 \times Y + T_4 \times X,$$

are 4 Williamson type (symmetric) matrices of order $2uv$. If the matrices of orders u and v were circulant or type 1, these matrices are type 1. □

8.2. Miyamoto's Theorem and Corollaries via M-Structures

In this section, we reformulate Miyamoto's [64] results so that symmetric Williamson-type matrices can be obtained. The results given here are due to Miyamoto, Seberry, and Yamada.

Lemma 8.6 (Miyamoto's Lemma Reformulated by Seberry-Yamada [87]). *Let U_i, V_j, $i, j = 1,2,3,4$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. *U_i, U_j are pairwise amicable, $i \neq j$;*
2. *V_i, V_j are pairwise amicable, $i \neq j$;*
3. *$U_i \pm V_i$ are $(+1, -1)$ matrices, $i = 1,2,3,4$;*
4. *the row sum of U_1 is 1, and the row sum of U_j is zero, $i = 2,3,4$;*
5. $\sum_{i=1}^{4} U_i U_i^T = (2n+1)I - 2J$, $\sum_{i=1}^{4} V_i V_i^T = (2n+1)I$.

Then there are four Williamson type matrices of order $2n+1$. Hence, there is a Williamson-type Hadamard matrix of order $4(2n+1)$. If U_i and V_i are symmetric, $i = 1,2,3,4$, then the Williamson type matrices are symmetric.

Proof. Let S_1, S_2, S_3, S_4 be 4 $(+1,-1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_1 = 2$ and of $S_i = 0$, $i = 2,3,4$. Now define

$$X_1 = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_1 \end{bmatrix} \quad \text{and} \quad X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \qquad i = 2,3,4.$$

First, note that since U_i, U_j, $i \neq j$, and V_i, V_j, $i \neq j$, are pairwise amicable,

$$\begin{aligned} S_i S_j^T &= \left(U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\left(U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \\ &= U_i U_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + V_i V_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= S_j S_i^T . \end{aligned}$$

(Note that this relationship is valid if and only if conditions (1) and (2) of the theorem are valid.)

$$\begin{aligned} \sum_{i=1}^{4} S_i S_i^T &= \sum_{i=1}^{4} U_i U_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^{4} V_i V_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= 2 \begin{bmatrix} 2(2n+1)I - 2J & -2J \\ -2J & 2(2n+1)I - 2J \end{bmatrix} \\ &= 4(2n+1)I_{2n} - 4J_{2n}. \end{aligned}$$

Next, we observe that

$$X_1 X_i^T = \begin{bmatrix} 1-2n & e_{2n} \\ e_{2n}^T & -J + S_1 S_i^T \end{bmatrix} = X_i X_1^T, \qquad i = 2,3,4,$$

and

$$X_i X_j^T = \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_i S_j^T \end{bmatrix} = X_j X_i^T, \qquad i \neq j, \quad i,j = 2,3,4.$$

Further,

$$\begin{aligned} \sum_{i=1}^{4} X_i X_i^T &= \begin{bmatrix} 1+2n & -3e_{2n} \\ -3e_{2n}^T & J + S_1 S_1^T \end{bmatrix} + \sum_{i=2}^{4} \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_i S_i^T \end{bmatrix} \\ &= \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J + 4(2n+1)I - 4J \end{bmatrix}. \end{aligned}$$

Thus, we have shown that X_1, X_2, X_3, X_4 are 4 Williamson-type matrices of order $2n+1$. Hence, there is a Williamson-type Hadamard matrix of order $4(2n+1)$. □

Many powerful corollaries which give many new results exist by suitable choices in the theorem. For example,

Corollary 8.7. *Let $q \equiv 1 \pmod 4$ be a prime power. Then there are symmetric Williamson-type matrices of order $q+2$ whenever $\frac{1}{2}(q+1)$ is a prime power or $\frac{1}{2}(q+3)$ is the order of a symmetric conference matrix. Also, there exists an Hadamard matrix of Williamson type of order $4(q+2)$.*

Corollary 8.8. *Let $q \equiv 1 \pmod 4$ be a prime power. Then*

1. *if there are Williamson type matrices of order $(q-1)/4$ or an Hadamard matrix of order $\frac{1}{2}(q-1)$, there exist Williamson type matrices of order q;*
2. *if there exist symmetric conference matrices of order $\frac{1}{2}(q-1)$ or a symmetric Hadamard matrix of order $\frac{1}{2}(q-1)$, then there exist symmetric Williamson type matrices of order q.*

Hence, there exists an Hadamard matrix of Williamson type of order $4q$.

Corollary 8.9. *Let $q \equiv 1 \pmod 4$ be a prime power or $q+1$ be the order of a symmetric conference matrix. Let $2q-1$ be a prime power. Then there exist symmetric Williamson type matrices of order $2q+1$ and an Hadamard matrix of Williamson type of order $4(2q+1)$.*

Note that this last corollary is a modified version of Miyamoto's Corollary 5 (original manuscript).

Theorem 8.10 (Miyamoto's Theorem [64] reformulated by Seberry-Yamada [87]). *Let U_{ij}, V_{ij}, $i,j = 1,2,3,4$, be $(0,+1,-1)$ matrices of order n that satisfy*

1. *U_{ki}, U_{kj} are pairwise amicable, $k = 1,2,3,4$, $i \neq j$;*
2. *V_{ki}, V_{kj} are pairwise amicable, $k = 1,2,3,4$, $i \neq j$;*
3. *$U_{ki} \pm V_{ki}$ are $(+1,-1)$ matrices, $i,k = 1,2,3,4$;*
4. *the row sum of U_{ii} is 1, and the row sum of U_{ij} is zero, $i \neq j$, $i,j = 1,2,3,4$;*
5. *$\sum_{i=1}^{4} U_{ji}U_{ji}^T = (2n+1)I - 2J$, $\sum_{i=1}^{4} V_{ji}V_{ji}^T = (2n+1)I$, $j = 1,2,3,4$;*
6. *$\sum_{i=1}^{4} U_{ji}U_{ki}^T = 0$, $\sum_{i=1}^{4} V_{ji}V_{ki}^T = 0$, $j \neq k$, $j,k = 1,2,3,4$.*

If conditions 1 to 5 hold, there are four Williamson-type matrices of order $2n+1$ and thus a Williamson type Hadamard matrix of order $4(2n+1)$. Furthermore, if the matrices U_{ki} and V_{ki} are symmetric for all $i,j = 1,2,3,4$, the Williamson matrices obtained of order $2n+1$ are also symmetric.

If conditions 3 to 6 hold, there is an M-structure Hadamard matrix of order $4(2n+1)$.

Proof. We prove the first assertion. Let S_{ij}, $i,j = 1,2,3,4$, be 16 $(+1,-1)$-matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_{ii} = 2$ and of $S_{ij} = 0$, $i \neq j$, $i, j = 1,2,3,4$. Now define

$$X_{11} = \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix}, \quad X_{12} = \begin{bmatrix} 1 & e \\ e^T & S_{12} \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix}, \quad X_{14} = \begin{bmatrix} -1 & e \\ e^T & S_{14} \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} 1 & e \\ e^T & S_{21} \end{bmatrix}, \quad X_{22} = \begin{bmatrix} -1 & -e \\ -e^T & S_{22} \end{bmatrix}, \quad X_{23} = \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix}, \quad X_{24} = \begin{bmatrix} -1 & e \\ e^T & S_{24} \end{bmatrix},$$

$$X_{31} = \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix}, \quad X_{32} = \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix}, \quad X_{33} = \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix}, \quad X_{34} = \begin{bmatrix} -1 & e \\ e^T & S_{34} \end{bmatrix},$$

$$X_{41} = \begin{bmatrix} -1 & e \\ e^T & -S_{41} \end{bmatrix}, \quad X_{42} = \begin{bmatrix} 1 & e \\ e^T & -S_{42} \end{bmatrix}, \quad X_{43} = \begin{bmatrix} -1 & e \\ e^T & -S_{43} \end{bmatrix}, \quad X_{44} = \begin{bmatrix} -1 & -e \\ -e^T & -S_{44} \end{bmatrix}.$$

Thus, X_{41}, X_{42}, X_{43}, X_{44} are 4 Williamson-type matrices of order $2n+1$, and thus a Williamson-type Hadamard matrix of order $4(2n+1)$ exists. □

Note that if we write our M-structure from the theorem as

$$\begin{matrix} -1 & 1 & 1 & -1 & -e & e & e & e \\ 1 & -1 & 1 & -1 & e & -e & e & e \\ 1 & 1 & -1 & -1 & e & e & -e & e \\ 1 & 1 & 1 & 1 & -e & -e & -e & e \\ -e^T & e^T & e^T & e^T & S_{11} & S_{12} & S_{13} & S_{14} \\ e^T & -e^T & e^T & e^T & S_{21} & S_{22} & S_{23} & S_{24} \\ e^T & e^T & -e^T & e^T & S_{31} & S_{32} & S_{33} & S_{34} \\ -e^T & -e^T & -e^T & e^T & S_{41} & S_{42} & S_{43} & S_{44} \end{matrix}$$

then we can see Yamada's matrix with trimming [131] or the J. Wallis-Whiteman [113] matrix with a border embodied in the construction.

Corollary 8.11. *Suppose that there exists a symmetric conference matrix of order $q+1 = 4t+2$ and an Hadamard matrix of order $4t = q-1$. Then there is an Hadamard matrix with M-structure of order $4(4t+1) = 4q$. Further, if the Hadamard matrix is symmetric, the Hadamard matrix of order $4q$ is of the form*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix},$$

where X, Y are amicable and symmetric.

In a similar fashion, we consider the following lemma so symmetric 8-Williamson-type matrices can be obtained.

Lemma 8.12 (Seberry-Yamada [87]). *Let U_i, V_j, $i, j = 1, \ldots, 8$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. *U_i, U_j, $i \neq j$ are pairwise amicable;*
2. *V_i, V_j, $i \neq j$ are pairwise amicable;*
3. *$U_i \pm V_i$ are $(+1, -1)$ matrices, $i = 1, \ldots, 8$;*
4. *the row (column) sums of U_1 and U_2 are both 1, and the row sum of U_i, $i = 3, \ldots, 8$ is zero;*
5. *$\sum_{i=1}^{8} U_i U_i^T = 2(2n+1)I - 4J$, $\sum_{i=1}^{8} V_i V_i^T = 2(2n+1)I$.*

Then there are 8-Williamson-type matrices of order $2n+1$. Furthermore, if the U_i and V_i are symmetric, $i = 1, \ldots, 8$, then the 8-Williamson-type matrices are symmetric. Hence, there is a block-type Hadamard matrix of order $8(2n+1)$.

Proof. Let $S_1, \ldots, S_8$ be 8 $(+1, -1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sums of S_1 and S_2 are both 2 and those of S_i are 0, $i = 3, \ldots, 8$. Now define

$$X_j = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_j \end{bmatrix}, \qquad j = 1, 2, \qquad \text{and}$$

$$X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \qquad i = 3, \ldots, 8.$$

Thus, we have that $X_1, \ldots, X_8$ are 8-Williamson type matrices of order $2n+1$.
Hence, there is a block-type Hadamard matrix of order $8(2n+1)$ obtained by replacing the variables of an orthogonal design $OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ by the 8-Williamson-type matrices. □

Some very powerful corollaries are

Corollary 8.13 [87]. *Let $q+1$ be the order of amicable Hadamard matrices $I+S$ and P. Suppose that there exist 4 Williamson-type matrices of order q. Then there exist Williamson-type matrices of order $2q+1$. Furthermore, there exists a 64 block M-structure Hadamard matrix of order $8(2q+1)$.*

Corollary 8.14. *Let q be a prime power and let $(q-1)/2$ be the order of (symmetric) 4 Williamson-type matrices. Then there exist (symmetric) 8 Williamson-type matrices of order q and a 64-block M-structure Hadamard matrix of order $8q$.*

Corollary 8.15. *Let $q \equiv 1 \pmod 4$ be a prime power or $q+1$ be the order of a symmetric conference matrix. Suppose that there exist (symmetric) 4 Williamson-type matrices of order q. Then there exist (symmetric) 8-Williamson-type matrices of order $2q+1$ and a 64-block M-structure Hadamard matrix of order $8(2q+1)$.*

Proof. Form the core Q. Thus, we choose

$$U_1 = I + Q, \quad U_2 = I - Q, \quad U_3 = U_4 = Q, \quad U_5 = U_6 = U_7 = U_8 = 0,$$
$$\text{and} \quad V_1 = V_2 = 0, \quad V_3 = V_4 = I, \quad V_{i+4} = W_i,$$

$i = 1,2,3,4$, where W_i are (symmetric) Williamson-type matrices. Then

$$\sum_{i=1}^{8} U_i U_i^T = 2(2q+1)I - 4J, \qquad \sum_{i=1}^{8} V_i V_i^T = 2(2q+1)I.$$

These U_i and V_i are then used in Lemma 8.12 to obtain the (symmetric) 8-Williamson-type matrices. □

This corollary gives 8-Williamson-type matrices for many new orders, but it does not give new Hadamard matrices for these orders.

Corollary 8.16 [87]. *Let $q = 9^t$, $t > 0$. There exist (symmetric) 4 Williamson-type matrices of order 9^t, $t > 0$. Hence, there exist (symmetric) 8-Williamson type matrices of order $2 \cdot 9^t + 1$, $t > 0$, and an Hadamard matrix of block structure of order $8(2 \cdot 9^t + 1)$.*

Also we have the following theorem:

Theorem 8.17 (Seberry-Yamada [87]). *Let U_{ij}, V_{ij}, $i,j = 1,\ldots,8$, be $(0,+1,-1)$ matrices of order n that satisfy*

1. *U_{ki}, U_{kj} are pairwise amicable, $k = 1,\ldots,8$, $i \neq j$;*
2. *V_{ki}, V_{kj} are pairwise amicable, $k = 1,\ldots,8$, $i \neq j$;*
3. *$U_{ki} \pm V_{ki}$ are $(+1,-1)$ matrices, $i,k = 1,\ldots,8$;*
4. *the row (column) sum of U_{ab} is 1 for $(a,b) \in \{(i,i),(i,i+1),(i+1,i)\}$, $i = 1,3,5,7$; the row (column) sum of U_{aa} is -1 for $a = 2,4,6,8$; and otherwise, the row (column) sum of U_{ij}, $i \neq j$ is zero;*
5. *$\sum_{i=1}^{8} U_{ji} U_{ji}^T = 2(2n+1)I - 4J$, $\sum_{i=1}^{8} V_{ji} V_{ji}^T = 2(2n+1)I$, $j = 1,\ldots,8$;*
6. *$\sum_{i=1}^{8} U_{ji} U_{ki}^T = 0$, $\sum_{i=1}^{8} V_{ji} V_{ki}^T = 0$, $j \neq k$, $j,k = 1,\ldots,8$.*

If conditions 1 to 5 hold, there are 8-Williamson-type matrices of order $2n+1$ and thus a block-type Hadamard matrix of order $8(2n+1)$. Further, if U_{7i}, V_{7i} are symmetric, $1 \le i \le 8$, then the 8-Williamson-type matrices are symmetric.

If conditions 3 to 6 hold, there is a 64-block M-structure Hadamard matrix of order $8(2n+1)$.

Proof. Let S_{ij} be 64 $(+1,-1)$-matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row (column) sum of S_{ii}, $S_{i,i+1}$, $S_{i+1,i}$ $i = 1,3,5,7$, is 2, the row (column) sum of S_{ii} is -2 for (i,i), $i = 2,4,6,8$, and otherwise, the row (column) sum of $S_{ij} = 0$, $i \neq j$. Now define

$$X_{11} = \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix}, \quad X_{12} = \begin{bmatrix} -1 & -e \\ -e^T & S_{12} \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix}, \quad X_{14} = \begin{bmatrix} 1 & e \\ e^T & S_{14} \end{bmatrix},$$

$$X_{15} = \begin{bmatrix} 1 & e \\ e^T & S_{15} \end{bmatrix}, \quad X_{16} = \begin{bmatrix} 1 & e \\ e^T & S_{16} \end{bmatrix}, \quad X_{17} = \begin{bmatrix} -1 & e \\ e^T & S_{17} \end{bmatrix}, \quad X_{18} = \begin{bmatrix} -1 & e \\ e^T & S_{18} \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} -1 & -e \\ -e^T & S_{21} \end{bmatrix}, \quad X_{22} = \begin{bmatrix} 1 & e \\ e^T & S_{22} \end{bmatrix}, \quad X_{23} = \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix}, \quad X_{24} = \begin{bmatrix} -1 & -e \\ -e^T & S_{24} \end{bmatrix},$$

$$X_{25} = \begin{bmatrix} 1 & e \\ e^T & S_{25} \end{bmatrix}, \quad X_{26} = \begin{bmatrix} -1 & -e \\ -e^T & S_{26} \end{bmatrix}, \quad X_{27} = \begin{bmatrix} -1 & e \\ e^T & S_{27} \end{bmatrix}, \quad X_{28} = \begin{bmatrix} 1 & -e \\ -e^T & S_{28} \end{bmatrix},$$

$$X_{31} = \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix}, \quad X_{32} = \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix}, \quad X_{33} = \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix}, \quad X_{34} = \begin{bmatrix} -1 & -e \\ -e^T & S_{34} \end{bmatrix},$$

$$X_{35} = \begin{bmatrix} 1 & e \\ e^T & S_{35} \end{bmatrix}, \quad X_{36} = \begin{bmatrix} 1 & e \\ e^T & S_{36} \end{bmatrix}, \quad X_{37} = \begin{bmatrix} -1 & e \\ e^T & S_{37} \end{bmatrix}, \quad X_{38} = \begin{bmatrix} -1 & e \\ e^T & S_{38} \end{bmatrix},$$

$$X_{41} = \begin{bmatrix} 1 & e \\ e^T & S_{41} \end{bmatrix}, \quad X_{42} = \begin{bmatrix} -1 & -e \\ -e^T & S_{42} \end{bmatrix}, \quad X_{43} = \begin{bmatrix} -1 & -e \\ -e^T & S_{43} \end{bmatrix}, \quad X_{44} = \begin{bmatrix} 1 & e \\ e^T & S_{44} \end{bmatrix},$$

$$X_{45} = \begin{bmatrix} 1 & e \\ e^T & S_{45} \end{bmatrix}, \quad X_{46} = \begin{bmatrix} -1 & -e \\ -e^T & S_{46} \end{bmatrix}, \quad X_{47} = \begin{bmatrix} -1 & e \\ e^T & S_{47} \end{bmatrix}, \quad X_{48} = \begin{bmatrix} 1 & -e \\ -e^T & S_{48} \end{bmatrix},$$

$$X_{51} = \begin{bmatrix} 1 & e \\ e^T & S_{51} \end{bmatrix}, \quad X_{52} = \begin{bmatrix} 1 & e \\ e^T & S_{52} \end{bmatrix}, \quad X_{53} = \begin{bmatrix} 1 & e \\ e^T & S_{53} \end{bmatrix}, \quad X_{54} = \begin{bmatrix} 1 & e \\ e^T & S_{54} \end{bmatrix},$$

$$X_{55} = \begin{bmatrix} -1 & -e \\ -e^T & S_{55} \end{bmatrix}, \quad X_{56} = \begin{bmatrix} -1 & -e \\ -e^T & S_{56} \end{bmatrix}, \quad X_{57} = \begin{bmatrix} -1 & e \\ e^T & S_{57} \end{bmatrix}, \quad X_{58} = \begin{bmatrix} -1 & e \\ e^T & S_{58} \end{bmatrix},$$

$$X_{61} = \begin{bmatrix} 1 & e \\ e^T & S_{61} \end{bmatrix}, \quad X_{62} = \begin{bmatrix} -1 & -e \\ -e^T & S_{62} \end{bmatrix}, \quad X_{63} = \begin{bmatrix} 1 & e \\ e^T & S_{63} \end{bmatrix}, \quad X_{64} = \begin{bmatrix} -1 & -e \\ -e^T & S_{64} \end{bmatrix},$$

$$X_{65} = \begin{bmatrix} -1 & -e \\ -e^T & S_{65} \end{bmatrix}, \quad X_{66} = \begin{bmatrix} 1 & e \\ e^T & S_{66} \end{bmatrix}, \quad X_{67} = \begin{bmatrix} -1 & e \\ e^T & S_{67} \end{bmatrix}, \quad X_{68} = \begin{bmatrix} 1 & -e \\ -e^T & S_{68} \end{bmatrix},$$

$$X_{71} = \begin{bmatrix} 1 & -e \\ -e^T & S_{71} \end{bmatrix}, \quad X_{72} = \begin{bmatrix} 1 & -e \\ -e^T & S_{72} \end{bmatrix}, \quad X_{73} = \begin{bmatrix} 1 & -e \\ -e^T & S_{73} \end{bmatrix}, \quad X_{74} = \begin{bmatrix} -1 & -e \\ -e^T & S_{74} \end{bmatrix},$$

$$X_{75} = \begin{bmatrix} 1 & -e \\ -e^T & S_{75} \end{bmatrix}, \quad X_{76} = \begin{bmatrix} 1 & -e \\ -e^T & S_{76} \end{bmatrix}, \quad X_{77} = \begin{bmatrix} 1 & e \\ e^T & S_{77} \end{bmatrix}, \quad X_{78} = \begin{bmatrix} 1 & e \\ e^T & S_{78} \end{bmatrix},$$

$$X_{81} = \begin{bmatrix} 1 & -e \\ -e^T & S_{81} \end{bmatrix}, \quad X_{82} = \begin{bmatrix} -1 & e \\ e^T & S_{82} \end{bmatrix}, \quad X_{83} = \begin{bmatrix} 1 & -e \\ -e^T & S_{83} \end{bmatrix}, \quad X_{84} = \begin{bmatrix} -1 & e \\ e^T & S_{84} \end{bmatrix},$$

$$X_{85} = \begin{bmatrix} 1 & -e \\ -e^T & S_{85} \end{bmatrix}, \quad X_{86} = \begin{bmatrix} -1 & e \\ e^T & S_{86} \end{bmatrix}, \quad X_{87} = \begin{bmatrix} 1 & e \\ e^T & S_{87} \end{bmatrix}, \quad X_{88} = \begin{bmatrix} -1 & -e \\ -e^T & S_{88} \end{bmatrix}.$$

Then provided conditions 1 to 5 hold, and $S_{7i}^T = S_{7i}$, $i = 1,\ldots,8$, are symmetric, X_{7i}, $i = 1,\ldots,8$, are symmetric 8-Williamson-type matrices. Otherwise, X_{7i}, $i = 1,\ldots,8$, are 8-Williamson-type matrices. This can be verified by straightforward checking. Hence, there is an Hadamard matrix of block structure of order $8(2n+1)$.

If conditions 3 to 6 hold, then straightforward verification shows the 64-block M-structure X_{ij} is an Hadamard matrix of order $8(2n+1)$. □

Corollary 8.18. *Let q be an odd prime power, and suppose that there exist Williamson-type matrices of order $\frac{1}{2}(q-1)$. Then there exists an M-structure Hadamard matrix of order $8q$.*

Corollary 8.19. *Let $q = 2m+1 \equiv 9 \pmod{16}$ be a prime power. Suppose that there are Williamson-type matrices of order q. Then there is a M-structure Hada-*
mard matrix of order $8(2q+1)$.

The analogous Yamada-J. Wallis-Whiteman structure to Theorem 8.17 is

$$\begin{array}{cccccccccccccccc}
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -e & -e & e & e & e & e & e & e \\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -e & e & e & -e & e & -e & e & -e \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & e & e & -e & -e & e & e & e & e \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & e & -e & -e & e & e & -e & e & -e \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & e & e & e & e & -e & -e & e & e \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & e & -e & e & -e & -e & e & e & -e \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -e & -e & -e & -e & -e & -e & e & e \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -e & e & -e & -e & e & e & -e & e \\
-e^T & -e^T & e^T & e^T & e^T & e^T & e^T & e^T & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\
-e^T & e^T & e^T & -e^T & e^T & -e^T & e^T & -e^T & S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\
e^T & e^T & -e^T & -e^T & e^T & e^T & e^T & e^T & S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\
e^T & -e^T & -e^T & e^T & e^T & -e^T & e^T & -e^T & S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\
e^T & e^T & e^T & e^T & -e^T & -e^T & e^T & e^T & S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} \\
e^T & -e^T & e^T & -e^T & -e^T & e^T & e^T & -e^T & S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} \\
-e^T & -e^T & -e^T & -e^T & -e^T & -e^T & e^T & e^T & S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} \\
-e^T & e^T & -e^T & e^T & -e^T & e^T & e^T & -e^T & S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88}
\end{array}$$

With some trimming, we can see Yamada's matrix [131] or the J. Wallis-Whiteman [113] matrix with a border embodied in the construction. Miyamoto has done further work using the quaternions rather than the complex numbers to build bigger M-structures [64]. This work is probably further extendable.

9 WILLIAMSON AND WILLIAMSON-TYPE MATRICES

In the previous section, we saw many constructions for Williamson-type matrices using M-structures. *Williamson matrices* and *Williamson-type matrices* were defined in Section 3. They are the most used "plug in" matrices and give many previously unknown Hadamard matrices.

Williamson's famous theorem is

Theorem 9.1 (Williamson [128]). *Suppose that there exist four symmetric* $(1,-1)$ *matrices* A,B,C,D *of order* n *that commute in pairs. Further, suppose that*

$$A^2 + B^2 + C^2 + D^2 = 4nI_n.$$

Then

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \tag{9}$$

is an Hadamard matrix of order $4n$ *of* Williamson type *or* quaternion type.

Theorem 9.2 (Williamson). *If there exist solutions to the equations*

$$\mu_i = 1 + 2\left\{\sum_{j=1}^{s} t_{ij}(w^j + w^{n-j})\right\}, \qquad i = 1,2,3,4$$

where $s = \frac{1}{2}(n-1)$, w *is an nth root of unity, exactly one of* $t_{1j}, t_{2j}, t_{3j}, t_{4j}$ *is nonzero and equals* ± 1 *for each* $j = 1,2,\ldots,s$, *and*

$$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 4n,$$

then there exist matrices A,B,C,D *satisfying Theorem* 9.1 *of the form*

$$A = \sum_{i=0}^{n-1} a_i T^i, \qquad a_0 = 1, \quad a_i = a_{n-i} = \pm 1;$$

$$B = \sum_{i=0}^{n-1} b_i T^i, \qquad b_0 = 1, \quad b_i = b_{n-i} = \pm 1;$$

$$C = \sum_{i=0}^{n-1} c_i T^i, \qquad c_0 = 1, \quad c_i = c_{n-i} = \pm 1;$$

$$D = \sum_{i=0}^{n-1} d_i T^i, \qquad d_0 = 1, \quad d_i = d_{n-1} = \pm 1.$$

where T is the matrix whose (i,j) entry is 1 if $j - i \equiv 1 \pmod n$ and 0 otherwise. Hence, there exists an Hadamard matrix of order $4n$.

Table 9.1 shows the μ_i found by Williamson [128], Baumert and Hall [5], Djokovic [18], Koukouvinos and Kounias [52], and Sawade [74]. We write w_j for $w^j + w^{n-j}$ and w_{2j} for $w^{2^j} + w^{n-2^j}$. Williamson found the results for 148 and 172, Baumert and Hall for 92, Baumert for 116, Sawade for 100 and 108, Koukouvinos and Kounias for 132, and Djokovic for 156. Results have also appeared in Baumert [3, 4], Koukouvinos [49], and Yamada [130].

Note: The sums of squares in Table 9.1 are not necessarily those of the corresponding ± 1 matrix. For example, the ± 1 matrices corresponding to $92 = 1^2 + 1^2 + (-3)^2 + 9^2$ have row sums $3, 3, 7, -5$.

Example 9.1. How to turn the formulas in Table 9.1 into Williamson matrices? Let $t = 13$, $n = 52$, $\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1^2 + 1^2 + 1^2 + 7^2$. Form four sums:

$$\sigma_1 = -\mu_1 + \mu_2 + \mu_3 + \mu_4 = 2 + 2w_1 - 2w_2 - 2w_3 - 2w_4 + 2w_5 - 2w_6,$$

$$\sigma_2 = \mu_1 - \mu_2 + \mu_3 + \mu_4 = 2 + 2w_1 - 2w_2 - 2w_3 - 2w_4 + 2w_5 - 2w_6,$$

$$\sigma_3 = \mu_1 + \mu_2 - \mu_3 + \mu_4 = 2 - 2w_1 - 2w_2 - 2w_3 + 2w_4 - 2w_5 + 2w_6,$$

$$\sigma_4 = \mu_1 + \mu_2 + \mu_3 - \mu_4 = 2 + 2w_1 + 2w_2 + 2w_3 - 2w_4 + 2w_5 - 2w_6.$$

Then, recalling $w_i = w^i + w^{n-i}$, we use $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to form the first rows (coefficients of T^i) of the circulant matrices A, B, C, D, respectively. σ_1 gives $a_0, a_2, \ldots, a_{12}$ as

$$a_0 = 1, \quad a_1 = a_{12} = 1, \quad a_2 = a_{11} = -1, \quad a_3 = a_{10} = -1,$$

$$a_4 = a_9 = -1, \quad a_5 = a_8 = 1, \quad a_6 = a_7 = -1$$

so the first row of A is

$$1\,1\,-\,-\,-\,1\,-\,-\,1\,-\,-\,-\,1 \quad \text{and} \quad \tfrac{1}{4}\sigma_1^2 = (-3)^2.$$

For B,C,D, we have

$$1\,1\,-\,-\,-\,1\,-\,-\,1\,-\,-\,-\,1 \quad \text{and} \quad \tfrac{1}{4}\sigma_2^2 = (-3)^2,$$
$$1\,-\,-\,-\,1\,-\,1\,1\,-\,1\,-\,-\,- \quad \text{and} \quad \tfrac{1}{4}\sigma_3^2 = (-3)^2,$$
$$1\,1\,1\,1\,-\,1\,-\,-\,1\,-\,1\,1\,1 \quad \text{and} \quad \tfrac{1}{4}\sigma_4^2 = 5^2,$$

where $4n = 52 = 3^2 + 3^2 + 3^2 + 5^2$.

We now introduce some matrices that were first used by Seberry and Whiteman [85] in the construction of conference matrices. Matrices obeying the same equations are constructed using auxilliary matrices from projective planes in [80].

Suppose that $B_1, B_2, \ldots, B_s$ are square $(1,-1)$ matrices of order b that satisfy

$$\begin{aligned} B_i^2 &= B_iB_j = J, && i,j \in \{1,2,\ldots,s\}; \\ B_iB_j^T &= B_j^TB_i = J, && i \neq j, \quad i,j \in \{1,2,\ldots,s\}; \\ B_iJ &= aJ, && a \in Z^+; \\ \sum_{i=1}^{s} B_iB_i^T &+ B_i^TB_i = 2sbI_b. \end{aligned} \tag{10}$$

Call s matrices satisfying equations (10) a *regular s-set* of matrices. Define, in particular,

$$\begin{aligned} A_i &= B_i \times \tfrac{1}{2}(B + B^T) + B_{i+1} \times \tfrac{1}{2}(B - B^T), \qquad i = 1,3,\ldots,s-1, \\ A_{i+1} &= -B_i \times \tfrac{1}{2}(C - C^T) + B_{i+1} \times \tfrac{1}{2}(C + C^T), \end{aligned}$$

where B,C is a regular 2-set and B_j, $j = 1,\ldots,s$, is a regular s-set of matrices. Then $A_1,\ldots,A_s$ is a regular s-set of matrices. Thus, we have

Lemma 9.3. *If there exists a regular s-set of matrices of order a, and a regular 2-set of order b, then there exists a regular s-set of order ab.*

So in the special case $s = t = 2$, if A_1, A_2 is a regular 2-set of order a and B_1, B_2 is a regular 2-set of order b, then C_1, C_2 is a regular 2-set of order $c = ab$.

In Seberry and Whiteman [85], it is shown that

Theorem 9.4 (Seberry-Whiteman). *If $n \equiv 3 \pmod 4$ is a prime power, then there exists a regular $\frac{1}{2}(n+1)$-set of matrices of order n^2.*

In particular, if $n = 3$, there is a regular 2-set of matrices of order 9. Hence, using Lemma 9.3, we have a regular 2-set of matrices of order 9^t, $t > 0$. Thus, we have another proof of Turyn's theorem.

TABLE 9.1 Hadamard Matrices from Williamson Matrices

t	n	$\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2$	μ_1	μ_2	μ_3	μ_4
3	12	$1^2+1^2+1^2+3^2$	1	1	1	$1-2w_1$
5	20	$1^2+1^2+3^2+3^2$	1	1	$1-2w_1$	$1-2w_2$
7	28	$1^2+3^2+3^2+3^2$	1	$1-2w_1$	$1-2w_2$	$1-2w_3$
7	28	$1^2+1^2+1^2+5^2$	1	1	$1+2w_1-2w_2$	$1+2w_3$
9	36	$3^2+3^2+3^2+3^2$	$1-2w_1$	$1-2w_2$	$1-2w_3$	$1-2w_4$
9	36	$1^2+1^2+3^2+5^2$	1	$1+2w_1-2w_2$	$1-2w_4$	$1+2w_3$
			1	1	$1-2w_2$	$1+2w_1+2w_3-2w_4$
11	44	$1^2+3^2+3^2+5^2$	$1+2w_1-2w_2$	$1-2w_4$	$1-2w_5$	$1+2w_3$
13	52	$1^2+1^2+1^2+7^2$	1	1	$1+2w_1-2w_4+2w_5-2w_6$	$1-2w_2-2w_3$
			1	$1+2w_4-2w_5$	$1-2w_1-2w_6$	$1-2w_2-2w_3$
13	52	$3^2+3^2+3^2+5^2$	$1-2w_2$	$1-2w_4$	$1-2w_1-2w_3+2w_5$	$1+2w_6$
13	52	$1^2+1^2+5^2+5^2$	$1-2w_3+2w_4$	$1-2w_2+2w_6$	$1+2w_1$	$1+2w_5$
15	60	$1^2+3^2+5^2+5^2$	1	$1-2w_5$	$1+2w_6$	$1+2w_1-2w_2+2w_3+2w_4-2w_7$
			$1-2w_1+2w_7$	$1-2w_3$	$1+2w_2$	$1+2w_4+2w_5-2w_6$
			$1-2w_4+2w_6$	$1-2w_1-2w_3+2w_5$	$1+2w_7$	$1+2w_2$
15	60	$1^2+1^2+3^2+7^2$	1	1	$1-2w_1-2w_5+2w_7$	$1+2w_2-2w_3-2w_4-2w_6$
17	68	$3^2+3^2+5^2+5^2$	$1-2w_2$	$1-2w_8$	$1-2w_1+2w_5+2w_6$	$1+2w_3-2w_4+2w_7$
17	68	$1^2+3^2+3^2+7^2$	$1-2w_3-2w_5+2w_6+2w_7$	$1-2w_2$	$1-2w_8$	$1-2w_1-2w_4$
			1	$1-2w_4-2w_5+2w_6$	$1-2w_1-2w_3+2w_7$	$1-2w_2-2w_8$
17	68	$1^2+3^2+3^2+7^2$	1	$1-2w_2-2w_4+2w_5$	$1-2w_1+2w_3-2w_8$	$1-2w_6-2w_7$

19	76	$1^2+5^2+5^2+5^2$	1	$1+2w_1-2w_2+2w_4$	$1-2w_3+2w_6+2w_8$	$1-2w_5+2w_7+2w_9$
			$1-2w_3-2w_4+2w_5+2w_9$	$1+2w_2-2w_7+2w_2$	$1+2w_6$	$1+2w_1$
			1	$1-2w_3+2w_8+2w_9$	$1+2w_4-2w_5+2w_7$	$1+2w_1-2w_2+2w_6$
19	76	$3^2+3^2+3^2+7^2$	None			
19	76	$1^2+1^2+5^2+7^2$	1	1	$1+2w_1-2w_3+2w_8$	$1+2w_2-2w_4-2w_5+2w_6-2w_7-2w_9$
			$1-2w_2+2w_8$	$1-2w_4+2w_7$	$1+2w_3+2w_6-2w_9$	$1-2w_1-2w_5$
			$1+2w_4-2w_8$	$1+2w_2-2w_5$	$1+2w_1$	$1-2w_3-2w_6+2w_7-2w_9$
21	84	$3^2+5^2+5^2+5^2$	$1-2w_7$	$1+2w_3+2w_5-2w_8$	$1-2w_2+2w_4+2w_6$	$1+2w_1+2w_9-2w_{10}$
21	84	$1^2+1^2+1^2+9^2$	$1+2w_2-2w_3$	$1-2w_6+2w_{10}$	$1+2w_8-2w_9$	$1+2w_1+2w_4+2w_5-2w_7$
			1	1	$1-2w_5-2w_6+2w_7+2w_9$	$1+2w_1+2w_2-2w_3+2w_4+2w_8-2w_{10}$
			$1-2w_3+2w_9$	$1+2w_8-2w_{10}$	$1+2w_4-2w_5$	$1+2w_1+2w_2-2w_6+2w_7$
21	84	$1^2+3^2+5^2+7^2$	$1-2w_4+2w_5$	$1+2w_2-2w_6-2w_8-2w_9+2w_{10}$	$1+2w_1$	$1-2w_3-2w_7$
			$1-2w_5+2w_9$	$1+2w_2-2w_4-2w_{10}$	$1+2w_6+2w_7-2w_8$	$1-2w_1-2w_3$
			$1-2w_6+2w_8$	$1+2w_2-2w_4-2w_{10}$	$1+2w_5+2w_7-2w_9$	$1-2w_1-2w_3$
23	92	$1^2+1^2+3^2+9^2$	$1-2w_4-2w_8+2w_9+2w_{11}$	$1+2w_5-2w_7$	$1+2w_1-2w_3-2w_{10}$	$1+2w_2+2w_6$
23	92	$3^2+3^2+5^2+7^2$	None			
25	100	$1^2+3^2+3^2+9^2$	$1+2w_6-2w_{11}$	$1-2w_1+2w_3-2w_{12}$	$1+2w_4-2w_7-w_9$	$1+2w_2+2w_5-2w_8+2w_{10}$
25	100	$5^2+5^2+5^2+5^2$	$1+2w_1-2w_6+2w_9$	$1+2w_7-2w_8+2w_{12}$	$1+2w_2-2w_4+2w_5$	$1-2w_3+2w_{10}+2w_{11}$

TABLE 9.1 Hadamard Matrices from Williamson Matrices *(continued)*

t	n	$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2$	μ_1	μ_2	μ_3	μ_4
25	100	$1^2 + 1^2 + 7^2 + 7^2$	1	1	$1 - 2w_2 - 2w_3 - 2w_5 + 2w_6 - 2w_7 + 2w_{12}$	$1 - 2w_1 - 2w_4 + 2w_8 + 2w_9 - 2w_{10} - 2w_{11}$
			$1 + 2w_3 - 2w_7$	$1 - 2w_1 + 2w_4$	$1 + 2w_8 - 2w_9 - 2w_{10} - 2w_{11}$	$1 - 2w_2 - 2w_5 + 2w_6 - 2w_{12}$
			$1 + 2w_3 - 2w_9$	$1 + 2w_4 - 2w_{12}$	$1 - 2w_1 - 2w_7$	$1 + 2w_6 + 2w_8 - 2w_{11} - 2w_{10} - 2w_5 - 2w_2$
25	100	$1^2 + 5^2 + 5^2 + 7^2$	$1 + 2w_5 - 2w_{10}$	$1 + 2w_6 + 2w_{11} - 2w_2$	$1 + 2w_{12} + 2w_9 + 2w_4 - 2w_7 - 2w_8$	$1 - 2w_1 - 2w_3$
27	108	$1^2 + 1^2 + 9^2 + 5^2$	1	1	$1 - 2w_3 + 2w_4 + 2w_5 + 2w_7 - 2w_9 + 2w_{12}$	$1 - 2w_1 - 2w_2 + 2w_6 + 2w_8 + 2w_{10} - 2w_{11} + 2w_{13}$
27	108	$1^2 + 3^2 + 7^2 + 7^2$	$1 + 2w_5 + 2w_2 - 2w_8 - 2w_7$	$1 + 2w_9 - 2w_{10} - 2w_{11}$	$1 + 2w_3 - 2w_4 - 2w_{13} - 2w_6$	$1 - 2w_1 - 2w_{12}$
27	108	$3^2 + 3^2 + 3^2 + 9^2$	None			
27	108	$3^2 + 5^2 + 5^2 + 7^2$	$1 + 2w_1 - 2w_4 - 2w_6$	$1 + 2w_{10} + 2w_{13} - 2w_{11}$	$1 + 2w_5 + 2w_2 - 2w_{12}$	$1 + 2w_7 - 2w_8 - 2w_3 - 2w_9$
29	116	$1^2 + 3^2 + 5^2 + 9^2$	$1 + 2w_2 - 2w_4 + 2w_6 - 2w_9 - 2w_{11} + 2w_{12}$	$1 - 2w_3 - 2w_5 + 2w_7 - 2w_8 + 2w_{10}$	$1 + 2w_1$	$1 + 2w_{13} + 2w_{14}$
31	124	$1^2 + 1^2 + 1^2 + 11^2$	1	1	$1 + 2w_3 + 2w_4 + 2w_5 - 2w_6 - 2w_8 - 2w_{12}$	$1 - 2w_1 - 2w_2 + 2w_7 - 2w_9 + 2w_{10} - 2w_{11} - 2w_{13} - 2w_{14} + 2w_{15}$
31	124	$3^2 + 3^2 + 5^2 + 9^2$	$1 - 2w_2 + 2w_{13} - 2w_{14}$	$1 + 2w_4 - 2w_{10} - 2w_{15}$	$1 + 2w_1 + 2w_3 - 2w_5 - 2w_6 + 2w_7$	$1 + 2w_8 + 2w_9 + 2w_{11} - 2w_{12}$
33	132	$1^2 + 1^2 + 3^2 + 11^2$	$1 + 2w_2 + 2w_5 - 2w_6 - 2w_8 - 2w_9 + 2w_{11}$	$1 + 2w_1 - 2w_{13} + 2w_{14} - 2w_{16}$	$1 - 2w_3 - 2w_7 + 2w_{12}$	$1 - 2w_4 - 2w_{10} - 2w_{15}$
33	132	$1^2 + 1^2 + 7^2 + 9^2$	$1 - 2w_6 - 2w_8 + 2w_{11} + 2w_{16}$	$1 - 2w_2 + 2w_3 - 2w_{10} + 2w_{14}$	$1 + 2w_1 - 2w_5 - 2w_{12} - 2w_{15}$	$1 + 2w_4 - 2w_7 + 2w_9 + 2w_{13}$

33	132	$1^2+5^2+5^2+9^2$	$1+2w_5-2w_7+2w_{12}-2w_{15}$	$1-2w_2+2w_{10}+2w_{16}$	$1-2w_1+2w_4+2w_6+2w_9-2w_{13}$	$1+2w_3+2w_8+2w_{11}-2w_{14}$
33	132	$1^2+5^2+5^2+9^2$	$1+2w_1-2w_8+2w_{10}-2w_{15}$	$1+2w_4-2w_7+2w_{13}$	$1+2w_7-2w_{12}-2w_{14}$	$1+2w_3+2w_5-2w_6-2w_9+2w_{11}+2w_{16}$
33	132	$3^2+5^2+7^2+7^2$	$1-2w_7-2w_{11}+2w_{12}$	$1-2w_5+2w_{14}+2w_{15}$	$1+2w_1-2w_3-2w_6-2w_8+2w_9-2w_{13}$	$1+2w_2-2w_4-2w_{10}-2w_{16}$
35	140	Any decomposition	None			
37	148	$1^2+1^2+5^2+11^2$	1	1	$1-2w_2-2w_6-2w_7-2w_8+2w_{11}-2w_{13}+2w_{14}$	$1+2w_1+2w_3-2w_4+2w_5-2w_9+2w_{10}-2w_{12}-2w_{15}-2w_{16}+2w_{17}+2w_{18}$
37	148	$5^2+5^2+7^2+7^2$	$1-2w_1+2w_3+2w_4+2w_7-2w_{11}$	$1+2w_5-2w_6-2w_8+2w_{13}+2w_{18}$	$1+-2w_2-2w_9-2w_{10}+2w_{16}$	$1-2w_{12}-2w_{14}+2w_{15}-2w_{17}$
37[a]	148	$1^2+7^2+7^2+7^2$	1	$1-2\alpha_0-2\alpha_1-2\alpha_5$	$1-2\alpha_3-2\alpha_4+2\alpha_8$	$1+2\alpha_2-2\alpha_6-2\alpha_7$
39	156	$3^2+7^2+7^2+7^2$	$1-2w_{13}$	$1-2w_3+2w_5-2w_{10}-2w_{11}+2w_{17}-2w_{18}$	$1-2w_6+2w_7-2w_8-2w_{12}-2w_{14}+2w_{16}$	$1+2w_1+2w_2-2w_4-2w_9-2w_{15}-2w_{19}$
43[b]	172	$1^2+1^2+1^2+13^2$	$1+2\alpha_0-2\alpha_2$	$1-2\alpha_1+2\alpha_3$	$1+2\alpha_4-2\alpha_6$	$1+2\alpha_5$
61	244	$1^2+1^2+11^2+11^2$	1	1	$1+2w_1-2w_6+2w_7-2w_9-2w_{10}+2w_{13}+2w_{17}+2w_{18}-2w_{19}-2w_{22}-2w_{26}+2w_{27}-2w_{28}-2w_{29}-2w_{30}$	$1-2w_2-2w_3+2w_4-2w_5+2w_8+2w_{11}-2w_{12}-2w_{14}+2w_{15}+2w_{16}-2w_{20}+2w_{21}-2w_{23}-2w_{25}-2w_{26}$

[a] $\alpha_j = w_{2^j} + w_{2^{9+j}}$.
[b] $\alpha_j = w_{3^j} + w_{3^{7+j}} + w_{3^{14+j}}$.

Corollary 9.5 (Turyn [109]). *There are Williamson-type matrices of order* 9^t, $t > 0$, *that pairwise satisfy* $XY = XY^T = J$, $XJ = 3J$.

Example 9.2. The regular 2-set of matrices of order 9 can be written as B, C where writing a, b, c, W for the circulant matrices with first rows

$$[0++]\ [-+-]\ [--+]\ [0+-],$$

respectively, we have

$$c = b^T, \qquad b + c = -2I.$$

The matrix B is

$$\begin{bmatrix} -c & a-I & -b \\ a-I & -b & -c \\ -b & -c & a-I \end{bmatrix}.$$

It should be noted that B is a block back-circulant matrix whose elements are circulant matrices. Hence, B is neither a type one nor a type two matrix over $Z_3 \times Z_3$ (perhaps it should be referred to as a type three matrix over $Z_3 \times Z_3$), but it can still be defined as a group matrix over $Z_3 \times Z_3$.

The matrix B may also be written in the form

$$B = \begin{bmatrix} M & MT & MT^2 \\ MT & MT^2 & M \\ MT^2 & M & MT \end{bmatrix} \quad \text{or} \quad M\begin{bmatrix} I & T & T^2 \\ T & T^2 & I \\ T^2 & I & T \end{bmatrix},$$

where $M = I + W$, W is as before, and T is the circulant matrix (shift matrix) with first row $[0+0]$. Note that

$$T^2 = T^T, \qquad T^3 = I, \qquad I + T + T^2 = J.$$

The matrix C is constructed as follows:

$$\begin{bmatrix} + & + & - & + & + & - & + & + & - \\ + & + & - & + & + & - & + & + & - \\ + & + & - & + & + & - & + & + & - \\ - & + & + & - & + & + & - & + & + \\ - & + & + & - & + & + & - & + & + \\ - & + & + & - & + & + & - & + & + \\ + & - & + & + & - & + & + & - & + \\ + & - & + & + & - & + & + & - & + \\ + & - & + & + & - & + & + & - & + \end{bmatrix}.$$

The construction of the matrix C is an ingenious idea of Mathon. Note that C is *not* composed of circulants or back circulants.

The matrix C may also be written in the form

$$C = \begin{bmatrix} N & N & N \\ NT & NT & NT \\ NT^2 & NT^2 & NT^2 \end{bmatrix} \quad \text{or} \quad N \begin{bmatrix} I & I & I \\ T & T & T \\ T^2 & T^2 & T^2 \end{bmatrix},$$

where

$$N = \begin{bmatrix} + & + & - \\ + & + & - \\ + & + & - \end{bmatrix}.$$

Note that each row of N is the same as the top row of M.

Corollary 9.6. *Since there is a regular 4-set of regular matrices of order 49 and a regular 2-set of regular matrices of order 9^t, $t > 0$, there is a regular 4-set of regular matrices of order $49 \cdot 9^t$. Hence, there are 8-Williamson-type matrices of order $49 \cdot 9^t$, $t \geq 0$.*

Using the OD(8;1,1,1,1,1,1,1) and the Plotkin OD(24;3,3,3,3,3,3,3,3), we have

Corollary 9.7. *There is an Hadamard matrix of order $8 \cdot 49 \cdot 3^t$, $t \geq 0$.*

In general, we have

Corollary 9.8. *If $n \equiv 3 \pmod 4$ is a prime power, there is a regular $\frac{1}{2}(n+1)$-set of regular matrices of order n^2. Hence, there are $(n+1)$-Williamson-type matrices of order $n^2 \cdot 9^t$, $t \geq 0$ each with row sum $3^t n$.*

This also means that we have

Corollary 9.9. *If $n \equiv 3 \pmod 4$ is a prime power, there is an Hadamard matrix of order $n^2(n+1) \cdot 9^t$, $t \geq 0$.*

Proof. Choose a Latin square of size $n+1$ and an Hadamard matrix $H = (h_{ij})$ of order $n+1$. Replace the $1, 2, 3, \ldots, \frac{1}{2}(n+1)$th elements of the Latin square by $B_1, B_2, \ldots, B_{(n+1)/2}$ and the $\frac{1}{2}(n+3)$rd$, \ldots, (n+1)$th elements by B_1^T, $B_2^T, \ldots, B_{(n+1)/2}^T$. We now have a block matrix (B_{ij}). The required Hadamard matrix is $(h_{ij} B_{ij})$. □

This method is considered further in [80], where it is used to show

Theorem 9.10 (Seberry). *Let q be a prime power. Then there are Williamson-type matrices of order*

1. $\frac{1}{2}q^2(q+1)$ *when* $q \equiv 1 \pmod 4$,
2. $\frac{1}{4}q^2(q+1)$ *when* $q \equiv 3 \pmod 4$, *and there are Williamson-type matrices of order* $\frac{1}{4}(q+1)$.

Example 9.3. Let $B_1, B_2, \ldots, B_6$ be the matrices constructed by Seberry and Whiteman [85] or Seberry [80] of order 121. Write $S_1 = B_1$, $S_7 = B_1^T$, $S_2 = B_2$, $S_8 = B_2^T, \ldots, S_6 = B_6$, $S_0 = B_6^T$.

Let

$$W_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad W_2 = W_3 = W_4 = \begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} S_1 & S_2 & S_3 \\ S_3 & S_1 & S_2 \\ S_2 & S_3 & S_1 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} S_4 & S_5 & S_6 \\ S_6 & S_4 & S_5 \\ S_5 & S_6 & S_4 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} S_7 & S_8 & S_9 \\ S_9 & S_7 & S_8 \\ S_8 & S_9 & S_7 \end{bmatrix}, \qquad Y_4 = \begin{bmatrix} S_{10} & S_{11} & S_0 \\ S_0 & S_{10} & S_{11} \\ S_{11} & S_0 & S_{10} \end{bmatrix},$$

and

$$X_1 = Y_1, \qquad X_2 = \begin{bmatrix} S_4 & -S_5 & -S_6 \\ -S_6 & S_4 & -S_5 \\ -S_5 & -S_6 & S_4 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} S_7 & -S_8 & -S_9 \\ -S_9 & S_7 & -S_8 \\ -S_8 & -S_9 & S_7 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} S_{10} & -S_{11} & -S_0 \\ -S_0 & S_{10} & -S_{11} \\ -S_{11} & -S_0 & S_{10} \end{bmatrix}.$$

Now the S_i are 12 $(1,-1)$ matrices of order 11^2, satisfying

$$S_i S_j^T = J, \qquad i \neq j,$$

$$\sum_{i=0}^{11} S_i S_i^T = 11^2 \cdot 12 I \times I.$$

Thus, $X_1 X_j^T = -J \times J$, $j = 2,3,4$,

$$X_i X_j^T = \begin{bmatrix} 3J & -J & -J \\ -J & 3J & -J \\ -J & -J & 3J \end{bmatrix}, \qquad i,j = 2,3,4,$$

and

$$\sum_{i=1}^{4} X_i X_i^T = \sum_{j=0}^{11} S_i S_i^T \times I = 11^2 \cdot 12 I \times I.$$

Hence, X_1, X_2, X_3, X_4 are Williamson-type matrices of order 363.

9.1. New Difference Sets

M. Xia and G. Liu [129] have recently announced the existence of 4-$\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets for $q \equiv 1 \pmod 4$ a prime power. A. L. Whiteman has also given the following set of 4-$\{9;3;3\}$ supplementary difference sets:

$$\{0,1,2\}, \qquad \{0,x,2x\}, \qquad \{0,x+1,2x+2\}, \qquad \{0,x+2,2x+1\},$$

whose incidence matrices A_i, $i = 1,2,3,4$, satisfy $A_i A_j = J$, $i \neq j$, and he has given 4-$\{25;10;15\}$ supplementary difference sets

$$\{2,3,x+1,x+2,x+3,2x+4,3x+1,4x+2,4x+3,4x+4\},$$

$$\{1,2,3,4,x,x+4,2x+4,3x+1,4x,4x+1\},$$

$$\{1,4,x+2,2x+1,2x+2,2x+4,3x+1,3x+3,3x+4,4x+3\},$$

$$\{1,2,3,4,x+2,2x,2x+3,3x,3x+2,4x+3\}.$$

The Xia-Liu result means the following:

Theorem 9.11 (Xia-Liu). *There exist four Williamson matrices of order q^2 for all $q \equiv 1 \pmod 4$ a prime power. The negation of each matrix has row sum q.*

This also gives Williamson matrices of orders p^4 for $p \equiv 3 \pmod 4$ a prime because then $p^2 \equiv 1 \pmod 4$. Thus,

Corollary 9.12. *There exist four Williamson matrices of orders 3^4, 5^4, and p^4, $p \equiv 3 \pmod 4$ a prime.*

Now OD$(4t;t,t,t,t)$ exist for $t = 3,9,27,5,25,125,7,49,11,121$, for all $t \equiv 1 \pmod 4$, t prime $\in \{13,17,29,37,41,53,61,101,\ldots\}$, and for t prime of the form $1 + 2^a 10^b 26^c$, $a,b,c \geq 0$. This gives

Corollary 9.13. *There exist Hadamard matrices of order $4 \cdot 3^r$, $4 \cdot 5^r$, $4 \cdot 13^r$, $4 \cdot 17^r$, $4 \cdot 29^r$, $4 \cdot 37^r$, $4 \cdot 41^r$, $4 \cdot 53^r$, $4 \cdot 61^r$, $4 \cdot 101^r$, $r \geq 0$; $4 \cdot g^{4i}$, $4 \cdot g^{4i+1}$, $4 \cdot g^{4i+2}$, $8 \cdot g^{4i+3}$, $i \geq 0$, $g = 7,11$; and $4 \cdot p^r$ whenever $p = 1 + 2^a 10^b 26^c$ is prime, $a,b,c \geq 0$.*

9.2. Other Results

We define a *complete regular 4-set of regular* matrices of order q^2 as four matrices satisfying

$$\begin{aligned}
A_i^T &= A_i, \\
A_iA_j &= pJ, \quad p \text{ constant}, \quad i \neq j, \quad i,j = 1,2,3,4, \\
\sum_{i=1}^{4} A_iA_i^T &= 4q^2I, \\
A_iJ &= qJ.
\end{aligned}$$

These are a special form of Williamson type matrices and exist for at least orders 9^i, $i = 1,2$.

As with regular 2-sets of regular matrices, we have

Theorem 9.14 (Seberry). *If there exist complete regular 4-sets of regular matrices of orders s^2 and t^2 respectively there exists a complete regular 4-set of regular matrices of order s^2t^2.*

Proof. Let the complete regular 4-sets of regular matrices of order s^2 and t^2 be A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4, respectively. Then

$$\begin{aligned}
C_1 &= \tfrac{1}{2}[A_1 \times (B_1 + B_2) + A_2 \times (B_1 - B_2)], \\
C_2 &= \tfrac{1}{2}[-A_1 \times (B_3 - B_4) + A_2 \times (B_3 + B_4)], \\
C_3 &= \tfrac{1}{2}[A_3 \times (B_1 + B_2) - A_4 \times (B_1 - B_2)], \\
C_4 &= \tfrac{1}{2}[A_3 \times (B_3 - B_4) + A_4 \times (B_3 + B_4)],
\end{aligned}$$

is a complete regular 4-set of regular matrices of order s^2t^2. □

Corollary 9.15. *If there exist complete regular 4-sets of regular matrices of orders $q_1, q_2, \ldots$, then there exists a complete regular 4-set of regular matrices of order $q_1.q_2.q_3\ldots$, and Williamson-type matrices.*

Many authors have found suitable and near suitable matrices of Williamson type, and this will be pursued in a later article. Appendix A.2 gives a summary of orders for which Williamson and Williamson-type matrices exist plus a list of known orders < 2000.

10 SBIBD AND THE EXCESS OF HADAMARD MATRICES

10.1. SBIBD$(4t, 2t-1, t-1)$

Every Hadamard matrix H of order $4t$ is associated in a natural way with an SBIBD with parameters $(4t-1, 2t-1, t-1)$, and with its complement, an SBIBD$(4t-1, 2t, t)$. To obtain the SBIBD, we first normalize H and write the resultant matrix in the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix}.$$

Then

$$AJ = JA = -J \qquad \text{and} \qquad AA^T = 4tI - J.$$

So $B = \frac{1}{2}(A + J)$ satisfies

$$BJ = JB = (2t-1)J \qquad \text{and} \qquad BB^T = tI + (t-1)J.$$

Thus, B is a $(0,1)$ matrix satisfying the equations for the incidence matrix of an SBIBD with parameters $(4t-1, 2t-1, t-1)$. Similarly, $C = \frac{1}{2}(J - A)$ is the incidence matrix of an SBIBD with parameters $(4t-1, 2t, t)$. Clearly, if we start with the incidence matrix of an SBIBD with parameters $(4t-1, 2t-1, t-1)$ or $(4t-1, 2t, t)$ and replace all the 0 elements by -1, we form either A or $-A$. Thus,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & & & \\ \vdots & & -A & \\ -1 & & & \end{bmatrix}$$

are Hadamard matrices of order $4t$ obtained from these SBIBD.

Thus, we have shown

Theorem 10.1. *There exists an Hadamard matrix of order* $4t$ *if and only if there exists an SBIBD*$(4t-1, 2t-1, t-1)$.

Since a $(4t-1, 2t-1, t-1)$ difference set yields an SBIBD we have

Corollary 10.2. *If there exists a* $(4t-1, 2t-1, t-1)$ *difference set, then there exists an Hadamard matrix of order* $4t$.

In view of the Seberry theorem [121] (see Section 3) we have that

Theorem 10.3. *Let q be any odd natural number. Then there exists a t ($\leq [2\log_2(q-3)]$) so that there is an SBIBD($2^t q-1, 2^{t-1}q-1, 2^{t-2}q-1$).*

Constructions given above indicate that for small q ($< 10{,}000$) $t = 2$ in about 97% of cases, and $t = 3,4,5$ in about 2% of further cases. So for $q <$ 10,000 most SBIBD($4q-1, 2q-1, q-1$) exist. Table A.2 in Appendix A.3 illustrates this point.

10.2. The Equivalence Theorem

The main theorem of this section deals with the equivalence among Hadamard matrices with maximal excess, regular Hadamard matrices, and certain SBIBDs. We begin with the definition of excess of a Hadamard matrix.

Definition 10.1. Let H be an Hadamard matrix of order n. The sum $\sigma(H)$ of the elements of H is called the *excess* of H. The maximum excess of H, over all Hadamard matrices of order n, is denoted by $\sigma(n)$; i.e.,

$$\sigma(n) = \max\{\sigma(H) : H \text{ an Hadamard matrix of order } n\}.$$

An equivalent notion is the *weight* of H, denoted $w(H)$, which is defined as the number of 1's in H. It follows that $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$ (see [8]).

Theorem 10.4. *There is an Hadamard matrix of order $n = 4s^2$ with maximal excess $n\sqrt{n} = 8s^3$ if and only if there is an SBIBD($4s^2, 2s^2+s, s^2+s$).*

In (Seberry) Wallis [114, p. 343], it is pointed out that Goethals and Seidel [25] and Shrikhande and Singh [92] have established

Theorem 10.5. *If there exists a BIBD($2k^2-k, 4k^2-1, 2k+1, k, 1$), then there exists a symmetric Hadamard matrix of order $4k^2$ with constant diagonal.*

Moreover, Shrikhande [90] has studied these designs and shown they exist for all $k = 2^t$, $t \geq 1$. They are also known for $k = 3,5,6,7$ [114].

In (Seberry) Wallis [114, pp. 344–346], it is established that symmetric Hadamard matrices of order h with constant diagonal exist for $h = 2^{2t}$ for all $t \geq 1$, and for $h = 36, 100, 144, 196$ (after Theorem 5.15 of [114]). Using results of (Seberry) Wallis-Whiteman [113] and Szekeres [99], they are shown to exist with the extra property of regularity (constant row sum) for $h = 4\cdot 5^2, 4\cdot 13^2, 4\cdot 29^2$, $4\cdot 51^2$, and $h = 4(2((p-3)/4)+1)^2$, for $p \equiv 3 \pmod 4$ a prime power (after Theorem 5.15 of [114]).

Remark 10.1. A theorem of Goethals and Seidel [25] (see Geramita and Seberry [23]) tells us that if there is an Hadamard matrix with constant diagonal of order $4k$, then there is a regular symmetric Hadamard matrix with constant diagonal of order $4(2k)^2$. So an Hadamard matrix of order $4t$ gives a regular symmetric Hadamard matrix with constant diagonal of order $4k^2$, $k = 2t$. In particular, known results give these matrices for $2t \le 210$.

Remark 10.2. We note that regular symmetric Hadamard matrices with constant diagonal of orders $4s^s$ and $4t^2$ give a regular symmetric Hadamard matrix with constant diagonal with order $(2st)^2$.

Theorem 10.6 (J. Wallis [114]). *A regular Hadamard matrix H of order $4k^2$ with row sum $\pm 2k$ exists if and only if there exists an SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$.*

We observe that the stipulation that the row sum is $\pm 2k$ is unnecessary for the following reason: If the matrix is regular, it must have constant row sum, say x. Thus, $eH^T = (x, \ldots, x)$, where e is the $1 \times 4k^2$ matrix of ones. Now $H^T H = 4k^2 I$, so

$$16k^4 = 4k^2 ee^T = eH^T He^T = (x, \ldots, x)(x, \ldots, x)^T = 4k^2 x^2.$$

Thus, $x = \pm 2k$. The matrix with constant row sum $-2k$ is the negative of the matrix with constant row sum $2k$.

We can now combine the results obtained so far as

Theorem 10.7 (Equivalence Theorem). *The following are equivalent:*

1. *There exists an Hadamard matrix of order $4k^2$ with maximal excess $8k^3$.*
2. *There exists a regular Hadamard matrix of order $4k^2$.*
3. *There is an SBIBD$(4k^2, 2k^2 + k, k^2 + k)$ (and its complement the SBIBD$(4k^2, 2k^2 - k, k^2 - k)$).*

Part of this result was also observed by Brown and Spencer [9] and Best [8].

We also note the following consequence of the Liu-Xia result mentioned in Section 9. In the next theorem, we need the notion of a proper n-dimensional Hadamard matrix. This is defined to be an n-dimensional array (with entries -1 and 1) such that every two-dimensional face is an Hadamard matrix.

Theorem 10.8. *Suppose that there exist 4-$\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets. Then*

1. *there is a regular symmetric Hadamard matrix with constant diagonal of order $4q^2$ with maximal excess $8q^3$;*
2. *there is an SBIBD$(4q^2, 2q^2 \pm q, q^2 \pm q)$;*
3. *there is a proper n-dimensional Hadamard matrix of order $(4q^2)^n$.*

10.3. Excess

In this section, we present several results dealing with the excess of a Hadamard matrix and the excess of an orthogonal design. We begin with an example.

Example 10.1. The excess of the following Hadamard matrices

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix},$$

is easily determined. We see that $\sigma(H_2) = 2$, $\sigma(H_4) = 4$, $\sigma(R_4) = 8$. Since R_4 has the maximal excess of all Hadamard matrices of order 4, $\sigma(4) = 8$. We can find the Hadamard matrix of maximal excess of order 8 quite easily. We note that if H and K are Hadamard matrices, then so is

$$\begin{bmatrix} H & H \\ K & -K \end{bmatrix}$$

and, in particular,

$$H_8 = \begin{bmatrix} R_4 & R_4 \\ H_4 & -H_4 \end{bmatrix}, \qquad \sigma(H_8) = 16.$$

Now H_8 has its fifth column $(-, 1, 1, 1, -, -, -, -)^T$. Negating this column gives R_8 where $\sigma(R_8) = 20$.

This construction yields

Lemma 10.9. $\sigma(2n) \geq 2\sigma(n) + 4$.

Noting that the Kronecker product of two Hadamard matrices is an Hadamard matrix, we have

Lemma 10.10. $\sigma(mn) \geq \sigma(m)\sigma(n)$.

We define the *excess of the orthogonal design* $D = x_1A_1 + \cdots + x_uA_u$ as

$$\sigma(D) = \sigma(A_1) + \cdots + \sigma(A_u),$$

where $\sigma(A_i)$ is the sum of the entries of A_i. This is equivalent to putting all the variables equal to $+1$.

The concept of excess of orthogonal designs is used by Hammer-Levingston-Seberry [34] to obtain bounds on the excess of Hadamard matrices and by Seberry [82], Koukouvinos and Kounias [54] and Koukouvinos, Kounias, and Seberry [55] to find Hadamard matrices of order $4k^2$ with maximal excess and equivalently SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$.

Example 10.2. The excesses of the OD$(4;1,1,1,1)$

$$D_1 = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}, \qquad D_2 = \begin{bmatrix} -A & B & C & D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix},$$

are

$$\sigma(D_1) = \sigma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \end{bmatrix}$$

$$+ \sigma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \\ - & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & - & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix}$$

$$= 4 + 0 + 0 + 0 = 4,$$

$$\sigma(D_2) = \sigma \begin{bmatrix} - & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$+ \sigma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= 2 + 2 + 2 + 2 = 8.$$

Constructions that give OD's of larger order with large excess could lead to a construction such as that of Seberry Wallis [121] for Hadamard matrices of large excess.

10.4. Bounds on the Excess of Hadamard Matrices

Many authors, including Brown and Spencer [9], Best [8], Enomoto and Miyamoto [21], Farmakis and Kounias [22, 61], Hammer, Levingston, and Seberry [34], Jenkins, Koukouvinos, Kounias, J. Seberry, and R. Seberry [39], Kharaghani [41], Koukouvinos and Kounias [54], Koukouvinos, Kounias, and Seberry [56], Koukouvinos and Seberry [59], Sathe and Shenoy [73], Schmidt and Wang [76], Seberry [82], Wallis [122] and Yamada [131] have found the excess of Hadamard matrices for particular orders or families of orders. Lower and upper bounds have been given [8, 61, 34, 56]. Here, we are interested in the upper bound, which is surveyed in Jenkins et al. [39].

The most encompassing upper bound is that of Brown and Spencer [9] and later by Best [8].

Brown-Spencer-Best Bound: $\sigma(n) \le n\sqrt{n}$ Now, in the case of $n = 4k^2$, we can restate this bound as $\sigma(4k^2) \le 8k^3$. Hadamard matrices with maximal excess meeting this bound have been found by Koukouvinos, Kounias, Seberry, and Yamada [54, 56, 82, 131] for $n = 4k^2$ with even k when there is an Hadamard matrix of order $2k$ (in particular, for all $2k \le 210$) and also for $k \in \{1, 3, 5, \ldots, 45, 49, \ldots, 69, 73, 75, 81, \ldots, 101, 105, 109, 125, 625\} \cup \{3^{2m}, 25 \cdot 3^{2m} : m \ge 0\}$.

Let a_i, $1 \le i \le n$, be the ith row sum of an Hadamard matrix of order n. Denote the integer part of z by $[z]$. Then, with

$$a_1 = a_2 = \cdots = a_i = t,$$

$$a_{i+1} = a_{i+2} = \cdots = a_n = t + 4,$$

where $t = [\sqrt{n}]$ when $[\sqrt{n}]$ is even and $t = [\sqrt{n}] - 3$ when $[\sqrt{n}]$ is odd, and i is the integer part of $(n((t+4)^2 - n)/8(t+2)$, the Brown-Spencer-Best bound can be refined to the HLS bound (see [34]).

Hammer-Levingston-Seberry (HLS) Bound: $\sigma(n) \le n(t+4) - 4i$ Jenkins et al. [39] lists a number of cases where this bound is satisfied. The HLS bound has been improved for some orders by Farmakis and Kounias [22]. Write $n = (2x+1)^2 + 3$. Then $[\sqrt{n}] = 2x + 1$. From HLS bound, putting $t = [\sqrt{n}] - 3 = 2x - 2$, $i = x^2 + x + 1$,

$$\sigma(n) \le n(2x+2) - 4(x^2 + x + 1) = n(2x+1) = n\sqrt{n-3}.$$

Thus, we have the Farmakis-Kounias bound.

Farmakis-Kounias (KF) Bound: $\sigma(n) \le n\sqrt{n-3}$ ***for*** $n = (2x+1)^2 + 3$ In some special cases, the HLS and KF bound are identical. If $n = (2x+1)^2 + 3$, both give $\sigma(n) \le n\sqrt{n-3}$. Hadamard matrices of order $n = (2x+1)^2 + 3$

satisfying the bound $\sigma(n) \leq n\sqrt{n-3}$ with equality are known for

$$x = 0, 1, \ldots, 7, 9, 11, 16, 18, 22, 25, 26, 29, 36, 37, 49.$$

There is also the Kharaghani-Kounias-Farmakis bound.

Kharaghani-Kounias-Farmakis Bound: $\sigma(n) \leq 4(m-1)^2(2m+1)$ ***for*** $n = 4m(m-1)$ Hadamard matrices are known that meet this bound for some values of m where m is the order of a skew Hadamard matrix, the order of a conference matrix, or the order of a skew complex Hadamard matrix [60, 56]. The precise details of the constructions used to find the Hadamard matrices of maximal excess and order $4k^2$ can be found in Koukouvinos, Kounias, and Sotirakoglou [51], Koukouvinos, Kounias, and Seberry [56], and Seberry [83].

Using all the known results we have the following:

Theorem 10.11. *Hadamard matrices of order* $4k^2$ *with maximal excess* $8k^3$ *exist for*

1. k *even,* $k \leq 210$*, or if an Hadamard matrix of order* $2k$ *exists;*
2. $k \in \{1, 3, 5, \ldots, 45, 49, \ldots, 57, 61, \ldots, 69, 75, 81, \ldots, 95, 99, 115, 117, 625\} \cup \{3^{2m}, 5^2 \cdot 3^{2m} : m \geq 0\}$;
3. $k = qs$, $q \in \{q : q \equiv 1 \pmod 4$ *is a prime power*$\}$, $s \in \{1, \ldots, 33, 37, \ldots, 41, 45, \ldots, 59\} \cup \{2g+1 : g$ *the length of a Golay sequence*$\}$.

It follows from the equivalence theorem (Theorem 10.7) that regular Hadamard matrices of order $4k^2$ and *SBIBD*$(4k^2, 2k^2 \pm k, k^2 \pm k)$ also exist for these k values.

11 COMPLEX HADAMARD MATRICES

Complex Hadamard matrices were first introduced by Richard J. Turyn [104] who showed how they could be used to construct Hadamard matrices. These matrices are very important for they exist for orders for which symmetric conference matrices cannot exist. Complex Hadamard matrices also give powerful "multiplication" theorems. They are conjectured to exist for all even orders [114], a conjecture that implies the Hadamard conjecture.

Known small orders and a list of classes of complex Hadamard matrices are given in this section. This section is not a complete study of complex Hadamard matrices; it just gives some interesting constructions.

Theorem 11.1 (Turyn [104]). *If* C *is a complex Hadamard matrix of order* c *and* H *is a real Hadamard matrix of order* h*, then there exists a real Hadamard matrix of order* hc.

We note a connection between complex Hadamard matrices and matrices to "plug into."

Lemma 11.2. *If there is a complex Hadamard matrix, $C = H + iK$ of order n, then H and K are amicable, disjoint, suitable matrices of total weight n.*

Lemma 11.3. *If there is a complex Hadamard matrix, $C = H + iK$ of order n, then there is an orthogonal design* $\mathrm{OD}(2n; n, n)$ *and amicable orthogonal designs* $\mathrm{AOD}(2n; (n, n); (n, n))$.

Proof. Let a, b be commuting variables and use

$$\begin{bmatrix} aH - bK & aH + bK \\ -aH - bK & aH - bK \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} aH + bK & aH - bK \\ aH - bK & -aH - bK \end{bmatrix}. \qquad \square$$

11.1. Constructions for Complex Hadamard Matrices

Theorem 11.4 (Turyn [104]). *If C and D are complex Hadamard matrices of orders r and q, then $C \times D$ (where $\times$ is the Kronecker product) is a complex Hadamard matrix of order rq.*

Proof. $CC^* = rI$ and $DD^* = qI$, so $(C \times D)(C^* \times D^*) = rqI$. $\square$

Theorem 11.5 (Turyn [104]). *If $I + N$ is a symmetric conference matrix, then $iI + N$ is a (symmetric) complex Hadamard matrix and $I + iN$ is a complex skew Hadamard matrix.*

Adapting a theorem of Turyn [104], Kharaghani and Seberry [43] showed

Theorem 11.6. *There is an Hadamard matrix of order $4m$ of the form*

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

if and only if there is a complex Hadamard matrix of order $2m$ of the form

$$\begin{bmatrix} S & T \\ -\overline{T} & \overline{S} \end{bmatrix},$$

where $\overline{T}$ denotes the complex conjugate of T.

This theorem and the next lemma show complex Hadamard matrices are also related to matrices to "plug in."

Lemma 11.7 (Kharaghani and Seberry [42]). *Suppose that A, B, C, D are four Williamson-type matrices of order m with constant row and column sum a, a, b, b. Then there exists a regular complex Hadamard matrix of order $2m$, with row sum $a + ib$.*

Proof. We form $X = \frac{1}{2}(A + B)$, $Y = \frac{1}{2}(A - B)$, $W = \frac{1}{2}(C + D)$ and $V = \frac{1}{2}(C - D)$, which have row sums $a, 0, b, 0$. Then

$$E = \begin{bmatrix} X + iY & V + iW \\ -V + iW & X - iY \end{bmatrix}$$

is the required regular complex Hadamard matrix with row and column sum $a + ib$. □

Lemma 11.8 (Kharaghani and Seberry [42]). *Let g be the length of a pair of Golay sequences U and V. Suppose that the row sums of U and V are a and b, so $2g = a^2 + b^2$. Then there is a regular complex Hadamard matrix of order $2g$, with row sum $a + ib$.*

Proof. Use U and V as the first rows of circulant matrices X and Y of order g. Then

$$C = \begin{bmatrix} X & iY \\ iY^T & X^T \end{bmatrix}$$

is the required regular complex Hadamard matrix. □

Lemma 11.9 (Kharaghani and Seberry [42]). *Suppose that there is a regular complex Hadamard matrix C of order $4c$, with row sum $a + ib$ and of the form*

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix},$$

where A and B are real. Then $D = \frac{1}{2}(-i + 1)(A + iB)$ is a regular complex Hadamard matrix of order $2c$ with row sum $\frac{1}{2}(a + b) + \frac{1}{2}(a - b)i$.

Lemma 11.10 (Kharaghani-Seberry [42]). *Let $c_1, c_2, \ldots, c_{2c}$ be the columns of a complex Hadamard matrix C. Define C_i to be the $2c \times 2c$ matrix $C_i = c_i c_i^*$ (where $*$ is the hermitian conjugate). Then*

1. $C_i = C_i^*$; $C_i C_j = 0$, $i \neq j$;
2. $\sum_{i=1}^{2c} C_i = 2cI_{2c}$; $\sum_{i=1}^{2c} C_i C_i^* = 4c^2 I_{2c}$.

The next four results, found by Kharaghani and Seberry [42], are based on the work of Kharaghani:

Theorem 11.11 (Kharaghani-Seberry [42]). *Let C be a complex Hadamard matrix of order c. Then there is a regular complex Hermitian Hadamard matrix, D of order c^2 with constant diagonal and with row (and column) sum c. Hence D has element sum c^3.*

Proof. Form $C_1, \ldots, C_c$ of order c as in the Lemma 11.5. Now from condition 1, $\sum_{i=1}^{c} C_i = cI_c$, and from condition 2, $C_i C_j^* = 0$.

Form the block back-circulant complex Hadamard matrix

$$D = \begin{bmatrix} C_1 & C_2 & \cdots & C_c \\ C_2 & C_3 & \cdots & C_1 \\ \vdots & & & \vdots \\ C_c & C_1 & \cdots & C_{c-1} \end{bmatrix}$$

of order c^2 which has row and column sum c and hence element sum c^3. The diagonal of each C_j, $j = 1, \ldots, c$, is one by condition 1 of Lemma 11.10, so D has diagonal one. Moreover, each C_j is hermitian, $C_j^* = C_j$, so D is hermitian. □

Lemma 11.12 (Kharaghani-Seberry [42]). *Let $H, C_1, C_2, \ldots, C_n$ be $(1, -1, i, -i)$ matrices of order n satisfying*

1. $HH^* = nI_n$; $HC_j = C_jH^*$;
2. $C_j^* = C_j$; $C_jC_k = 0$, $k \neq j$; $\sum_{j=1}^{n} C_j^2 = n^2 I_n$.

Then there is a complex Hadamard matrix of order $2n(n+1)$ of the form

$$D = \begin{bmatrix} A & iB \\ iB^* & A^* \end{bmatrix}$$

where A and B are block circulant. Furthermore, if $H, C_1, C_2, \ldots, C_n$ are real and H is regular, then D is regular.

Corollary 11.13. *For each positive integer n, there is a regular complex Hadamard matrix of order $4^n(4^n + 1)$.*

The next result is based on a similar theorem for real Hadamard matrices by Mukopadhyay [65].

Theorem 11.14 (Kharaghani-Seberry). *Suppose that there exists a skew-type complex Hadamard matrix $C = I + U$ of order $p + 1$, where $U^* = -U$. Further, suppose that there exist two $(1, -1, i, -i)$ matrices A_r, B_r of order q satisfying*

1. $A_rB_r^* = B_rA_r^*$,
2. $A_rA_r^* + pB_rB_r^* = q(1 + p)I_q$.

Then there are two $(1,-1,i,-i)$ *matrices of order* $p^j q$, $j \geq 0$, *satisfying*

$$A_{r+j}B^*_{r+j} = B_{r+j}A^*_{r+j},$$

$$A_{r+j}A^*_{r+j} + pB_{r+j}B^*_{r+j} = qp^j(p+1)I.$$

Also, there exists a complex Hadamard matrix of order $qp^j(p+1)$ *for every* $j \geq 0$.

Corollary 11.15. *Let* $n+1$ *be the order of a symmetric conference matrix. Then there is a complex Hadamard matrix of order* $n^j(n+1)$ *for every* $j \geq 0$.

A result analogous to the next one was also found by R. Turyn [104].

Lemma 11.16 (Miyamoto [64]). *If there is an Hadamard matrix of order* $4t$ *with structure*

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

then there is a complex Hadamard matrix of order $2t$.

Proof. From the Hadamard matrix $AB^T = BA^T$ and $AA^T + BB^T = 4tI_{2t}$. Let

$$E = \frac{1}{2}(A+B) - \frac{i}{2}(A-B).$$

Then the elements of E are $1, -1, i, -i$ and

$$EE^* = \frac{1}{2}(AA^T + BB^T) + \frac{i}{2}(-AB^T + BA^T) = 2tI_{2t}.$$

Thus, E is the desired complex Hadamard matrix. Clearly, E will be a real matrix if and only if $A = B$. □

This lemma, in view of many recent results on Williamson-type matrices gives us many new complex Hadamard matrices:

Corollary 11.17. *Let* w *be the order of a Williamson-type matrix. Then there exists a complex Hadamard matrix of order* $2w$. *In particular, there are complex Hadamard matrices for orders* $2c$, $c \in \{33, 39, 53, 73, 81, 83, 89, 93, 101, 105, 109, 113, 125, 137, 149, 153, 173, 189, 193, 197, 233, 241, 243, 257, 277, 281, 293\}$.

Kharaghani and Seberry went on to show how certain complex Hadamard matrices were extremely powerful in the construction of real Hadamard matrices with large excess.

Seberry and Whiteman [84] have also found complex weighing matrices analogous to the real matrices of Goethals and Seidel [25], and these matrices give some of the unsolved complex orthogonal designs of Geramita and Geramita [24].

11.2. Constructions Using Amicable Hadamard Matrices

Theorem 11.18 (Seberry-Wallis [114]). *Let $W = I + C$ be a complex skew Hadamard matrix of order w. Let $M = I + U$ and N be complex amicable orthogonal designs* CAOD$(m;(1,m-1),(m))$ *of order m satisfying $U^* = -U$ and $N^* = N$. Further, let X,Y,Z be pairwise amicable complex matrices of order p that are suitable matrices for a complex orthogonal design,* COD$(wm;1,m-1,(w-1)m)$:

$$XX^* + (m-1)YY^* + (w-1)mZZ^* = wpmI.$$

Then there is a complex Hadamard matrix of order wpm.

Proof. Use $K = I \times I \times X + I \times U \times Y + C \times N \times Z$. □

Corollary 11.19. *Let $I + C$ be a complex skew Hadamard matrix of order w. Let X,Z be amicable complex matrices of order p that are suitable matrices for a* COD$(w;1,w-1)$. *Then there is a complex Hadamard matrix of order pw.*

Proof. Put $m = 1$ in the theorem. □

Corollary 11.20 [89]. *Let $S = I + C$ be a complex skew Hadamard matrix of order w. Then there is a complex Hadamard matrix of order $w(w-1)$.*

We can use this corollary to form complex Hadamard matrices. In Table 11.1, the * signifies that a symmetric conference matrix for this order is not possible as $w(w-1)$ is not the sum of two squares. A number of other similar constructions are discussed in Seberry-Wallis [114, pp. 349–353], but we will not pursue them here.

TABLE 11.1

w	Complex Hadamard order	Comment
18	306	
26	$650 = 59 \times 11 + 1$	*
30	$870 = 789 \times 11 + 1$	*
38	$1406 = 281 \times 5 + 1$	
50	$2450 = 79 \times 31 + 1$	*
62	$3782 = 199 \times 19 + 1$	*

Seberry and Zhang [89] have constructed amicable, disjoint $W(4mn, 2mn)$ U and V from Hadamard matrices of orders $4m$ and $4n$. Thus, we have

Theorem 11.21 (Seberry-Zhang [88]). *Suppose that $4m$ and $4n$ are the orders of Hadamard matrices. Then $U + iV$ (U, V above) is a complex Hadamard matrix of order $4mn$.*

The strong Kronecker product is used to prove Theorem 3.4.

11.3. Orders for Which Complex Hadamard Matrices Exist

We noted in Theorem 11.5 that symmetric conference matrices N always give a complex Hadamard matrix $iI + N$. So in Table 11.2 of complex Hadamard matrices, ci refers to the construction in Appendix A.1 for conference matrices. The construction $x2$ refers to Turyn's theorem [104], as well as to that of Kharaghani and Seberry [42] that Williamson-type matrices of order w give complex Hadamard matrices of order $2w$.

APPENDIX

A.1. Hadamard Matrices

One of us (Seberry) has a table containing odd integers $q < 40{,}000$ for which Hadamard matrices orders $2^t q$ exist. In Appendix A.3, we give this table for $q \leq 3000$. The key for the methods of construction follows: Note that not all construction methods appear, only those that, in the opinion of the authors, enabled us to compile the tables efficiently.

Amicable Hadamard Matrices

Key	Method	Explanation
$a1$	$p^r + 1$	$p^r \equiv 3 \pmod 4$ is a prime power [110]
$a2$	$2(q+1)$	$2q+1$ is a prime power; $q \equiv 1 \pmod 4$ is a prime [114]
$a5$	nh	n, h, are amicable Hadamard matrices [110]

Skew Hadamard Matrices

Key	Method	Explanation
$s1$	$2^t \prod k_i$	t all positive integers; $k_i - 1 \equiv 3 \pmod 4$ a prime power [66]
$s2$	$(p-1)^u + 1$	p is a skew Hadamard matrix; $u > 0$ is an odd integer [105]
$s3$	$2(q+1)$	$q \equiv 5 \pmod 8$ is a prime power [98]

TABLE 11.2 Complex Hadamard Matrices

q	How	q	How	q	How	q	How	q	How
1		89	$x2$	177	$c1$	265	$c1$	353	$x2$
3	$c1$	91	$c1$	179		267		355	$c1$
5	$c1$	93	$x2$	181	$c1$	269		357	
7	$c1$	95	$x2$	183		271	$c1$	359	
9	$c1$	97	$c1$	185		273		361	$x2$
11	$x2$	99	$c1$	187	$c1$	275		363	$x2$
13	$c1$	101	$x2$	189	$x2$	277	$x2$	365	$c1$
15	$c1$	103		191		279	$c1$	367	$c1$
17	$x2$	105	$x2$	193	$x2$	281	$x2$	369	
19	$c1$	107		195	$c1$	283		371	
21	$c1$	109	$x2$	197	$x2$	285	$c1$	373	$x2$
23	$c3$	111		199	$c1$	287		375	$x2$
25	$c1$	113	$c2$	201	$c1$	289	$c1$	377	
27	$c1$	115	$c1$	203		291		379	$c1$
29	$x2$	117	$c1$	205	$c1$	293	$x2$	381	$c1$
31	$c1$	119		207		295		383	
33	$x2$	121	$c1$	209		297	$c1$	385	$c1$
35		123	$c3$	211	$c1$	299		387	$c1$
37	$c1$	125	$x2$	213		301	$c1$	389	$x2$
39	$x2$	127		215		303		391	
41	$c1$	129	$c1$	217	$c1$	305		393	
43	$x2$	131		219		307	$c1$	395	
45	$c1$	133		221		309	$c1$	397	$x2$
47		135	$c1$	223		311		399	$c1$
49	$c1$	137	$x2$	225	$c1$	313	$c1$	401	$x2$
51	$c1$	139	$c1$	227		315	$x2$	403	
53	$x2$	141	$c1$	229	$c1$	317	$x2$	405	$c1$
55	$c1$	143		231	$c1$	319		407	
57	$c1$	145	$c1$	233	$x2$	321	$c1$	409	$x2$
59		147	$c1$	235		323		411	$c1$
61	$c1$	149	$x2$	237		325	$x2$	413	
63	$c1$	151		239		327	$c1$	415	$c1$
65		153	$x2$	241	$x2$	329		417	
67		155		243	$x2$	331	$c1$	419	
69	$c1$	157	$c1$	245		333		421	$c1$
71		159	$c1$	247		335		423	$x2$
73	$x2$	161		249		337	$c1$	425	
75	$c1$	163		251		339	$c1$	427	$c1$
77		165		253		341		429	$c1$
79	$c1$	167		255	$c1$	343		431	
81	$x2$	169	$c1$	257	$x2$	345		433	$x2$
83	$x2$	171		259		347		435	$x2$
85	$c1$	173	$x2$	261	$c1$	349	$x2$	437	
87	$c1$	175	$c1$	263		351	$c1$	439	$c1$

TABLE 11.2 Complex Hadamard Matrices *(continued)*

q	How	q	How	q	How	q	How	q	How
441	$c1$	529	$x2$	617	$x2$	705	$c1$	793	
443		531	$c1$	619	$c1$	707		795	
445		533		621		709	$x2$	797	$x2$
447		535	$c1$	623		711		799	$c1$
449	$x2$	537		625	$c1$	713		801	$c1$
451	$x2$	539		627	$x2$	715	$c1$	803	
453		541	$x2$	629		717	$c1$	805	$c1$
455		543	$x2$	631		719		807	$c1$
457	$x2$	545	$c3$	633		721		809	$x2$
459	$x2$	547	$c1$	635		723		811	$c1$
461	$x2$	549	$c1$	637		725		813	
463		551		639	$c1$	727	$c1$	815	
465	$c1$	553		641	$x2$	729	$x2$	817	
467		555	$c1$	643		731		819	$c1$
469	$c1$	557	$x2$	645	$c1$	733		821	$x2$
471	$c1$	559	$c1$	647		735	$x2$	823	
473	$x2$	561	$x6$	649	$c1$	737		825	
475		563		651	$c1$	739		827	
477	$c1$	565	$c1$	653		741	$c1$	829	$c1$
479		567		655		743		831	
481	$c1$	569	$x2$	657		745	$c1$	833	
483		571		659		747	$c1$	835	$c1$
485		573		661	$c1$	749		837	
487		575		663	$x2$	751		839	
489	$c1$	577	$c1$	665		753		841	$c1$
491		579	$x2$	667		755		843	$x2$
493		581		669		757	$x2$	845	
495		583		671		759	$x2$	847	$c1$
497		585		673	$x2$	761	$c2$	849	$c1$
499	$c1$	587		675	$x2$	763		851	
501		589		677	$x2$	765		853	
503		591	$c1$	679		767		855	$c1$
505	$c1$	593	$x2$	681	$c1$	769	$x2$	857	
507	$c1$	595		683		771		859	
509		597	$c1$	685	$c1$	773	$x2$	861	$c1$
511	$c1$	599		687	$c1$	775	$c1$	863	
513		601	$c1$	689		777	$c1$	865	
515		603		691	$c1$	779		867	$c1$
517	$c1$	605		693		781		869	
519		607	$c1$	695		783		871	$c1$
521	$x2$	609	$c1$	697		785		873	
523		611		699		787		875	
525	$c1$	613	$c2$	701	$x2$	789		877	$c1$
527		615	$c1$	703	$x2$	791		879	$x2$

TABLE 11.2 Complex Hadamard Matrices *(continued)*

q	How	q	How	q	How	q	How	q	How
881	$x2$	905		929	$x2$	953	$x2$	977	$x2$
883		907		931	$c1$	955		979	
885	$x2$	909		933		957	$c1$	981	
887		911		935		959		983	
889	$c1$	913		937	$c1$	961	$x2$	985	
891		915		939	$c1$	963		987	$c1$
893		917		941		965		989	
895	$c1$	919		943		967	$c1$	991	
897		921		945	$c1$	969		993	
899		923		947		971		995	
901	$c1$	925	$c1$	949		973		997	$c1$
903		927		951	$c1$	975	$c1$	999	$c1$

Skew Hadamard Matrices (continued)

$s4$	$2(q+1)$	$q = p^t$ is a prime power where $p \equiv 5 \pmod 8$ and $t \equiv 2 \pmod 4$ [99, 125]
$s5$	$4m$	$3 \le m \le 33,127$ [35, 100, 18a] $m \in \{37, 43, 67, 113, 127, 157, 163, 181, 241\}$ [17, 16]
$s6$	$4(q+1)$	$q \equiv 9 \pmod{16}$ is a prime power [113]
$s7$	$(\lvert t\rvert + 1)(q+1)$	$q = s^2 + 4t^2 \equiv 5 \pmod 8$ is a prime power; $\lvert t\rvert + 1$ is a skew Hadamard matrix [117]
$s8$	$4(q^2+q+1)$	q is a prime power, $q^2+q+1 \equiv 3, 5, 7 \pmod 8$ a prime, or $2(q^2+q+1)+1$ is a prime power [94]
$s0$	hm	h is a skew Hadamard matrix; m is an amicable Hadamard matrix [114]

Spence Hadamard Matrices

Key	Method	Explanation
$p1$	$4(q^2+q+1)$	$q^2+q+1 \equiv 1 \pmod 8$ is a prime [94]
$p2$	$4n$ or $8n$	n, $n-2$ are prime powers; if $n \equiv 1 \pmod 4$, there exists a Hadamard matrix of order $4n$; if $n \equiv 3 \pmod 4$, there exists a Hadamard matrix of order $8n$ [93]
$p3$	$4m$	m is an odd prime power for which an integer $s \ge 0$ such that $(m - (2^{s+1}+1))/2^{s+1}$ is an odd prime power [93]

Conference Matrices That Give Symmetric Hadamard Matrices The following methods give symmetric Hadamard matrices of order $2n$ and conference matrices of order n with the exception of $c6$ which produces an Hadamard matrix. The order of the Hadamard matrix is given in the column headed "Method."

Key	Method	Explanation
$c1$	$2(p^r + 1)$	$p^r \equiv 1 \pmod 4$ is a prime power [66, 25]
$c2$	$2((h-1)^2 + 1)$	h is a skew Hadamard matrix [7]
$c3$	$2(q^2(q-2) + 1)$	$q \equiv 3 \pmod 4$ is a prime power $q - 2$ is a prime power [63]
$c4$	$2(5 \cdot 9^{2t+1} + 1)$	$t \geq 0$ [85]
$c5$	$2((n-1)^s + 1)$	n is a conference matrix $s \geq 2$ [105]
$c6$	nh	n is a conference matrix h is a Hadamard matrix [25]

Note: A conference matrix of order $n \equiv 2 \pmod 4$ exists only if $n - 1$ is the sum of two squares.

Hadamard Matrices Obtained from Williamson Matrices If a Williamson matrix of order $2^t q$ exists, then there is a Hadamard matrix of order $2^{t+2}q$, the same key as in the Index of Williamson Matrices in Appendix A.2 is used to index the Hadamard matrices produced from them.

OD Hadamard Matrices

Key	Method	Explanation
$o1$	$2^{t+2}q$	If a T-matrix of order $2^t q$ exists, then there is a Hadamard matrix of order $2^{t+2}q$ [12, 108]
$o2$	ow	o is an OD-Hadamard matrix; w is a Williamson matrix [6, 12, 115]
$o3$	$8pw$	an $OD(8p; p, p, p, p, p, p, p, p)$ exists for $p = 1, 3$; there exist 8-Williamson matrices of order w [67]

Yamada Hadamard Matrices

Key	Method	Explanation
$y1$	$4q$	$q \equiv 1 \pmod 8$ is a prime power; $(q-1)/2$ is a Hadamard matrix [132]
$y2$	$4(q+2)$	$q \equiv 5 \pmod 8$ is a prime power; $(q+3)/2$ is a skew Hadamard matrix [132]
$y3$	$4(q+2)$	$q \equiv 1 \pmod 8$ is a prime power; $(q+3)/2$ is a conference matrix [132]

Miyamoto Hadamard Matrices

Key	Method	Explanation
$m1$	$4q$	$q \equiv 1 \pmod 4$ is a prime power; $q-1$ is a Hadamard matrix [64]
$m2$	$8q$	$q \equiv 3 \pmod 4$ is a prime power; $2q-3$ is a prime power [64]

Koukouvinos and Kounias

Key	Method	Explanation
$k1$	$2^t q$	$2^t q = g_1 + g_2$, where g_1 and g_2 are the lengths of Golay sequences [53]

Agayan Multiplication

Key	Method	Explanation
$d1$	$2^{t+s-1}pq$	$2^t p$ and $2^s q$ are the orders of Hadamard matrices [1]

Seberry

Key	Method	Explanation
se	$2^t q$	t is the smallest integer such that for given odd q, $a(q+1)+b(q-3)=2^t$ has a solution for a, b nonnegative integers [121]

Craigen-Seberry-Zhang

Key	Method	Explanation
cz	$2^{t+s+u+w-4}$	$2^t a, 2^s b, 2^u c, 2^w d$ are the orders of Hadamard matrices [14]

A.2. Index of Williamson Matrices

One of us (Seberry) has a list on the computer of odd integers $q < 40{,}000$ for which Williamson or Williamson type matrices exist. The following legend gives a list of constructions for these matrices, the method used, and the discoverer—with apologies to anyone excluded:

Key	Method	Explanation
$w1$	$\{1,\ldots,33,37,39,41,43\}$	[52, 18, 130]
$w2$	$(p+1)/2$	$p \equiv 1 \pmod 4$ a prime power [26, 106, 126]
$w3$	3^d	d a natural number [65, 109]
$w4$	$[p(p+1)]/2$	$p \equiv 1 \pmod 4$ a prime power [112, 127]
$w5$	$s(4s+3), s(4s-1)$	$s \in \{1,3,5,\ldots,31\}$ [120]
$w6$	93	[120]
$w7$	$[(f-1)(4f+1)]/4$	$p = 4f+1$, f odd, is a prime power of the form $1+4t^2$; $(f-1)/8$ is the order of a good matrix [118]
$w8$	$[(f+1)(4f+1)]/4$	$p = 4f+1$, f odd, is a prime power of the form $25+4t^2$; $(f+1)/8$ is the order of a good matrix [118]
$w9$	$[p(p-1)]/2$	$p = 4f+1$ is a prime power; $(p-1)/4$ is the order of a good matrix [118]
$w0$	$(p+2)(p+1)$	$p \equiv 1 \pmod 4$ a prime power; $p+3$ is the order of a symmetric Hadamard matrix [118]
wa	$[(f+1)(4f+1)]/2$	$p = 4f+1$, f odd, is a prime power of the form $9+4t^2$; $(f-1)/2 \equiv 1 \pmod 4$ a prime power [118]
wb	$[(f-1)(4f+1)]/2$	$p = 4f+1$, f odd, is a prime power of the form $49+4t^2$; $(f-3)/2 \equiv 1 \pmod 4$ a prime power [118]
wc	$2p+1$	$q = 2p-1$ is a prime power, p is a prime [64, 87]
wd	7.3^i	$i \geq 0$ [65]
$w\#e$	$7^{i+1}, 11 \cdot 7^i$	$i \geq 0$ (gives 8-Williamson matrices) [78]
wf	$q^d(q+1)/2$	$q \equiv 1 \pmod 4$ is a prime power, $d \geq 2$ [65, 95a]
wg	$p^2(p+1)/2$	$p \equiv 1 \pmod 4$ is a prime power [80]
wh	$p^2(p+1)/4$	$p \equiv 3 \pmod 4$ is a prime power; $(p+1)/4$ is the order of a Williamson-type matrix [80]
wi	$q+2$	$q \equiv 1 \pmod 4$ is a prime power; $(q+1)/2$ is a prime power [64]
wj	$q+2$	$q \equiv 1 \pmod 4$ is a prime power; $(q+3)/2$ is the order of a symmetric conference matrix [64]

wk	q	$q \equiv 1 \pmod 4$ is a prime power; $(q-1)/2$ is the order of a symmetric conference matrix or the order of a symmetric Hadamard matrix [64]
wl	q	$q \equiv 1 \pmod 4$ is a prime power; $(q-1)/4$ is the order of a Williamson-type matrix [64]
wm	q	$q \equiv 1 \pmod 4$ is a prime power; $(q-1)/2$ is the order of a Hadamard matrix [87]
wn	wn	w is the order of a Williamson-type matrix; n is the order of a symmetric conference matrix
wo	$2wu$	w and u are the orders of Williamson-type matrices [87]
$w\#p$	$2q+1$	$q+1$ is the order of an amicable Hadamard matrix; q is the order of a Williamson type matrix [87]
$w\#q$	q	q is a prime power; and $(q-1)/2$ is the order of a Williamson-type matrix [87]
$w\#r$	$2q+1$	$q+1$ is the order of a symmetric conference matrix; q is the order of a Williamson-type matrix [87]
$w\#s$	2.9^t+1	$t > 0$ [87]

$S = \{1, \ldots, 31\}$ is the set of good matrices.

Note: The fact that if there is a Williamson matrix of order n, then there is a Williamson matrix of order $2n$, is used in the calculation of wh.

We now give in Table A.1 known Williamson-type matrices of orders < 2000. The order in which the algorithms were applied was $w_1, w_2, w_3, w_4, w_5, w_6, wi, wj, wk, wl, wn, w\#p, w\#q, w\#r$, and then others if it appeared they might give a new order. To interpret the results in the table, we note that if there is an Hadamard matrix of order $4q$, then it can be a Williamson-type matrix, but this was not included. A notation $w\#x$ means that 8-Williamson matrices are known, but not four, so an $OD(8s; s, s, s, s, s, s, s, s)$ is needed to get an Hadamard matrix. The notation $47, 3, w\#p$ means that there are 8-Williamson matrices of order 47, and thus an Hadamard matrix of order $8 \cdot 47$. A notation with wn of 3 indicates that there are four Williamson-type matrices but they are of even order. The notation $35, 3, wn$ means that there are four Williamson-type matrices of order 70 and an Hadamard matrix of order 280.

TABLE A.1 Williamson and Williamson-Type Matrices

q	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How
1		*w*1	85		*w*2	169		*w*2	253	3	*wn*	337		*w*2
3		*w*1	87		*w*2	171	3	*wn*	255		*w*2	339		*w*2
5		*w*1	89		*wl*	173		*wl*	257		*wl*	341	3	*wn*
7		*w*1	91		*w*2	175		*w*2	259	3	*wn*	343	3	*wn*
9		*w*1	93		*w*5	177		*w*2	261		*w*2	345	3	*wn*
11		*w*1	95		*w*6	179	3	*w#q*	263			347	3	*w#q*
13		*w*1	97		*w*2	181		*w*2	265		*w*2	349		*wk*
15		*w*1	99		*w*2	183	3	*wn*	267	3	*wn*	351		*w*2
17		*w*1	101		*wk*	185	3	*wn*	269			353		*wl*
19		*w*1	103	3	*w#q*	187		*w*2	271		*w*2	355		*w*2
21		*w*1	105	3	*wn*	189		*w*5	273	3	*wn*	357	3	*wn*
23		*w*1	107	3	*w#q*	191	3	*w#p*	275	3	*wn*	359		
25		*w*1	109		*wk*	193		*wk*	277		*wk*	361		*wk*
27		*w*1	111	3	*wn*	195		*w*2	279		*w*2	363		*wi*
29		*w*1	113		*wk*	197		*wk*	281		*wl*	365		*w*2
31		*w*1	115		*w*2	199		*w*2	283	3	*w#q*	367		*w*2
33		*w*1	117		*w*2	201		*w*2	285		*w*2	369	3	*wn*
35	3	*wn*	119	3	*wn*	203	3	*w*9	287	3	*wn*	371	3	*wn*
37		*w*1	121		*w*2	205		*w*2	289		*w*2	373		*wl*
39		*w*1	123		*wi*	207	3	*wn*	291	3	*wn*	375		*wf*
41		*w*1	125		*wk*	209	3	*wn*	293		*wl*	377	3	*wn*
43		*w*1	127	3	*w#p*	211		*w*2	295			379		*w*2
45		*w*2	129		*w*2	213			297		*w*2	381		*w*2
47	3	*w#p*	131			215	3	*wn*	299	3	*wn*	383		
49		*w*2	133	3	*wn*	217		*w*2	301		*w*2	385		*w*2
51		*w*2	135		*w*2	219	3	*wn*	303	3	*w*7	387		*w*2
53		*wk*	137		*wl*	221	3	*wn*	305	3	*wn*	389		*wk*
55		*w*2	139		*w*2	223			307		*w*2	391	3	*wn*
57		*w*2	141		*w*2	225		*w*2	309		*w*2	393		
59	3	*w#q*	143	3	*wn*	227	3	*w#q*	311			395	3	*wn*
61		*w*2	145		*w*2	229		*w*2	313		*w*2	397		*wk*
63		*w*2	147		*w*2	231		*w*2	315		*w*5	399		*w*2
65	3	*wn*	149		*wk*	233		*wl*	317		*wk*	401		*wk*
67	3	*w#q*	151	3	*w#q*	235			319	3	*wo*	403	3	*wn*
69		*w*2	153		*w*4	237	3	*wn*	321		*w*2	405		*w*2
71			155	3	*wn*	239			323	3	*wn*	407	3	*wn*
73		*wk*	157		*w*2	241		*wk*	325		*w*4	409		*wk*
75		*w*2	159		*w*2	243		*wj*	327		*w*2	411		*w*2
77	3	*wn*	161	3	*wn*	245	3	*wn*	329			413		
79		*w*2	163	3	*w#q*	247	3	*wn*	331		*w*2	415		*w*2
81		*w*3	165	3	*wn*	249	3	*wn*	333	3	*w*9	417	3	*wn*
83		*wi*	167	3	*w#p*	251	3	*w#q*	335			419		

TABLE A.1 Williamson and Williamson-Type Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
421		w2	513	3	wn	605	3	wn	697	3	wn	789		
423		wi	515	3	w#r	607		w2	699	3	wn	791	3	wn
425	3	wn	517		w2	609		w2	701		wk	793	3	wn
427		w2	519	3	wn	611			703		w4	795	3	wn
429		w2	521		wl	613		wl	705		w2	797		wk
431			523	3	w#q	615		w2	707	3	wn	799		w2
433		wk	525		w2	617		wl	709		wk	801		w2
435		w4	527	3	wn	619		w2	711	3	wn	803	3	wo
437	3	wn	529		wl	621	3	wn	713	3	wn	805		w2
439		w2	531		w2	623	3	wn	715		w2	807		w2
441		w2	533	3	wn	625		w2	717		w2	809		wl
443			535		w2	627		wi	719			811		w2
445	3	wn	537			629	3	wn	721			813	3	wn
447	3	wn	539	3	wn	631	3	w#q	723	3	wn	815		
449		wk	541		wk	633	3	wn	725	3	wn	817	3	wn
451		wj	543		wi	635	3	w#r	727		w2	819		w2
453			545	3	wn	637	3	wn	729		w3	821		wk
455	3	wn	547		w2	639		w2	731	3	wo	823	3	w#q
457		wk	549		w2	641		wk	733	3	w#q	825	3	wn
459		wi	551	3	wn	643	3	w#q	735		wi	827		
461		wk	553	3	wn	645		w2	737			829		w2
463	3	w#q	555		w2	647			739			831	3	wn
465		w2	557		wk	649		w2	741		w2	833	3	wn
467	3	w#q	559		w2	651		w2	743			835		w2
469		w2	561	3	wn	653			745		w2	837	3	wn
471		w2	563	3	w#q	655	3	w#p	747		w2	839		
473		w5	565		w2	657	3	wn	749			841		w2
475	3	wn	567	3	wn	659			751	3	w#q	843		wi
477		w2	569		wm	661		w2	753			845	3	wn
479			571	3	w#q	663		w5	755			847		w2
481		w2	573			665	3	wn	757		wl	849		w2
483	3	wn	575	3	wn	667	3	wn	759		wi	851	3	wn
485	3	wn	577		w2	669			761		wl	853		
487	3	w#p	579		wj	671	3	wn	763			855		w2
489		w2	581	3	wn	673		wk	765	3	wn	857		
491			583	3	wo	675		wi	767			859	3	w#q
493	3	wo	585	3	wn	677		wk	769		wk	861		w2
495	3	wn	587	3	w#q	679	3	wn	771	3	wn	863		
497			589	3	wn	681		w2	773		wl	865	3	wn
499		w2	591		w2	683			775		w2	867		w2
501			593		wk	685		w2	777		w2	869	3	wn
503			595	3	wn	687		w2	779	3	wn	871		w2
505		w2	597		w2	689	3	w9	781			873	3	wn
507		w2	599			691		w2	783	3	wn	875	3	wn
509			601		w2	693	3	wn	785	3	wn	877		w2
511		w2	603	3	wn	695	3	wn	787			879		wi

TABLE A.1 Williamson and Williamson-Type Matrices *(continued)*

q	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How
881		*wk*	973	3	*wn*	1065		*w2*	1157	3	*wn*	1249		*wl*
883	3	*w#q*	975		*w2*	1067	3	*wn*	1159	3	*wn*	1251		*wi*
885		*w5*	977		*wk*	1069		*w2*	1161	3	*wn*	1253		
887			979	3	*wo*	1071		*w2*	1163			1255		
889		*w2*	981	3	*wn*	1073	3	*wn*	1165	3	*wn*	1257		
891	3	*wn*	983			1075	3	*wn*	1167		*w2*	1259		
893			985	3	*wn*	1077		*w2*	1169			1261		*w2*
895		*w2*	987		*w2*	1079	3	*wn*	1171		*w2*	1263	3	*wn*
897	3	*wn*	989	3	*wn*	1081		*w2*	1173	3	*wn*	1265	3	*wn*
899	3	*wn*	991			1083	3	*wn*	1175			1267	3	*wn*
901		*w2*	993	3	*wn*	1085	3	*wn*	1177			1269	3	*wn*
903	3	*wn*	995	3	*wn*	1087	3	*w#p*	1179		*w2*	1271	3	*wn*
905	3	*wn*	997		*w2*	1089	3	*wn*	1181			1273		
907			999		*w2*	1091			1183		*wf*	1275		*w2*
909	3	*wn*	1001	3	*wn*	1093	3	*w#q*	1185	3	*wn*	1277	3	*w#q*
911			1003			1095		*wi*	1187	3	*w#q*	1279		*w2*
913	3	*wo*	1005	3	*wn*	1097		*wl*	1189		*w2*	1281	3	*wn*
915	3	*w9*	1007	3	*wn*	1099		*w2*	1191		*w2*	1283	3	*w#q*
917			1009		*w2*	1101	3	*wn*	1193		*wl*	1285	3	*wn*
919	3	*w#q*	1011	3	*wn*	1103			1195		*w2*	1287	3	*wn*
921	3	*wn*	1013	3	*wn*	1105		*w2*	1197		*w2*	1289		*wl*
923	3	*w#r*	1015		*w2*	1107		*w2*	1199	3	*wo*	1291	3	*w#q*
925		*w2*	1017	3	*wn*	1109		*wl*	1201		*w2*	1293		
927	3	*wn*	1019	3	*w#r*	1111		*w2*	1203		*wi*	1295	3	*wn*
929		*wl*	1021		*wk*	1113	3	*wn*	1205	3	*wn*	1297		*w2*
931		*w2*	1023	3	*wn*	1115	3	*w#r*	1207		*w5*	1299	3	*wn*
933			1025	3	*wn*	1117		*wk*	1209		*w2*	1301		*wl*
935	3	*wn*	1027		*w2*	1119		*w2*	1211	3	*wn*	1303	3	*w#q*
937		*w2*	1029	3	*wn*	1121			1213	3	*w#q*	1305		*w2*
939		*w2*	1031			1123			1215		*wi*	1307	3	*w#r*
941			1033		*wk*	1125	3	*wn*	1217		*wk*	1309		*w2*
943	3	*wn*	1035		*w2*	1127	3	*wn*	1219		*w2*	1311		*w2*
945		*w2*	1037	3	*wn*	1129		*wk*	1221		*w2*	1313	3	*wn*
947	3	*w#q*	1039			1131	3	*wn*	1223			1315		
949	3	*wn*	1041		*w2*	1133			1225		*w4*	1317		*w2*
951		*w2*	1043	3	*wn*	1135		*w2*	1227	3	*wn*	1319		
953		*wl*	1045		*w2*	1137		*w2*	1229		*wk*	1321		*wl*
955			1047	3	*wn*	1139		*w5*	1231	3	*w#p*	1323		*wi*
957		*w2*	1049		*wm*	1141		*w2*	1233	3	*wn*	1325	3	*wn*
959	3	*wn*	1051	3	*w#q*	1143	3	*wn*	1235	3	*wn*	1327	3	*w#p*
961		*wk*	1053	3	*wn*	1145	3	*wn*	1237		*w2*	1329		*w2*
963	3	*wn*	1055	3	*wn*	1147		*w2*	1239		*w2*	1331	3	*wn*
965	3	*wn*	1057		*w2*	1149		*w2*	1241	3	*wo*	1333	3	*wn*
967		*w2*	1059	3	*wn*	1151			1243	3	*wn*	1335	3	*wn*
969	3	*wn*	1061		*wk*	1153		*wk*	1245	3	*wn*	1337		
971			1063	3	*w#q*	1155		*w2*	1247	3	*wo*	1339		*w2*

TABLE A.1 Williamson and Williamson-Type Matrices *(continued)*

q	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How
1341		*wb*	1433			1525		*w2*	1617	3	*wn*	1709		*wk*
1343	3	*wn*	1435	3	*wn*	1527			1619	3	*w#q*	1711		
1345		*w2*	1437			1529	3	*wn*	1621		*wk*	1713		
1347		*w2*	1439			1531		*w2*	1623		*wi*	1715	3	*w#r*
1349			1441			1533	3	*wn*	1625	3	*wn*	1717		*w2*
1351	3	*wn*	1443	3	*wn*	1535	3	*wn*	1627		*w2*	1719		
1353	3	*wn*	1445	3	*wn*	1537	3	*wo*	1629		*w2*	1721		*wl*
1355	3	*wn*	1447			1539	3	*wn*	1631	3	*wn*	1723	3	*w#q*
1357		*w2*	1449		*w2*	1541			1633			1725		*w2*
1359			1451			1543			1635	3	*wn*	1727	3	*wn*
1361		*wk*	1453		*wl*	1545		*w2*	1637		*wl*	1729		*w2*
1363			1455		*w2*	1547	3	*wn*	1639	3	*wo*	1731		*w2*
1365		*w2*	1457			1549		*wk*	1641	3	*wn*	1733		*wl*
1367			1459		*w2*	1551	3	*wn*	1643	3	*wn*	1735		*w2*
1369		*wl*	1461			1553		*wk*	1645			1737	3	*wn*
1371		*w2*	1463	3	*wn*	1555		*w2*	1647	3	*wn*	1739		
1373	3	*w#q*	1465	3	*wn*	1557	3	*wn*	1649	3	*wn*	1741		*w2*
1375		*w2*	1467	3	*wn*	1559			1651		*w2*	1743		*w5*
1377		*w2*	1469	3	*wn*	1561		*w2*	1653	3	*wn*	1745	3	*wn*
1379	3	*wn*	1471	3	*w#p*	1563		*w2*	1655	3	*wn*	1747		
1381	3	*w#q*	1473			1565	3	*wn*	1657		*w2*	1749	3	*wn*
1383		*wi*	1475			1567			1659		*wi*	1751		
1385	3	*wn*	1477		*w2*	1569		*w2*	1661			1753		*wl*
1387	3	*wn*	1479		*w2*	1571			1663			1755		*wi*
1389		*w2*	1481		*wl*	1573	3	*wn*	1665		*w2*	1757		
1391			1483	3	*w#q*	1575	3	*wn*	1667			1759		*w2*
1393	3	*wn*	1485		*w2*	1577	3	*wn*	1669	3	*w#q*	1761		
1395		*w2*	1487			1579			1671	3	*wn*	1763	3	*wn*
1397			1489		*wl*	1581	3	*wn*	1673			1765		*w2*
1399		*w2*	1491			1583			1675			1767		*w2*
1401		*w2*	1493		*wl*	1585		*w2*	1677	3	*wn*	1769	3	*wn*
1403	3	*wn*	1495	3	*wn*	1587		*wh*	1679	3	*wn*	1771		*w2*
1405		*w2*	1497	3	*wn*	1589			1681		*w2*	1773	3	*wn*
1407	3	*wn*	1499			1591		*w2*	1683		*wj*	1775	3	*wn*
1409		*wl*	1501		*w2*	1593	3	*wn*	1685	3	*wn*	1777		*wl*
1411	3	*wo*	1503			1595	3	*wn*	1687		*w2*	1779		*w2*
1413	3	*wn*	1505	3	*wn*	1597		*wk*	1689			1781	3	*wn*
1415			1507	3	*wo*	1599	3	*wn*	1691	3	*wn*	1783		
1417		*w2*	1509			1601		*wk*	1693		*wl*	1785	3	*wn*
1419		*w2*	1511			1603	3	*wn*	1695		*w2*	1787		
1421	3	*wn*	1513	3	*wn*	1605		*w2*	1697		*wl*	1789	3	*w#q*
1423			1515	3	*wn*	1607			1699	3	*w#q*	1791		*w2*
1425		*w5*	1517	3	*wn*	1609		*w2*	1701	3	*wn*	1793		
1427			1519		*w2*	1611		*w2*	1703			1795		
1429		*w2*	1521		*w2*	1613	3	*w#q*	1705	3	*wn*	1797		*w2*
1431		*w2*	1523	3	*w#q*	1615		*w2*	1707		*w2*	1799	3	*wn*

TABLE A.1 Williamson and Williamson-Type Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
1801		*wk*	1841			1881		*w*2	1921	3	*wn*	1961	3	*wn*
1803	3	*wn*	1843	3	*wn*	1883	3	*w#r*	1923	3	*wn*	1963		
1805		*wh*	1845	3	*wn*	1885		*w*2	1925	3	*wn*	1965		*w*2
1807		*w*2	1847			1887	3	*wn*	1927		*w*2	1967	3	*wn*
1809		*w*2	1849		*w*2	1889		*wm*	1929			1969		
1811			1851		*w*2	1891		*w*4	1931			1971	3	*wn*
1813	3	*wn*	1853	3	*wo*	1893			1933	3	*w#q*	1973	3	*w#q*
1815	3	*wn*	1855		*w*2	1895	3	*wn*	1935		*wi*	1975	3	*wn*
1817	3	*wn*	1857	3	*wn*	1897		*w*2	1937	3	*wn*	1977		
1819		*w*2	1859	3	*wn*	1899		*w*2	1939		*w*2	1979		
1821	3	*wn*	1861		*w*2	1901	3	*w#q*	1941		*w*2	1981		
1823	3	*wn*	1863	3	*wn*	1903	3	*wo*	1943			1983	3	*wn*
1825	3	*wn*	1865	3	*wn*	1905	3	*wn*	1945		*w*2	1985	3	*wn*
1827		*w*5	1867		*w*2	1907	3	*w#q*	1947	3	*wn*	1987		
1829			1869	3	*wn*	1909	3	*wn*	1949			1989	3	*wn*
1831			1871			1911		*w*2	1951	3	*w#p*	1991	3	*wn*
1833	3	*wn*	1873		*wk*	1913			1953	3	*wn*	1993		*wl*
1835	3	*wn*	1875		*wf*	1915			1955	3	*wn*	1995		*w*2
1837		*w*2	1877		*wk*	1917		*w*2	1957			1997		*wk*
1839		*w*2	1879	3	*w#q*	1919	3	*wn*	1959		*w*2	1999		*w#p*

A.3. Tables of Hadamard matrices

Table A.2 gives the orders of known Hadamard matrices. The table gives the odd part q of an order, the smallest power of two, t, for which the Hadamard matrix is known and a construction method. If there is no entry in the t column the power is two. Thus, there are Hadamard matrices known of orders $2^2 \cdot 105$ and $2^3 \cdot 107$. We see at a glance, therefore, that the smallest order for which an Hadamard matrix is not yet known is $4 \cdot 107$. Since the theorem of Seberry ensures that a t exists for every q, there is either a t entry for each q, or $t = 2$ is implied.

With the exception of order $4 \cdot 163$, marked dj, which was announced recently [16], the method of construction used is indicated. The order in which the algorithms were applied reflects the fact that other tables were being constructed at the same time. Hence, the "Amicable Hadamard," "Skew Hadamard," "Conference Matrix," "Williamson Matrix," direct "Complex Hadamard" were implemented first (in that order). The tables reflect this and not the priority in time of a construction or its discoverer.

Next the "Spence," "Miyamoto," and "Yamada" direct constructions were applied because they were noticed to fill places in the table. The methods $o1$ and of Koukouvinos and Kounias were now applied as lists of ODs were constructed. These were then used to "plug in" the Williamson-type matrices implementing methods $o2$ and $o3$.

Finally, the multiplication theorems of Agayan, Seberry, and Zhang were applied. The Craigen, Seberry, and Zhang theorem was applied to the table that one of us (Seberry), had in the computer. The method and order of application was by personal choice to improve the efficiency of implementation. This means that some authors, for example, Baumert, Hall, Turyn, and Whiteman, who have priority of construction are not mentioned by name in the final table.

TABLE A.2 Orders of Known Hadamard Matrices

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
1		$a1$	69		$c1$	137		$a1$	205		$c1$	273		$a1$
3		$a1$	71		$a1$	139		$c1$	207		$a1$	275		$o1$
5		$a1$	73		wk	141		$a1$	209		$o1$	277		wk
7		$c1$	75		$c1$	143		$a1$	211		$c1$	279		$c1$
9		$c1$	77		$a1$	145		$c1$	213		$o2$	281		$a1$
11		$a1$	79		$c1$	147		$a1$	215		$a1$	283	3	$w\#q$
13		$c1$	81		$w3$	149		wk	217		$c1$	285		$c1$
15		$a1$	83		$a1$	151		$y2$	219		$o2$	287		$o1$
17		$a1$	85		$c1$	153		$w4$	221		$a1$	289		$c1$
19		$c1$	87		$a1$	155		$a1$	223	3	$a1$	291		$a1$
21		$a1$	89		$a2$	157		$c1$	225		$c1$	293		$a1$
23		$w1$	91		$c1$	159		$c1$	227		$a1$	295		$o1$
25		$c1$	93		$w5$	161		$a1$	229		$c1$	297		$a1$
27		$a1$	95		$a1$	163		dj	231		$c1$	299		$o1$
29		$a2$	97		$c1$	165		$a1$	233		$a2$	301		$c1$
31		$c1$	99		$c1$	167	3	$w\#p$	235		$o1$	303		$w7$
33		$a1$	101		wk	169		$c1$	237		$a1$	305		$o1$
35		$a1$	103		$y2$	171		$a1$	239	4	$a1$	307		$c1$
37		$c1$	105		$a1$	173		$a1$	241		wk	309		$c1$
39		wi	107	3	$w\#q$	175		$c1$	243		$a1$	311	3	$m3$
41		$a1$	109		wk	177		$c1$	245		$o1$	313		$c1$
43		$w1$	111		$a1$	179	3	$w\#q$	247		$o1$	315		$a1$
45		$a1$	113		$a2$	181		$c1$	249		$o2$	317		wk
47		$o1$	115		$c1$	183		$o2$	251	3	$w\#q$	319		$o1$
49		$c1$	117		$a1$	185		$a1$	253		$o1$	321		$a1$
51		$c1$	119		$o1$	187		$c1$	255		$a1$	323		$a1$
53		$a1$	121		$c1$	189		$w5$	257		wl	325		$w4$
55		$c1$	123		$a1$	191	3	$w\#p$	259		$o1$	327		$a1$
57		$a1$	125		$a1$	193		wk	261		$c1$	329		$o1$
59		$o1$	127		$y2$	195		$c1$	263		$a1$	331		$c1$
61		$c1$	129		$c1$	197		$a1$	265		$c1$	333		$w9$
63		$a1$	131		$a1$	199		$c1$	267		$o2$	335		$o1$
65		$o1$	133		$o1$	201		$c1$	269		$m2$	337		$c1$
67		$o1$	135		$c1$	203		$a1$	271		$c1$	339		$c1$

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
341		o1	433		wk	525		a1	617		a1	709		wk
343		o2	435		w4	527		o1	619		c1	711		a1
345		o1	437		a1	529		wl	621		o1	713		a1
347	3	w#q	439		c1	531		c1	623		o2	715		c1
349		wk	441		c1	533		a1	625		c1	717		c1
351		c1	443	3	m3	535		c1	627		wi	719	4	a1
353		wl	445		o2	537	3	o3	629		o1	721	3	d1
355		c1	447		a1	539		o2	631	3	w#q	723		o2
357		a1	449		wk	541		wk	633		a1	725		o1
359	4	a1	451		wj	543		wi	635		a1	727		c1
361		wk	453		a1	545		a1	637		o2	729		w3
363		a1	455		o1	547		c1	639		c1	731		o2
365		a1	457		wk	549		c1	641		a2	733		m2
367		c1	459		wi	551		a1	643	3	w#q	735		a1
369		o1	461		wk	553		o2	645		a1	737		o2
371		a1	463	3	w#q	555		c1	647	3	m3	739	16	se
373		wl	465		c1	557		wk	649		c1	741		a1
375		a1	467		a1	559		c1	651		c1	743		a1
377		o1	469		c1	561		a1	653		a2	745		c1
379		c1	471		c1	563		a1	655		y2	747		c1
381		a1	473		w5	565		c1	657		o2	749	4	d1
383		a1	475		o1	567		a1	659	17	se	751	3	a1
385		c1	477		a1	569		wm	661		c1	753		a1
387		c1	479	16	se	571	3	a1	663		w5	755		a1
389		wk	481		c1	573	3	a1	665		a1	757		wl
391		o1	483		a1	575		o1	667		o2	759		wi
393		a1	485		o2	577		c1	669	3	a1	761		a2
395		a1	487	3	w#p	579		wj	671		a1	763		o2
397		wk	489		c1	581		o2	673		wk	765		o1
399		c1	491	15	se	583		o1	675		a1	767		a1
401		wk	493		o1	585		a1	677		a1	769		wk
403		o1	495		a1	587		a1	679		o2	771		a1
405		a1	497		a1	589		o2	681		c1	773		wl
407		a1	499		c1	591		c1	683		a1	775		c1
409		wk	501		a1	593		a1	685		c1	777		c1
411		c1	503		a1	595		o1	687		c1	779		o1
413		o1	505		c1	597		c1	689		w9	781		o2
415		c1	507		a1	599	8	a1	691		c1	783		o1
417		a1	509		a2	601		c1	693		o1	785		o2
419	4	a1	511		c1	603		a1	695		o2	787	3	m3
421		c1	513		o1	605		o2	697		o1	789	3	a1
423		wi	515	3	w#r	607		c1	699		o2	791		a1
425		a1	517		c1	609		c1	701		a1	793		o2
427		c1	519		o2	611		o1	703		w4	795		o1
429		c1	521		a1	613		wl	705		a1	797		a1
431		a1	523	3	w#q	615		a1	707		o2	799		c1

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
801		a1	893		a1	985		o2	1077		c1	1169	4	k1
803		o2	895		c1	987		a1	1079		o2	1171		c1
805		c1	897		o2	989		o2	1081		c1	1173		a1
807		c1	899		o2	991	3	a1	1083		o2	1175		o1
809		a2	901		c1	993		o2	1085		a1	1177	4	d1
811		c1	903		o1	995		o2	1087	3	w#p	1179		c1
813		a1	905		o2	997		c1	1089		o1	1181		a1
815		a1	907	3	m3	999		c1	1091		a1	1183		wf
817		o2	909		o1	1001		a1	1093	3	w#q	1185		o2
819		c1	911		a1	1003		o1	1095		wi	1187	3	w#q
821		wk	913		o2	1005		a1	1097		wl	1189		c1
823	3	w#q	915		a1	1007		a1	1099		w2	1191		c1
825		a1	917	3	o2	1009		c1	1101		o2	1193		wl
827		a1	919	3	a1	1011		o2	1103	3	m3	1195		c1
829		c1	921		o2	1013		a1	1105		c1	1197		a1
831		a1	923		a1	1015		c1	1107		c1	1199		o2
833		a1	925		c1	1017		o2	1109		wl	1201		c1
835		c1	927		o2	1019	3	w#r	1111		c1	1203		wi
837		a1	929		wl	1021		wk	1113		a1	1205		o2
839	8	a1	931		c1	1023		a1	1115	3	w#r	1207		w5
841		c1	933	4	d1	1025		a1	1117		wk	1209		c1
843		a1	935		a1	1027		c1	1119		c1	1211		o2
845		o1	937		c1	1029		o2	1121		a1	1213		m2
847		c1	939		c1	1031	6	a1	1123	3	m3	1215		wi
849		c1	941		m2	1033		wk	1125		o1	1217		wk
851		o2	943		o1	1035		a1	1127		a1	1219		c1
853	3	a1	945		a1	1037		o2	1129		wk	1221		c1
855		c1	947	3	w#q	1039	3	a1	1131		a1	1223	8	a1
857		m2	949		o2	1041		c1	1133	3	d1	1225		w4
859	3	a1	951		a1	1043		o2	1135		c1	1227		o2
861		c1	953		a2	1045		c1	1137		a1	1229		a2
863	3	m3	955	3	a1	1047		o2	1139		w5	1231		y2
865		o2	957		c1	1049		a2	1141		c1	1233		a1
867		a1	959		o2	1051	3	w#q	1143		o2	1235		o1
869		o2	961		wk	1053		a1	1145		o2	1237		c1
871		c1	963		a1	1055		a1	1147		c1	1239		c1
873		a1	965		o2	1057		c1	1149		c1	1241		o2
875		a1	967		c1	1059		o2	1151		a1	1243		o2
877		c1	969		o2	1061		a1	1153		wk	1245		o2
879		wi	971	6	a1	1063	3	w#q	1155		c1	1247		a1
881		wk	973		o2	1065		a1	1157		o2	1249		wl
883	3	w#q	975		c1	1067		o2	1159		o2	1251		a1
885		a1	977		a1	1069		c1	1161		a1	1253		a1
887		a1	979		o2	1071		a1	1163		a1	1255	3	a1
889		c1	981		a1	1073		o2	1165		o2	1257	4	o2
891		o1	983		a1	1075		o2	1167		c1	1259	4	a1

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
1261		c1	1353		o1	1445		a1	1537		o1	1629		c1
1263		a1	1355		a1	1447	19	se	1539		o1	1631		o2
1265		a1	1357		c1	1449		c1	1541		a1	1633		o2
1267		o2	1359	3	d1	1451	6	a1	1543	3	a1	1635		o2
1269		o1	1361		a1	1453		wl	1545		c1	1637		a1
1271		o1	1363		o2	1455		c1	1547		o2	1639		o2
1273		o2	1365		c1	1457		a1	1549		wk	1641		a1
1275		a1	1367	3	m3	1459		c1	1551		a1	1643		a1
1277		a1	1369		wl	1461		a1	1553		a1	1645		o1
1279		c1	1371		a1	1463		a1	1555		c1	1647		o2
1281		o1	1373		m2	1465		o2	1557		o2	1649		o2
1283	3	w#q	1375		c1	1467		a1	1559	4	a1	1651		c1
1285		o1	1377		a1	1469		o2	1561		c1	1653		o2
1287		a1	1379		o2	1471	3	w#p	1563		w2	1655		a1
1289		a2	1381		m2	1473	3	a1	1565		o2	1657		c1
1291	3	w#q	1383		a1	1475		o1	1567	19	se	1659		wi
1293		a1	1385		o2	1477		c1	1569		c1	1661	3	d1
1295		a1	1387		o2	1479		c1	1571	18	se	1663	3	m3
1297		c1	1389		c1	1481		a1	1573		o2	1665		a1
1299		o2	1391		a1	1483	3	a1	1575		a1	1667	3	m3
1301		wl	1393		o2	1485		a1	1577		o2	1669	3	w#q
1303	3	w#q	1395		c1	1487	3	m3	1579	5	a1	1671		o2
1305		c1	1397	3	d1	1489		wl	1581		a1	1673		a1
1307		a1	1399		c1	1491		o2	1583	3	m3	1675		o2
1309		c1	1401		c1	1493		wl	1585		c1	1677		o1
1311		c1	1403		o2	1495		o1	1587		wh	1679		o2
1313		o2	1405		c1	1497		a1	1589	3	d1	1681		c1
1315	3	d1	1407		o1	1499	18	se	1591		c1	1683		wj
1317		c1	1409		a2	1501		c1	1593		o1	1685		o2
1319	18	se	1411		o2	1503		a1	1595		a1	1687		c1
1321		wl	1413		a1	1505		o2	1597		wk	1689	3	o3
1323		wi	1415		a1	1507		o2	1599		o2	1691		a1
1325		o1	1417		c1	1509	3	a1	1601		a2	1693		wl
1327	3	w#p	1419		c1	1511		a1	1603		o2	1695		a1
1329		c1	1421		a1	1513		o2	1605		c1	1697		wl
1331		a1	1423	3	a1	1515		o2	1607		a1	1699	3	a1
1333		o2	1425		w5	1517		a1	1609		c1	1701		a1
1335		o2	1427	3	m3	1519		c1	1611		c1	1703	3	o2
1337		a1	1429		c1	1521		c1	1613		a1	1705		o2
1339		c1	1431		c1	1523		a1	1615		c1	1707		a1
1341		wb	1433		m2	1525		c1	1617		o2	1709		wk
1343		o2	1435		o1	1527	3	d1	1619	3	w#q	1711		o2
1345		c1	1437	6	a1	1529		o2	1621		wk	1713	3	o3
1347		a1	1439	19	se	1531		c1	1623		a1	1715		o2
1349		o2	1441	3	a1	1533		a1	1625		o1	1717		c1
1351		o2	1443		o2	1535		o2	1627		c1	1719	3	a1

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
1721		$a1$	1813		$o2$	1905		$o2$	1997		wk	2089		$c1$
1723	3	$w\#q$	1815		$o1$	1907	3	$w\#q$	1999		$y2$	2091		$a1$
1725		$a1$	1817		$o2$	1909		$o2$	2001		$c1$	2093		$w5$
1727		$a1$	1819		$c1$	1911		$a1$	2003		$a1$	2095	3	$a1$
1729		$c1$	1821		$a1$	1913	19	se	2005		$o1$	2097		$a1$
1731		$c1$	1823	3	wn	1915	3	$a1$	2007		$c1$	2099	3	$w\#r$
1733		$a2$	1825		$o2$	1917		$c1$	2009		$o2$	2101		$c1$
1735		$c1$	1827		$a1$	1919		$o2$	2011		$c1$	2103		$o2$
1737		$a1$	1829		$o2$	1921		$o2$	2013		$o2$	2105		$a1$
1739		$o2$	1831	3	$m3$	1923		$a1$	2015		$a1$	2107		$o2$
1741		$c1$	1833		$a1$	1925		$a1$	2017		wk	2109		$c1$
1743		$a1$	1835		$o2$	1927		$c1$	2019		wi	2111		$a1$
1745		$o2$	1837		$c1$	1929	3	$o3$	2021		$o2$	2113		wl
1747	3	$m3$	1839		$c1$	1931		$a1$	2023		$o2$	2115		$c1$
1749		$o1$	1841	3	$d1$	1933	3	$w\#q$	2025		$c1$	2117		$a1$
1751	3	$d1$	1843		$o2$	1935		wi	2027	3	$w\#r$	2119	3	$d1$
1753		wl	1845		$o1$	1937		$o2$	2029		$c1$	2121		$c1$
1755		$a1$	1847	3	$m3$	1939		$c1$	2031		$a1$	2123		$o2$
1757		$a1$	1849		$c1$	1941		$c1$	2033	4	$d1$	2125		$o1$
1759		$c1$	1851		$c1$	1943		$o2$	2035		$o2$	2127		$c1$
1761		$a1$	1853		$a1$	1945		$c1$	2037		$a1$	2129		$a2$
1763		$o2$	1855		$c1$	1947		$o1$	2039	20	se	2131		$c1$
1765		$c1$	1857		$o2$	1949	4	$a1$	2041		$o2$	2133		$o2$
1767		$c1$	1859		$o2$	1951		$y2$	2043		$a1$	2135		$a1$
1769		$o2$	1861		$c1$	1953		$o1$	2045		$a1$	2137		$c1$
1771		$c1$	1863		$a1$	1955		$o1$	2047		$c1$	2139		wi
1773		$o2$	1865		$a1$	1957	3	$d1$	2049		$o1$	2141		$a1$
1775		$o2$	1867		$c1$	1959		$c1$	2051		$o2$	2143	3	$w\#q$
1777		wl	1869		$o2$	1961		$o1$	2053	3	$w\#q$	2145		$c1$
1779		$c1$	1871	3	$m3$	1963	3	$d1$	2055		$a1$	2147		$o2$
1781		$o2$	1873		wk	1965		$c1$	2057		$o2$	2149		$c1$
1783	7	$a1$	1875		$a1$	1967		$a1$	2059		$o2$	2151		$o2$
1785		$o1$	1877		$a1$	1969	4	$o2$	2061		$a1$	2153		wm
1787	3	$m3$	1879	3	$a1$	1971		$a1$	2063	8	$a1$	2155	3	$a1$
1789	3	$w\#q$	1881		$a1$	1973		$a2$	2065		$c1$	2157		$a1$
1791		$c1$	1883	3	$w\#r$	1975		$o2$	2067		$c1$	2159	3	$d1$
1793	4	$a1$	1885		$c1$	1977		$a1$	2069		$a2$	2161		wk
1795	5	$d1$	1887		$a1$	1979	4	$a1$	2071		$o2$	2163		$o2$
1797		$a1$	1889		$a2$	1981	4	$d1$	2073		$a1$	2165		$o2$
1799		$o1$	1891		$w4$	1983		$o2$	2075		$o2$	2167		$o2$
1801		wk	1893	3	$o3$	1985		$o2$	2077		$c1$	2169		$c1$
1803		$a1$	1895		$o2$	1987	16	se	2079		$c1$	2171	4	$d1$
1805		$a1$	1897		$c1$	1989		$o1$	2081		wl	2173		$o1$
1807		$c1$	1899		$c1$	1991		$a1$	2083	3	$w\#q$	2175		$a1$
1809		$c1$	1901		$a1$	1993		wl	2085		$o1$	2177		$a1$
1811		$a1$	1903		$o2$	1995		$c1$	2087	4	$a1$	2179		$c1$

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	t	How	q	t	How	q	t	How	q	t	How	q	t	How
2181		o2	2273		a1	2365		c1	2457		w2	2549		a2
2183		a1	2275		c1	2367		a1	2459	3	w#q	2551		c1
2185		w5	2277		o2	2369	3	d1	2461	3	a1	2553		a1
2187		a1	2279		o2	2371	9	a1	2463		a1	2555		o2
2189		o2	2281		c1	2373		a1	2465		a1	2557		c1
2191		o2	2283		o2	2375		o2	2467		c1	2559		wi
2193		o1	2285		o2	2377		wk	2469		c1	2561		a1
2195		a1	2287	20	se	2379		o2	2471		a1	2563		o2
2197		wk	2289		o2	2381		m2	2473		wk	2565		a1
2199		c1	2291		o2	2383	3	w#q	2475		w5	2567		a1
2201		a1	2293	22	se	2385		a1	2477		a1	2569		o2
2203	3	a1	2295		o1	2387		a1	2479		c1	2571	3	d1
2205		a1	2297		a1	2389		wk	2481		a1	2573		o2
2207	4	a1	2299		c1	2391		o2	2483		a1	2575		w5
2209		wk	2301		a1	2393		a2	2485		c1	2577		c1
2211		c1	2303		o2	2395		c1	2487		c1	2579	3	w#q
2213		m2	2305		o2	2397		a1	2489	3	o2	2581		o2
2215	4	d1	2307		a1	2399	8	a1	2491		o1	2583		a1
2217		a1	2309		wk	2401		c1	2493		o2	2585		o2
2219		o2	2311		c1	2403		wi	2495		o2	2587		o2
2221		c1	2313		o1	2405		a1	2497		c1	2589	4	d1
2223		o1	2315	4	a1	2407		c1	2499		o1	2591	3	m3
2225		o2	2317		o2	2409		c1	2501		o2	2593		wl
2227	3	o2	2319		c1	2411		a1	2503	3	a1	2595		c1
2229		c1	2321		a1	2413	3	d1	2505		c1	2597		o2
2231		a1	2323		o2	2415		o1	2507		o2	2599		c1
2233		o2	2325		c1	2417		wm	2509		o2	2601		wf
2235		o2	2327	4	o2	2419		o1	2511		c1	2603		o2
2237		wk	2329		c1	2421		o2	2513	4	k1	2605		c1
2239		y2	2331		a1	2423	4	a1	2515	3	d1	2607		a1
2241		a1	2333		m2	2425		o2	2517		a1	2609		wm
2243		a1	2335	3	a1	2427		o2	2519		o2	2611		o2
2245		c1	2337		c1	2429	4	d1	2521		c1	2613		o1
2247		c1	2339	4	a1	2431		c1	2523		a1	2615		a1
2249		o2	2341	3	w#q	2433		o2	2525		a1	2617		c1
2251	5	a1	2343		a1	2435		a1	2527		o2	2619		c1
2253		a1	2345		o2	2437		wk	2529		o2	2621		m2
2255		o1	2347	3	m3	2439		c1	2531	3	m3	2623		o2
2257		c1	2349		o1	2441		wl	2533		o2	2625		a1
2259		c1	2351		a1	2443		o2	2535		a1	2627		o2
2261		a1	2353		o2	2445		c1	2537		o2	2629	3	a1
2263		o2	2355		a1	2447		a1	2539		c1	2631		c1
2265		a1	2357		m2	2449		o2	2541		a1	2633		a1
2267		a1	2359		o2	2451		a1	2543	6	a1	2635		o1
2269	3	w#q	2361		c1	2453		a1	2545	3	a1	2637		c1
2271		o2	2363		o2	2455		c1	2547		o2	2639		o2

TABLE A.2 Orders of Known Hadamard Matrices *(continued)*

q	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How	*q*	*t*	How
2641		*c1*	2713		*wk*	2785		*c1*	2857		*wl*	2929		*c1*
2643		*o2*	2715		*a1*	2787		*c1*	2859		*c1*	2931		*c1*
2645		*o2*	2717		*a1*	2789		*m2*	2861		*a1*	2933		*a1*
2647	3	*w#q*	2719		*c1*	2791		*c1*	2863		*o2*	2935		*c1*
2649		*c1*	2721		*a1*	2793		*a1*	2865	3	*d1*	2937		*o2*
2651		*o2*	2723		*a1*	2795		*o2*	2867		*a1*	2939	8	*a1*
2653		*o2*	2725		*c1*	2797		*m2*	2869		*c1*	2941		*c1*
2655		*c1*	2727		*o2*	2799		*wi*	2871		*a1*	2943		*o2*
2657		*a1*	2729		*wl*	2801		*wk*	2873		*a1*	2945		*a1*
2659	3	*w#q*	2731	3	*w#q*	2803	3	*w#q*	2875		*c1*	2947		*o2*
2661	3	*d1*	2733	3	*a1*	2805		*o1*	2877		*o2*	2949		*c1*
2663		*a1*	2735		*a1*	2807		*o1*	2879	21	*se*	2951	3	*d1*
2665		*c1*	2737		*o1*	2809		*wk*	2881		*o2*	2953		*wl*
2667		*a1*	2739		*c1*	2811		*a1*	2883		*o2*	2955		*o2*
2669		*o2*	2741		*a2*	2813		*a1*	2885		*o2*	2957		*a1*
2671	9	*a1*	2743		*o2*	2815	3	*d1*	2887	5	*a1*	2959		*y2*
2673		*a1*	2745		*a1*	2817		*o2*	2889		*w5*	2961		*o1*
2675		*o2*	2747		*a1*	2819	3	*w#q*	2891		*o2*	2963	3	*w#q*
2677	9	*a1*	2749		*wk*	2821		*c1*	2893	3	*a1*	2965		*o2*
2679		*o2*	2751		*a1*	2823	3	*d1*	2895		*a1*	2967		*a1*
2681		*a1*	2753		*a2*	2825		*a1*	2897		*a1*	2969		*a2*
2683	7	*a1*	2755		*o2*	2827		*c1*	2899	4	*d1*	2971	3	*a1*
2685		*a1*	2757		*a1*	2829		*c1*	2901		*c1*	2973	4	*d1*
2687	21	*se*	2759		*o2*	2831		*o2*	2903	4	*a1*	2975		*o1*
2689		*wk*	2761		*c1*	2833		*wk*	2905		*o2*	2977		*c1*
2691		*c1*	2763		*o2*	2835		*c1*	2907		*c1*	2979		*o2*
2693		*a1*	2765		*a1*	2837		*wl*	2909		*wk*	2981		*a1*
2695		*o2*	2767	3	*w#p*	2839		*y2*	2911		*c1*	2983		*o2*
2697		*c1*	2769		*o2*	2841	3	*a1*	2913	7	*d1*	2985		*a1*
2699	21	*se*	2771		*a1*	2843	3	*m3*	2915		*o1*	2987	3	*w#r*
2701		*w4*	2773	3	*d1*	2845		*c1*	2917		*wl*	2989		*o2*
2703		*o1*	2775		*o2*	2847		*c1*	2919		*wi*	2991		*c1*
2705		*o1*	2777		*m2*	2849		*o2*	2921	3	*d1*	2993		*a1*
2707		*c1*	2779		*c1*	2851		*c1*	2923		*o2*	2995	9	*a1*
2709		*c1*	2781		*o2*	2853		*a1*	2925		*a1*	2997		*a1*
2711	3	*m3*	2783		*a1*	2855	4	*d1*	2927	3	*m3*	2999	22	*se*

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