Chapter 1

Preliminaries

In this chapter, We provide some definitions, examples and properties of permanent of matrix that could be helpful to proceed further. Also, geometrical interpretation of determinant and permanent of matrix is provided.

1.1 Permanent

1.1.1 Definition

If N be an $n \times n$ matrix over a field F, and N_{ij}, its entries (with i ranging over a set I and J over a set J, each of n elements), then permanent of N is given by

$$per(N) = \sum_{\pi \in S} \prod_{i \in I} N_{i,\pi(i)}$$

S denote the set of all bijection $\pi : I \rightarrow J$ and is consists of the elements of the symmetric group S_n, the group of permutations of n objects. I and J are identified with set of natural numbers, *i.e,* $1, \cdots, n$.

This definition of permanent is closely related to the following definition of determinant of a matrix:

$$det(N) = \sum_{\pi \in S} sgn(\pi) \prod_{i \in I} N_{i,\pi(i)}$$

Note that if π is an even permutation then we use " $+$ " sign and for odd permutation, we use " $-$ " sign.

Example 1. *Suppose that N be a square matrix of order 2 as given below.*

$$N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$per(N) = ad + bc$$

and

$$det(N) = ad - bc$$

1.1.2 Basic properties:

In this section, we describe some properties of permanent of matrices.

1. Permanent have the property of homogeneous of degree n *i.e,* For matrix A

   $$per(kA) = k^n per(A)$$

   provided that k is scalar.

2. Let A be a square matrix and A^T be its transpose then

   $$per(A^T) = per(A).$$

3. Suppose that matrix A has only non-negative entries, then

   $$per(A) \geq det(A).$$

4. Permanent of matrix A (per(A)) is invariant under arbitrary permutations of rows and /or columns of matrix A. *i.e,*

   $$per(A) = per(PAQ)$$

   for any appropriately sized permutation matrices P and Q.

5. Let B be a matrix of the form

$$B = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

then

$$per \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = per(A) * per(D).$$

6. For an identity matrix I,

$$per(I) = 1.$$

7. If matrix A has a row that has all entries equal to zero, then

$$per(A) = 0.$$

1.1.3 Geometrical Interpretation

It is observed that permanent of matrix has not very clear geometrical interpretation like determinant of matrix but still permanent can be illustrated with the help of bipartite graph and perfect matchings.

The permanent of a (0, 1)-matrix can be interpreted as the number of perfect matchings in a bipartite graph. More precisely, given such a matrix

$$A = (a_{ij})_{1 \leq i,j \leq n}$$

we can define a bipartite graph G with two parts

$$U = \{u_1, u_1, \cdots, u_n\}$$

and

$$V = \{v_1, v_1, \cdots, v_n\}$$

and there is an edge between $u_i \in U$ and $v_j \in V$ if and only if $a_{ij} = 1$. It directly follows from the definition of the permanent that

$$perm(A) = pm(G)$$

Conversely, the number of perfect matchings of a bipartite graph is the permanent of its incidence matrix, *i.e*, if U and V are the two color classes, the matrix is

$$(a_{uv})_{(u,v) \in U \times V}$$

with a $uv = 1$ if uv is an edge, and 0 otherwise.

1.1.4 Bipartite Graphs and Perfect Matchings

In this section, we describe definition of bipartite graph and perfect matching. We explain relation between bipartite graph and perfect matching that helps in understanding the concept of permanent, perhaps, more clearly.

Matching:

In the mathematical discipline of graph theory, a matching or independent edge set in a graph is a set of edges without common vertices. It may also be an entire graph consisting of edges without common vertices. Bipartite matching is a special case of a network flow problem. Let G be a graph. A matching of G is a set M of edges of G such that no two edges in M are adjacent in G [Wik10, Hir08, GN72]. A matching of graph G is a subgraph of G such that every edge shares no vertex with any other edge. That is, each vertex in matching M has degree one.

The size of a matching is the number of edges in that matching. Consider the graph in Figure 1.1. Denote the edge that connects vertices i and j as (i, j). Note that

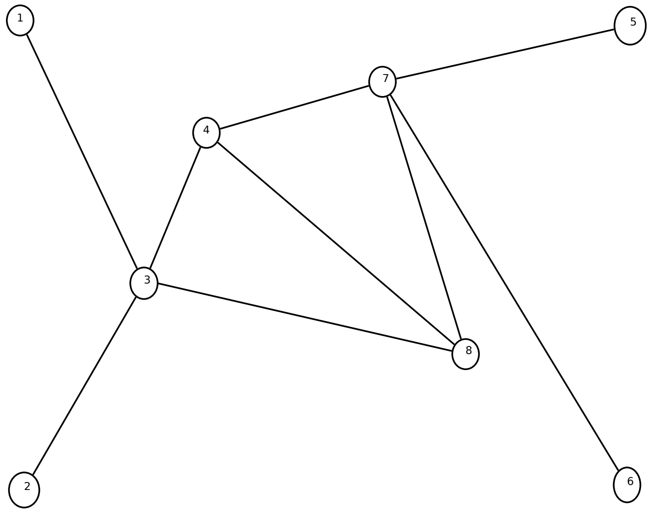


Figure 1.1: Example of Matching

$\{(3, 8)\}$ is a matching. Obviously we can get more. The pairs

$$\{(3, 8), (4, 7)\}$$

also make a matching. That is a matching of size two. A matching of size 3 is

$$\{(2, 3), (4, 8), (5, 7)\}$$

Perfect Matching:

A matching M is perfect if every vertex of G is incident to an edge of M. Simply, we can say that, a matching which matches all vertices of the graph, *i.e.* every vertex of the graph is incident to exactly one edge of the matching. The number of perfect matchings of G can be denoted by $pm(G)$. Figure 1.2 represents an example of perfect matching. Corresponding to the graph presented in figure 1.2, an adjacency matrix

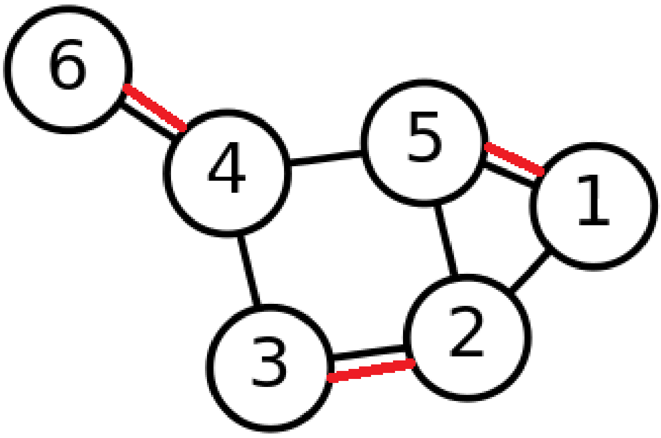


Figure 1.2: Perfect Matching

A is given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Number of perfect matchings in figure 1.2 is one that can be determined by computing permanent of matrix A.

$$\text{Permanent of matrix A} = 1$$

Bipartite Graph:

In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets U and V (that is, U and V are each independent sets) such that every edge connects a vertex in U to one in V. Vertex set U and V are often denoted as partite sets.

A bipartite graph is a special case of a graph with specific properties. Firstly, in a bipartite graph, every vertex can be sorted into one of two disjoint sets A and B. Secondly, each vertex in a set (for example, A) is connected to at least one vertex in the opposite set B by an edge, and vice versa. Lastly, for any given vertex in a set, it cannot be connected to any other vertex in the same set, thus the only connections between vertices are between the two disjoint sets [ADH98, Wik10]. An example of a simple bipartite graph is shown in figure 1.3. The three vertices on the top form one

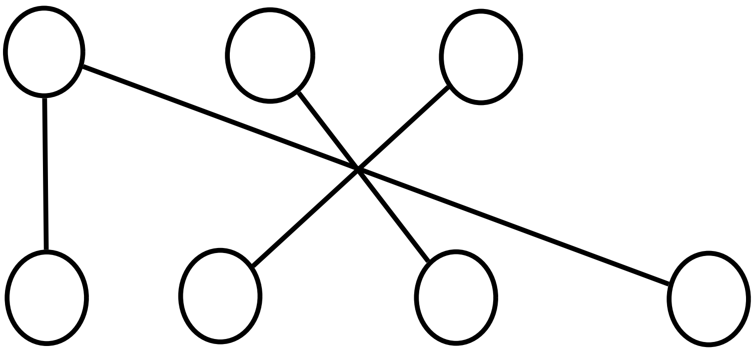


Figure 1.3: Simple bipartite graph

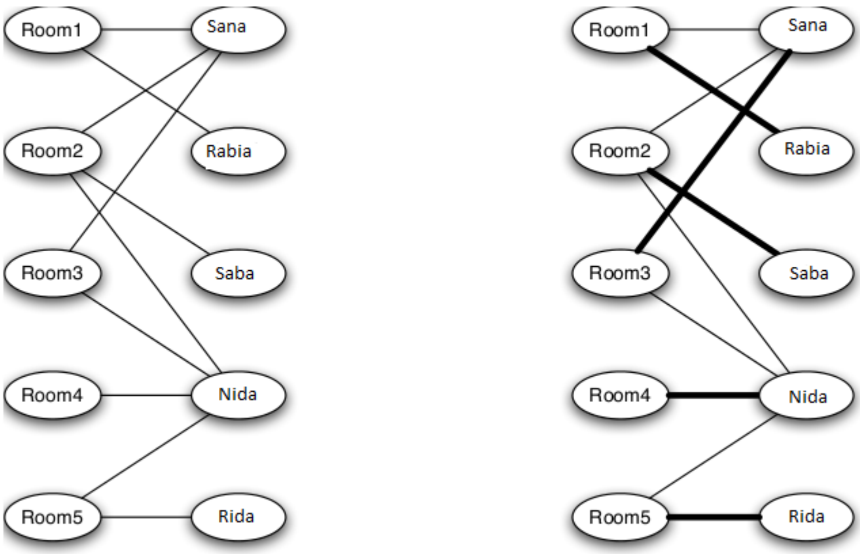
set and the bottom four vertices form the second disjoint set. Note that any graph whose edges have some numerical value assigned to them is called a weighted graph.

Example 2. *Suppose that administrators of a college dormitory need to assign rooms to students in such a way that each room assign to a single student. Now, students and rooms can be considered as two different categories or sets. If we denote these two sets by A and B, then*

$$A = \{Sana,\ Rabia,\ Saba,\ Nida,\ Rida\}$$

$$B = \{Room1,\ Room2,\ Room3,\ Room4,\ Room5\}$$

These two sets (A and B) are presented in figure 1.4. Figure 1.4 part (a) presents bipartite graph between two sets A and B, where as part (b) of the figure presents perfect matching between the sets. Note that in part (b) of the figure 1.4, each room is assigned to a single student where as in part (a) of the figure 1.4, a room is assigned to more than one students.



(a) A Bipartite Graph

(b) A Perfect Matching

Figure 1.4: Bipartite Graph and Perfect Matching

Further, If there are equal number of nodes on each side of a bipartite graph, a perfect matching is an assignment of nodes on the one side to nodes on the other side in such a way that

- Each node is connected by an edge to the node it is assigned to.
- No two nodes on the left are assigned to the same node on the right.

1.2 Determinant

Determinant of square matrices plays a fundamental role in linear algebra. It is a linear function on rows (and columns) of the matrix, and has several nice interpretations. Geometrically, it is the volume of the parallelepiped defined by rows (or columns) of the matrix, and algebraically, it is the product of all eigenvalues, with multiplicity, of the matrix. It also satisfies a number of other interesting properties, e.g., it is multiplicative, invariant under linear combinations of rows (and columns) etc [Agr06, vzG87, yC90].

1.2.1 Geometrical interpretation

Determinant can be interpreted as the volume of parallelepiped defined by rows (or columns) of matrix. Suppose that there are three vectors

$$\vec{a} = (a_1,\ a_2,\ a_3),$$

$$\vec{b} \ = \ (b_1,\ b_2,\ b_3)$$

and

$$\vec{c} \ = \ (c_1,\ c_2,\ c_3)$$

represent three edges that meet at one vertex of parallelepiped (see figure 1.5). Then volume of the parallelepiped is equal to the absolute value of the scalar triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

can be computed as follows.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Note that the volume V of parallelepiped is the absolute value of scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$.

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right|$$

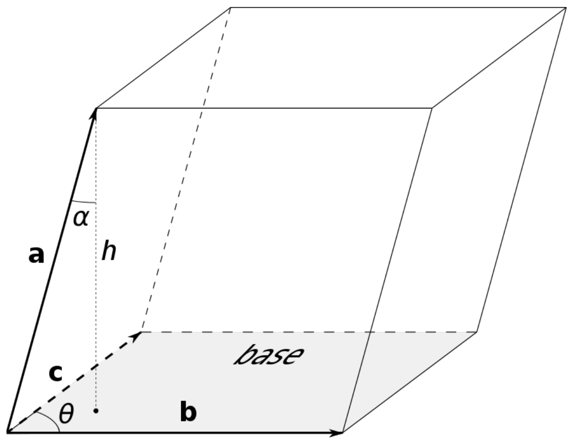


Figure 1.5: Parallelepiped

Example 3. *Let $\vec{a}$=$(2,0,2)^T$, $\vec{b}$=$(-1,0,1)^T$ and $\vec{c}$=$(0,3,1)^T$ be three vectors that represent three edges meet at one vertex of parallelepiped (see figure 1.6). Then volume of parallelepiped is given as*

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})| = \left|\det\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}\right| = 12$$

Now we can construct a matrix A by using above vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as rows

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Determinant of matrix A is given as

$$\det(A) = \det\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} = 12$$

1.3 Matrices

A matrix is a concise and useful way of uniquely representing and working with linear transformations. In particular, every linear transformation can be represented by a

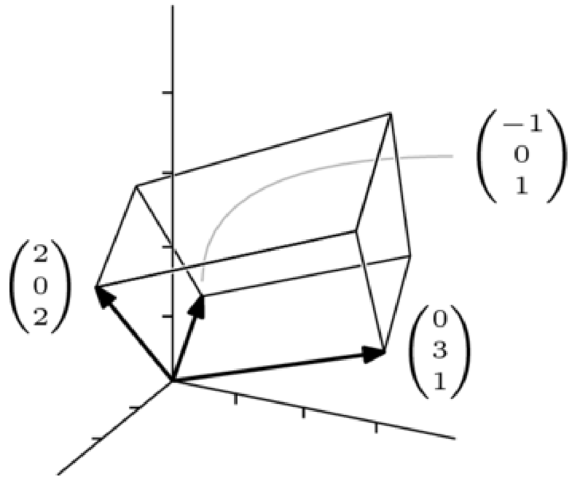


Figure 1.6: Second Example

matrix, and every matrix corresponds to a unique linear transformation. The matrix, and its close relative, the determinant, are extremely important concepts in linear algebra, and were first formulated by Sylvester (1851) and Cayley [Syl83]. In real life, a matrix is a rectangular array with prescribed numbers n of *rows* and m of columns ($n \times m$ matrix).

1.3.1 Adjacency matrix

The adjacency matrix for a graph with n vertices is an $n \times n$ matrix whose i, j-th entry is 1, if vertex i and vertex j are adjacent, and 0, if they are not. An adjacency matrix is a mean of representing the vertices of graph which are adjacent to other vertices. The adjacency matrix of bipartite graph whose parts have r and s vertices has the form [W^{+}01],

$$\begin{bmatrix} O & B \\ B^T & O \end{bmatrix}$$

where B is an $r \times s$ matrix and O is an all zero matrix. The matrix B represent a bipartite graph and it is also called biadjacency matrix. The adjacency matrix of an undirected simple graph is symmetric, and therefore has complete set of real eigenvalues and an orthogonal eigenvector basis.

Example 4. *Corresponding to the graph presented in figure 1.2, an adjacency matrix A is given below.*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

1.3.2 (0-1) matrix

A (0-1) matrix is a matrix whose entries are either 0 or 1. Such matrices arise frequently in combinatorics and graph theory. It is known that the largest number of ones in an n×n nonsingular (0-1) matrix is $n^2 - n + 1$ [HLZ05].

A (0, 1)-matrix (also identified as zero-one or Boolean) is a rectangular matrix for which each element of the matrix has the value of either one or zero. (0, 1)-matrices arise from problems in a variety of application areas. Prominent examples include:

1. adjacency matrix for simple graph, representing connectivity relationship between vertices.

2. Matrix calculus applications in statistics and econometrics which generate special (0,1)-matrices such as selection, permutation, commutation, elimination, duplication, and shifting matrices [And04].

Example 5. *The 0-1 matrix of size 5*

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

1.3.3 Biadjacency Matrix

Let $G = (V, E)$ be a bipartite graph. The biadjacency matrix B represents the bipartite graph.The biadjacency matrix $B = b_{ij}$ is r× s, (0-1) matrix in which $b_{ij} = 1$ if and only if $(u_i, v_j) \in E$ otherwise $b_{ij} = 0$.

A biadjacency matrix B of bipartite graph given in figure 1.4 is

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The permanent of biadjacency matrix B is given as

$$per(B) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = 1$$

As the perfect matching of given bipartite graph is unique and permanent of biadjacency matrix of bipartite graph is also one. Thus, permanent of the biadjacency matrix of a bipartite graph is exactly the number of perfect matchings in the graph.

Square matrices

Square matrices having non negative entries are considered to be very important as they played important role in probabilistic theory of finite Morkov chains and in the study of linear models. The properties of such matrices were first investigated by Perron [GD53, Ser10]. In mathematics, a square matrix is a matrix with the same number of rows and columns. An $n \times n$ matrix is known as a square matrix of order n.

Example 6.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 8 & 1 & 2 \end{bmatrix}$$

is a square matrix of size 3.

1.3.4 Finding Rank of Matrices

The rank of a matrix is equal to the number of linearly independent rows. A linearly independent row is one that is not a combination of other rows.

The following matrix has two linearly independent rows (1 and 2). However, when the third row is thrown into the mix, you can see that the first row is now equal to the sum of the second and third rows. Therefore, the rank of this particular matrix is 2, as there are only two linearly independent rows.

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

Another example is given below.

Example 7.

$$N = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$

Rank of matrix N is 2, because it has two linearly independent rows and the third row is the linear combination of first row and second row.

The matrix rank will always be less than the number of non-zero rows or the number of columns in the matrix. If all of the rows in a matrix are linearly independent, the matrix is full row rank. For a square matrix, it is only full rank if its determinant is zero.

Figuring out the rank of a matrix by trying to determine by sight only how many rows or columns are linearly independent can be practically impossible. An easier (and perhaps obvious) way is to convert to row echelon form. Finding the rank of a matrix is simple if we know how to find the row echelon matrix. Thus, to find the rank of any matrix:

- Find the row echelon matrix.
- Count the number of non-zero rows.

Example 8. *Consider the matrix*

$$K = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 6 & 12 \end{bmatrix}$$

Performing elementary row operations, matrix K is reduced to echelon form as given below

$$K \simeq \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix has been converted to row echelon form with two non-zero rows. Therefore, the rank of the matrix is 2 [Sch91, Sta15].

Identity matrix

The identity matrix I_n of size n is the matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

Example 9. *An identity matrix of size n can be written as*

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$