10.2 BLOCK DESIGNS

A generic *block design* can be regarded as a generalization of a graph, in which a *block* is a generalized edge.

DEF: A ***block design*** $\mathcal{B}$ has a non-empty domain

$$X = \{x_1, x_2, \ldots, x_v\}$$

whose elements are sometimes called *varieties* and a non-empty collection

$$B = \{B_1, B_2, \ldots, B_b\}$$

of subsets of X called ***blocks***. It is a ***simple design*** if no two blocks are identical.

DEF: The number of blocks in which an element x appears is called the ***valence of that element of the design***.

DEF: The number of blocks in which a pair of elements x and y appears is called the ***covalence of that pair***.

Thus, a graph is a block design in which every block has size 2. The valence of an element within the block design would be its degree as a vertex of the graph. The covalence of a pair of elements of the design would be their multiplicity of adjacency as vertices of the graph. To allow self-loops in a graph, one would allow the blocks to be multisets of elements of the design and make suitable revisions in the definition of valence and covalence.

DEF: A block design is ***regular*** if the following two conditions hold:

- every block is the same size $k \geq 2$, which is called the ***blocksize***;
- each element x_j has the same valence; that is, each appears in the same number r of blocks, which is called the ***replication number***.

Thus, a d-regular graph is a regular block design with blocksize 2 and replication number d.

Balanced Designs

The idea of *balancing* a design with *incomplete blocks* arose with Sir Ronald Fisher (1890-1962) in his theoretical study of the design of experiments in agriculture.

DEF: A regular block design $\mathcal{B}$ with v varieties and b blocks is ***balanced*** and is called either a ***(v, b, r, k, λ)-design*** or a ***(v, k, λ)-design*** if each pair of elements x_i and x_j has the same covalence, that is, if each pair appears in the same number λ of blocks, which is called the ***index of the design***.

A balanced design is ***complete*** if $k = v$, so that each block contains all of X. If $k < v$, then it is ***incomplete***.

TERMINOLOGY: A balanced incomplete block design is commonly called a ***BIBD***.

Example 10.2.1: For $X = \{0, 1, 2, 3\}$, the blocks

$$B_1 : 012 \quad B_2 : 013 \quad B_3 : 023 \quad B_4 : 123$$

form a $(4, 4, 3, 3, 2)$-design.

Example 10.2.2: For $X = \{0, 1, \ldots, 8, 9, A\}$, the blocks

$$\begin{array}{cccccc} 02348 & 13459 & 2456A & 35670 & 46781 & 57892 \\ 689A3 & 79A04 & 8A015 & 90126 & A1237 & \end{array}$$

form a $(v = 11,\ b = 11,\ r = 5,\ k = 5,\ \lambda = 2)$-design. In this design, the initial block generates all of the others, if we regard the elements of X as integers modulo 11, with a standing for 10 modulo 11. Then each other block is obtained by adding 1 modulo 11 to each of the elements of the previous block.

Example 10.2.3: For every $n \geq 2$, setting $X = [1 : n]$ and $B_1 = X$ yields a complete design with $v = n$, $b = 1$, $r = 1$, $k = n$, and $\lambda = 1$.

Example 10.2.4: For every $n \geq 2$, setting $X = [1 : n]$ and having the pairs of elements from X as blocks yields a balanced design with

$$v = n, \quad b = \binom{n}{2}, \quad r = n - 1, \quad k = 2, \quad \lambda = 1$$

Thus, the complete graph K_n is representable as a BIBD.

Example 10.2.5: When a simple graph is drawn on an arbitrary surface without crossings, each edge lies on exactly two faces. If the graph is K_n, and if all faces are k-sided, then this drawing may be regarded as a BIBD with $v = n$, blocksize k, and $\lambda = 2$, in which a block is the set of corners of a face.

Necessary Conditions

The examples above establish that BIBD's exist for certain combinations of the parameters v, b, r, k, and λ. However, there are no BIBD's for various other combinations. Our immediate concern is to derive some necessary conditions for the existence of a (v, b, r, k, λ)-design.

Prop 10.2.1. *For every non-empty (v, b, r, k, λ)-BIBD*

$$(a)\ \lambda \geq 1 \quad \text{and} \quad (b)\ k < v$$

Proof: Since there is at least one block, and since it has at least two elements, some pair has at least once occurrence. Since all pairs occur equally often, it follows that $\lambda \geq 1$.

Since a block is a subset of the domain, its size cannot exceed the size of the domain. Thus, $k \leq v$. Since a BIBD is *incomplete*, it follows that $k < v$. ♢

Proposition 10.2.2. *The parameters of a (v, b, r, k, λ)-design on*

$$X = \{ x_1, x_2, \ldots, x_v \}$$

satisfy the following two conditions:

$$(a) \qquad bk = vr$$
$$(b) \qquad r\,(k-1) = \lambda\,(v-1)$$

Proof: First consider the $v \times b$ incidence matrix

$$I = \begin{array}{c|ccc} & B_1 & \cdots & B_b \\ \hline x_1 & \iota_{1,1} & \cdots & \iota_{1,b} \\ \vdots & \vdots & \vdots & \vdots \\ x_v & \iota_{v,1} & \cdots & \iota_{v,b} \end{array} \qquad \iota_{i,j} = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{otherwise} \end{cases}$$

There are v rows, each with row-sum r, and there are b columns, each with column-sum k. Therefore, $bk = vr$.

Next consider the $\binom{v}{2} \times b$ pair-incidence matrix

$$I' = \begin{array}{c|ccc} & B_1 & \cdots & B_b \\ \hline x_1x_2 & \iota'_{12,1} & \cdots & \iota'_{12,b} \\ \vdots & \vdots & \vdots & \vdots \\ x_{v-1}x_v & \iota'_{(v-1)v,1} & \cdots & \iota'_{(v-1)v,b} \end{array}$$

with

$$\iota'_{ij,\ell} = \begin{cases} 1 & \text{if } x_ix_j \in B_\ell \\ 0 & \text{otherwise} \end{cases}$$

There are $\binom{v}{2}$ rows, each with row-sum λ, and there are b columns, each with column-sum $\binom{k}{2}$. Therefore,

$$\lambda\binom{v}{2} = b\binom{k}{2}$$

Accordingly,

$$\begin{aligned} \lambda v\,(v-1) &= bk\,(k-1) \\ \Rightarrow \lambda v\,(v-1) &= vr\,(k-1) \quad \text{since } bk = vr \\ \Rightarrow \lambda\,(v-1) &= r\,(k-1) \end{aligned} \qquad \diamondsuit$$

TERMINOLOGY NOTE: The inferrability (from Prop 10.2.2) of values of b and r from values of v, k, and λ justifies optionally calling a (v, b, r, k, λ)-design a (v, k, λ)-design.

Corollary 10.2.3. *For every non-empty BIBD,*

$$\lambda < r$$

Proof: Since $\lambda\,(v-1) = r\,(k-1)$ (from Thm 10.2.2) and $k < v$ (from Prop 10.2.1), it follows that $\lambda < r$. $\diamondsuit$

REVIEW FROM LINEAR ALGEBRA:

- If AB is the product of the matrices A and B then

$$rank(AB) \leq \min\{rank(A), rank(B)\}$$

NOTATION: The *transpose* of a matrix M is denoted M^T.

Thm 10.2.4 [Fisher's Ineq]. *In any BIBD, $b \geq v$.*

Proof: Let I be the incidence matrix of the BIBD. Then

$$I\,I^T = \begin{pmatrix} r & \lambda & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \lambda & \cdots & r \end{pmatrix}$$

Subtracting the first column of a matrix from the other columns does not change the determinant. Hence,

$$det(I\,I^T) \;=\; \begin{vmatrix} r & \lambda - r & \lambda - r & \lambda - r & \cdots & \lambda - r \\ \lambda & r-\lambda & 0 & 0 & \cdots & 0 \\ \lambda & 0 & r-\lambda & 0 & \cdots & 0 \\ \lambda & 0 & 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & 0 & \cdots & r-\lambda \end{vmatrix}$$

Adding the other rows of a matrix to the first row does not change the determinant. Hence,

$$det(I\,I^T) \;=\; \begin{vmatrix} r+(v-1)\lambda & 0 & 0 & 0 & \cdots & 0 \\ \lambda & r-\lambda & 0 & 0 & \cdots & 0 \\ \lambda & 0 & r-\lambda & 0 & \cdots & 0 \\ \lambda & 0 & 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & 0 & \cdots & r-\lambda \end{vmatrix}$$

Since the upper triangle of this matrix is all zeroes, the determinant is the product of the diagonal entries. Thus,

$$det(I\,I^T) \;=\; [r+(v-1)\lambda](r-\lambda)^{v-1}$$

By Corollary 10.2.3, $r - \lambda > 0$. Moreover, $r + (v-1)\lambda$ is positive. Thus, $det(I\,I^T)$ is non-zero. Accordingly, the rank of the $v \times v$-matrix $I\,I^T$ is v. Since the rank of the $v \times b$ incidence matrix I is at most b, and since the rank, v, of the product matrix $I\,I^T$ cannot exceed the rank of the matrix I, it follows that $v \leq b$. ♢

Steiner Triple Systems

DEF: A $(v, 3, 1)$-design is also called a ***Steiner triple system***.

Example 10.2.6: The complete balanced block design

$$\mathcal{A} = \begin{cases} \text{domain} & X = \{0,\ 1,\ 2\} \\ \text{1 block} & B = \{012\} \end{cases} \tag{10.2.1}$$

is a Steiner triple system. (A Steiner triple system on a domain with more than three elements is a BIBD.)

Example 10.2.7: The BIBD

$$\mathcal{B} = \begin{cases} \text{domain} & Y = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\} \\ \text{7 blocks} & C = \{013,\ 124,\ 235,\ 346,\ 450,\ 561,\ 602\} \end{cases}$$

is a $(7, 3, 1)$-design. As in Example 10.2.2, the first block generates the others.

Proposition 10.2.5. *In a $(v, 3, 1)$-design,*

$$(a)\ r = \frac{v-1}{2} \quad \text{and} \quad (b)\ b = \frac{v(v-1)}{6}$$

Proof: Part (a) follows from Proposition 10.2.2(b):

$$r\,(k-1) = \lambda\,(v-1)$$

Simply substitute $k = 3$ and $\lambda = 1$.

For part (b), start with the equation

$$bk = rv$$

from Proposition 10.2.2(a). Then substitute 3 for k and $(v-1)/2$ for r to obtain

$$3b = v\frac{v-1}{2}$$

which leads immediately to the desired formula. ♢

Corollary 10.2.6. *In a* $(v,3,1)$*-design,*

$$v \equiv 1 \text{ or } 3 \text{ modulo } 6$$

Proof: Prop 10.2.5(a) implies that v is odd. Thus,

$$v \equiv 1, 3 \text{ or } 5 \text{ modulo } 6$$

However, if $v \equiv 5$ modulo 6, then $v(v-1) \equiv 2$ modulo 6, contradicting Prop 10.2.5(b). ♢

Constructing Designs

Jakob Steiner (1796-1893) asked in 1853 whether for every positive v such that $v \equiv 1$ or 3 modulo 6, there exists a $(v,3,1)$-design. He was unaware that in 1847, the Rev. Thomas P. Kirkman (1806-1895) had proved they always exist. Kirkman's methods are beyond the present scope. We presently offer some elementary methods that can also be used for constructing BIBD's with larger block-size. The first such method generalizes Example 10.2.7.

DEF: A set of numbers

$$S = \{a_1,\ a_2,\ \ldots,\ a_k\}$$

in $\mathbb{Z}_n$ is a ***perfect difference set*** of index λ for $\mathbb{Z}_n$ if each non-zero number in $\mathbb{Z}_n$ occurs exactly λ times in the list

$$\langle x_{ij} \ = \ a_i - a_j \mid a_i, a_j \in S;\ \ i \neq j \rangle$$

It is simply called a ***perfect difference set*** if $\lambda = 1$.

Proposition 10.2.7. *A perfect difference set B of cardinality k and index λ for $\mathbb{Z}_v$ generates a (v, k, λ)-design.*

Proof: For $j = 0, \ldots, v-1$, let $B_j \ = \ \{j + b \mid b \in B\}$. By the definition of a perfect difference set, these blocks form a (v, k, λ)-design. ♢

Example 10.2.7, cont.: The set $\{0, 1, 3\} \subset \mathbb{Z}_7$ is a perfect difference set of index 1, since

$$\begin{array}{lll} 1 = 1-0 & 2 = 3-1 & 3 = 3-0 \\ 4 = 0-3 & 5 = 1-3 & 6 = 0-1 \end{array}$$

DEF: A family $\mathcal{S}$ of sets $S_1, \ldots, S_f \subset Z_n$ is a ***perfect difference family*** of index λ if each non-zero number in $\mathbb{Z}_n$ occurs exactly λ times in the list

$$\langle x_{ijk} \ = \ a_i - a_j \mid a_i, a_j \in S_k;\ \ i \neq j;\ 1 \leq k \leq f \rangle$$

It is called a ***perfect difference family*** if $\lambda = 1$.

Proposition 10.2.8. *If the sets of a perfect difference family of index λ for $\mathbb{Z}_v$ are all of the same size k, then they generate a (v, k, λ)-design.* ♢

Example 10.2.8: We construct a perfect difference family for $\mathbb{Z}_{13}$

$$\{0,\ 1,\ 4\} \text{ with differences } \{1,\ 3,\ 4,\ 9,\ 10,\ 12\}$$
$$\{0,\ 2,\ 8\} \text{ with differences } \{2,\ 5,\ 6,\ 7,\ 8,\ 11\}$$

These two blocks together generate the following $(13, 3, 1)$-design.

$$X = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ A,\ B,\ C\}$$

$$B = \left\{ \begin{array}{ccccccccc} 014 & 125 & 236 & 347 & 458 & 569 & 67A & 78B & 89C \\ & 9A0 & AB1 & BC2 & C03 & & & & \\ 028 & 139 & 24A & 35B & 46C & 570 & 681 & 792 & 8A3 \\ & 9B4 & AC5 & B06 & C17 & & & & \end{array} \right\}$$

Example 10.2.9: The set $\{0, 1, 4, 6\}$ is a perfect difference set for $\mathbb{Z}_{13}$. Thus, with the domain

$$X = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ A,\ B,\ C\}$$

the set of blocks

$$B = \left\{ \begin{array}{ccccccc} 0146 & 1257 & 2368 & 3479 & 458A & 569B & 67AC \\ 78B0 & 89C1 & 9A02 & AB13 & BC24 & C035 & \end{array} \right\}$$

forms a (13, 4, 1)-design.

The next example offers a way to construct a new Steiner triple system from two (possibly identical) smaller systems.

Example 10.2.10: The cartesian product of the domain of the $(3,3,1)$-design $\mathcal{A}$ of Example 10.2.6 and the domain of the $(7,3,1)$-design $\mathcal{B}$ of Example 10.2.7 is representable as the following array.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 00 & 01 & 02 & 03 & 04 & 05 & 06 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{pmatrix} \end{array}$$

To obtain a (21, 3, 1)-design $\mathcal{A} \times \mathcal{B}$ on the set of elements of that array, we choose as blocks

(i) every column;

(ii) from each row, each triple $\{ri,\ rj,\ rk\}$ such that $\{i,\ j,\ k\}$ is a block of $\mathcal{B}$;

(iii) each triple $\{0i,\ 1j,\ 2k\}$ such that $\{i,\ j,\ k\}$ is a block of $\mathcal{B}$.

Observe that the number of blocks we have chosen is

$$7 + 21 + 42 = 70$$

Two elements xy and $x'y'$ of $\mathcal{A}\times\mathcal{B}$ appear in one and only one block. There are three cases.

(i) $x \neq x'$ and $y = y'$: only in the block arising from column y.

(ii) $x = x'$ and $y \neq y'$: only in the block arising from row x and the unique block of $\mathcal{B}$ in which y and y' are paired.

(iii) $x \neq x'$ and $y \neq y'$: let x'' be the remaining row, and let y'' be the third entry in the unique block of $\mathcal{B}$ that contains both y and y'. Then $\{xy,\ x'y',\ x''y''\}$ is the unique block containing xy and $x'y'$.

DEF: The ***product of two Steiner triple systems*** $\mathcal{A}$ and $\mathcal{B}$ is the triple system whose domain is the product of the domains of $\mathcal{A}$ and $\mathcal{B}$, with blocks as follows:

(i) from each column of the product array $\mathcal{A} \times \mathcal{B}$, each triple $\{rc,\ sc,\ tc\}$ such that $\{r,\ s,\ t\}$ is a block of $\mathcal{A}$;

(ii) from each row of $\mathcal{A} \times \mathcal{B}$, each triple $\{ri,\ rj,\ rk\}$ such that $\{i,\ j,\ k\}$ is a block of $\mathcal{B}$;

(iii) each triple $\{ri,\ sj,\ tk\}$ such that $\{r,\ s,\ t\}$ is a block of $\mathcal{A}$ and $\{i,\ j,\ k\}$ is a block of $\mathcal{B}$.

Theorem 10.2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Steiner triple systems with u and v varieties, respectively. Then their product is a Steiner triple system with uv varieties.*

Proof: The proof for the general case is essentially the same as for Example 10.2.10. ♢

Remark: The definition and theorem just above are generalizable to a product of BIBD's and a theorem that the result is a new BIBD.

Isomorphism of Designs

DEF: A bijection $f : X \to Y$ of the domains of two block designs

$$\mathcal{B} = \langle X, \{B_i\}\rangle \quad \text{and} \quad \mathcal{C} = \langle Y, \{C_j\}\rangle$$

is called an ***isomorphism of block designs*** if for every block C_j of design $\mathcal{C}$, there is a block B_i of design $\mathcal{B}$, such that the restriction $f : B_i \to C_j$ is onto.

Proposition 10.2.10. *Let* $\mathcal{B} = \langle X, \{B_i\}\rangle$ *be a* (7, 3, 1) *Steiner system. Then* $\mathcal{B}$ *is isomorphic to the* (7, 3, 1) *Steiner system with elements* $0, 1, 2, 3, 4, 5, 6$ *and blocks*

$$013 \quad 124 \quad 235 \quad 346 \quad 450 \quad 561 \quad 602$$

Proof: Choose an arbitrary element of X and call it x_0. Since each of the six other elements of X must appear with x_0 exactly once, there must be exactly three blocks of $\mathcal{B}$ that contain x_0. Call the other two elements in one of these blocks x_1 and x_3, and call the other two in a second of these blocks x_2 and x_6. Partially specify the bijection f by

$$x_0 \mapsto 0 \quad x_1 \mapsto 1 \quad x_2 \mapsto 2 \quad x_3 \mapsto 3 \quad x_6 \mapsto 6$$

which ensures some block preservation, namely,

$$x_0x_1x_3 \mapsto 013 \quad x_0x_2x_6 \mapsto 026 \quad x_0x_4x_5 \mapsto 045$$

The elements x_1 and x_2 appear together in a unique block of $\mathcal{B}$. Since the third element of that block cannot be x_0, x_3, or x_6, each of which appears in another block with x_1 or x_2, it can be called x_4, with the remaining element of X to be x_5.

Completing the bijection specification with

$$x_4 \mapsto 4 \quad x_5 \mapsto 5$$

immediately ensures further block preservation

$$x_1x_2x_4 \mapsto 124$$

Moreover, given that $x_0x_1x_3$ and $x_1x_2x_4$ are blocks, it follows that the third block containing x_1 must be $x_1x_5x_6$. Similarly, the third block containing x_2 must be $x_2x_3x_5$. Since the elements x_3, x_4, and x_6 have so far appeared in only two blocks each, the seventh block must be $x_3x_4x_6$. Thus all blocks are preserved by the bijection f. ♢

Remark: There is essentially only one (7, 3, 1)-design, as established by Prop 10.2.10, and also only one (9, 3, 1)-design. There are two non-isomorphic (13, 3, 1)-designs and 80 mutually non-isomorphic (15, 3, 1)-designs. See the table on p764 of [CoDi2000a].

10.3 CLASSICAL FINITE GEOMETRY

Many properties of the Euclidean spaces $\mathbb{R}^n$ can be derived purely from a short list of axioms about points and lines, without consideration of distance or angles, and without consideration that a line in $\mathbb{R}^n$ contains infinitely many points. In this spirit, various kinds of combinatorial designs on a finite set of elements have been called ***finite geometries***. The elements of their domains are traditionally called the ***points of the geometry***, and their distinguished subsets are called the ***lines of the geometry***. The following two general axioms are standard for geometries.

G1. Two distinct points are contained in at most one line.

G2. Two distinct lines intersect in at most one point.

NOTATION: In view of Axiom G1, we may denote the line containing two distinct points u and v by uv.

TERMINOLOGY: Two disjoint lines of a geometry are often said to be ***parallel lines***.

DEF: The ***incidence matrix of a geometry*** $\langle X, L\rangle$ with p points

$$X = \{x_1, \ldots, x_p\}$$

and ℓ lines

$$L = \{L_1, \ldots, L_\ell\}$$

is the $p \times \ell$ matrix

$$M_{\langle X,L\rangle}[i,j] \;=\; \begin{cases} 1 & \text{if } x_i \in L_j \\ 0 & \text{otherwise} \end{cases}$$

A geometry is commonly specified by its incidence matrix.

Example 10.3.1: Figure 10.3.1 illustrates a geometry with a drawing of its four points and its six lines.

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccccc} L_0 & L_1 & L_2 & L_3 & L_4 & L_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

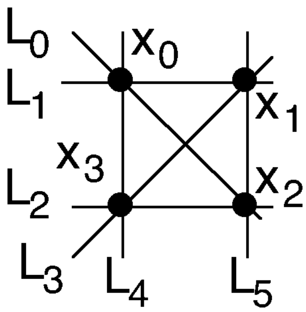

Fig 10.3.1 A geometry with 4 points and 6 lines.

DEF: The ***dual of a geometry*** $\langle X, L\rangle$ is the geometry $\langle X^*, L^*\rangle$ with

$$X^* \;=\; L \quad \text{and} \quad L^* \;=\; X$$

whose incidence matrix is the transpose of the incidence matrix of $\langle X, L\rangle$. (In view of the reciprocity of Axioms G1 and G2, the dual design satisfies both of them.)

Example 10.3.1, cont.: Figure 10.3.2 illustrates the dual of the geometry specified by Figure 10.3.1.

	x_0^*	x_1^*	x_2^*	x_3^*
L_0^*	1	0	1	0
L_1^*	1	1	0	0
L_2^*	0	0	1	1
L_3^*	0	1	0	1
L_4^*	1	0	0	1
L_5^*	0	1	1	0

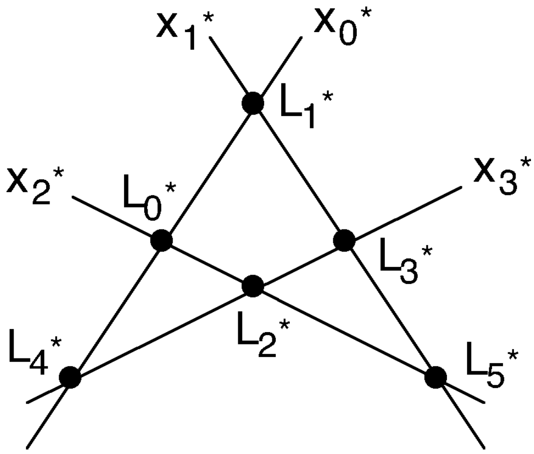

Fig 10.3.2 The dual geometry has 6 points and 4 lines.

The Fano Plane

A design named for the Italian geometer Gino Fano (1871-1952) is the first of three widely cited classical geometries that we now consider.

DEF: The ***Fano plane*** is defined by the incidence matrix

	L_0	L_1	L_2	L_3	L_4	L_5	L_6
0	0	0	0	1	0	1	1
1	1	1	0	1	0	0	0
2	1	0	1	0	0	0	1
3	0	0	1	1	1	0	0
4	1	0	0	0	1	1	0
5	0	1	1	0	0	1	0
6	0	1	0	0	1	0	1

It is depicted in the diagram in Figure 10.3.3, in which the line L_0 is represented by a circle.

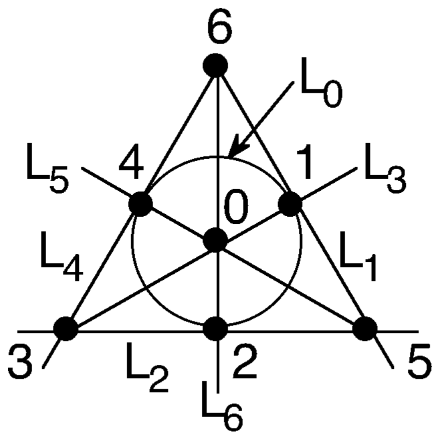

Fig 10.3.3 **The Fano plane.**

We observe that as a design, the Fano plane is precisely the Steiner triple system of Example 10.2.7.

The Pappus Geometry

A second classical geometry is named for Pappus of Alexandria (c. 300-350 C.E.), who proved the following theorem of Euclidean geometry.

Theorem of Pappus. *Let 0, 1, and 2 be three distinct points on a line* L_1 *and 3, 4, and 5 three distinct points on line* $L_2 \neq L_1$*, such that there are points of intersection*

$$6 = 04 \cap 13 \quad 7 = 05 \cap 23 \quad \text{and} \quad 8 = 15 \cap 24$$

Then the points 6, 7, and 8 are colinear. ♢

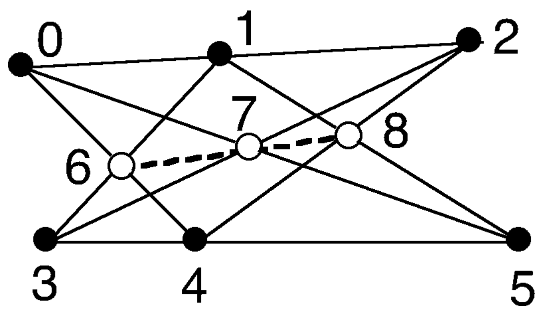

Fig 10.3.4 **The geometry of Pappus.**

DEF: The ***Pappus geometry*** is the following finite geometry

$$X = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$$
$$L = \{012,\ 345,\ 064,\ 075,\ 163,\ 185,\ 273,\ 284,\ 678\}$$

or any other geometry of the same isomorphism type.

The Pappus geometry has uniform blocksize 3 and uniform replication number 3. As in Euclidean plane geometry, no pair of points occurs more than once in a line. However, in the Pappus geometry, and unlike Euclidean geometry, some pairs of points do not lie on any line. This implies that the Pappus geometry is not a Steiner triple system or a BIBD. The Pappus geometry shares the following property with Euclidean plane geometry.

Proposition 10.3.1. *Let L_i be any line of the Pappus geometry, and let p be a point that is not on that line. Then there is a unique line L_j containing the point p and parallel to the line L_i.*

Proof: The lines of the Pappus geometry are resolvable into three classes of parallel lines.

$$
\begin{aligned}
C_1 &= \{012 \quad 345 \quad 678\} \\
C_2 &= \{064 \quad 185 \quad 273\} \\
C_3 &= \{075 \quad 163 \quad 284\}
\end{aligned}
$$

If the given line L_i lies in the class C_k, then choose line L_j to be the unique line in class C_k that contains point p. ♢

The Desargues Geometry

Another theorem of plane Euclidean geometry is due to Girard Desargues (1591-1661).

Theorem of Desargues. *Let 123 and 456 be triangles such that the lines 14, 25, and 36 meet at point 0. Let*

$$7 = 13 \cap 46 \quad 8 = 23 \cap 56 \quad \text{and} \quad 9 = 12 \cap 45$$

Then 7, 8, and 9 are colinear. ♢

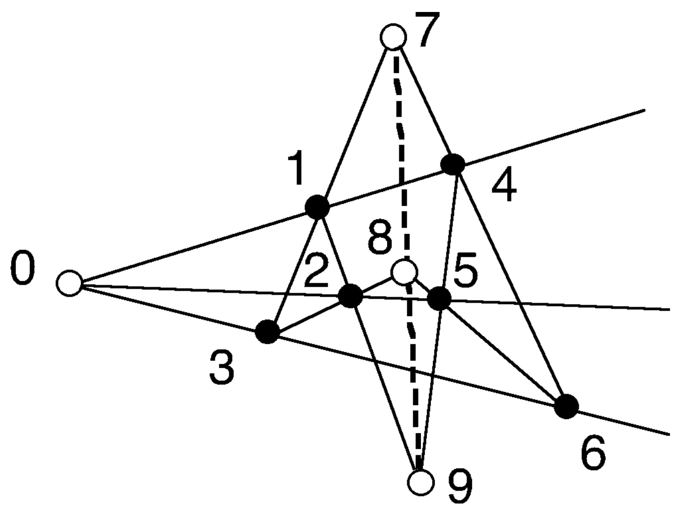

Fig 10.3.5 **The geometry of Desargues.**

DEF: The ***Desargues geometry*** is the following finite geometry

$$X = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$$
$$L = \{014,\ 025,\ 036,\ 137,\ 129,\ 238,\ 467,\ 459,\ 568,\ 789\}$$

or any other geometry of the same isomorphism type.

In the Desargues geometry, as in the Pappus geometry, there is a uniform blocksize of 3 and a uniform replication number of 3. As in Euclidean geometry and the Pappus geometry, no pair of points occurs more than once in a block. As in the Pappus geometry, and unlike Euclidean geometry, some pairs do not occur on any line. Accordingly, it is not a Steiner triple system or a BIBD.

Remark 1: Observe that Proposition 10.3.1 does not apply to the Desargues geometry. In fact, for every line L_i in the Desargues geometry, there is a point p such that no line containing p intersects the line L_i. Such a point p is called a *pole* of the line L_i.

Example 10.3.2: In the Desargues geometry, the point 8 is a pole of the line 014, and the point 1 is a pole of the line 568.

Remark 2: Another interesting property in which Desargues geometry differs from Euclidean geometry is that in the Desargues geometry, two lines that are parallel to the same line are *not* parallel to each other.

Example 10.3.3: The lines that are parallel to the line 789 of the Desargues geometry are 014, 036, and 025. Observe that any pair of them intersects in the point 0.

Partially Balanced Designs

DEF: A $(v, b, r, k; \lambda_1, \lambda_2)$-***PBIBD*** (stands for ***partially balanced incomplete block design***) is a design with v elements and b blocks, in which

(i) each element lies in exactly r blocks;

(ii) each block contains exactly k elements;

(iii) each pair of distinct elements occurs either in λ_1 or λ_2 blocks.

Example 10.3.4: The Pappus geometry is a $(9, 9, 3, 3; 1, 0)$-PBIBD.

Example 10.3.5: The Desargues geometry is a $(10, 10, 3, 3; 1, 0)$-PBIBD.