HIGHER-ORDER DIFFERENTIAL EQUATIONS

1. INTRODUCTION.

In Chapter 13, we analysed the methods of solving the first-order differential equations — both mathematical as well as economic. We have told that the first order differential equations involve the derivative or differential having the highest power 1. But there also exist differential equations of higher order, which we simply call n th order.

The general form of linear differential equation of n th–order is presented as:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots\dots\dots\dots + a_{n-1} \frac{dy}{dt} + a_n y = b.$$

or

$$y^n (t) + a_1 y^{(n-1)} (t) + \dots\dots\dots\dots + a_{n-1} y' (t) + a_n y = b.$$

This is n th order equation, because the n th derivative (the first term on the left) is the highest derivative in the equation. It is linear because all the derivatives as well as the dependent variable, y appear only in the first degree. Again the term in the form of multiplication of y and its derivative $\left(\frac{dy}{dt}\right)$ occurs. Moreover, this differential equation is furnished with constant–coefficients (the a's) and a constant term (b).

2. Second–Order Linear Differential Equations With Constant Coefficients and Constant Terms.

The most notable higher–order differential equation is Second–order differential equation. Accordingly, a second-order differential equation is the one in which highest order derivative or differential involved is second-order derivative. A specific second-order differential equation may be as: $\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 6y = 35$. In case of economics we may confront a function which is describing the rate of change of the rate of change of income variable (Y), as: $\frac{d^2Y}{dt^2} = kY$. From this differential equation we are required to find the time path of Y —— Y (t). The task of finding the time path is concerned with solving the second–order differential equation.

The general form of a second-order differential equation with constant coefficients and a constant term is as:

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = b \quad \text{or} \quad y'' (t) + a_1 y' (t) + a_2 y = b.$$

where a_1 and a_2 are the coefficients and b is the term and they all are constants. If the term b is zero we have a homogeneous equation, but if b is a non-zero constant, the equation is non-homogeneous. Here, we shall concentrate upon non-homogeneous equation. Again, when we solve the non-homogeneous differential equation, the solution of the homogeneous equation will emerge automatically as a by–product.

Solution. We have already told that if y_c is the complementary function, i.e., the general solution (with arbitrary constants) of the above reduced equation, and if y_p is the particular integral, i.e., any particular solution (with no arbitrary constants) of the above complete equation, then $y(t) = y_c + y_p$ will be the general solution of the complete equation. As discussed earlier that the y_p component provides us with the equilibrium value of the variable y in the inter-temporal sense of the term. While the y_c represents the deviation of the time path $y(t)$ from the equilibrium.

Finding The Particular Integral (Y_p). As the particular integral can be any solution of above equation—— any value of y that satisfies this non–homogeneous equation. To find it out we take

C-1: $y(t) = a$, then $y'(t) = 0$ and again $y''(t) = 0$. Then our general second order differential equation will be as:

$$y''(t) + a_1 y'(t) + a_2 y(t) = b \Rightarrow (0) + a_1(0) + a_2 y = b \Rightarrow a_2 y = b \Rightarrow y = \frac{b}{a_2}.$$

Thus the desired particular integral is : $\boxed{y_p = \frac{b}{a_2}}$, where $a_2 \neq 0$. It is proved as: Let the particular integral be: $y(t) = k$, then $y'(t) = y''(t) = 0$

By substituting into the general form: $y''(t) + a_1 y'(t) + a_2 y(t) = b \Rightarrow 0 + a_1(0) + a_2(k) = b$

$$\Rightarrow \quad a_2 k = b \quad \Rightarrow \quad k = \frac{b}{a_2}. \text{ Accordingly, } y_p = k = \frac{b}{a_2}.$$

Example 1. We find Y_p, given the differential equation: $y''(t) - 6y'(t) + 5y(t) = 15$.

Here $a_1 = -6$, $a_2 = 5$ and $b = 15$. Thus $y_p = \frac{b}{a_2} = \frac{15}{5} = 3$.

Example 2. We find Y_p of the equation: $y''(t) + y'(t) - 2y = -10$.

Thus the coefficient are: $a_2 = -2$ and $b = -10$. Therefore, $y_p = \frac{b}{a_2} = \frac{-10}{-2} = 5$.

C-2: ~~Example 3.~~ If $y(t) = kt$ and $a_2 = 0$. we find y_p. In case of $y(t) = kt$, $y'(t) = k$ and $y''(t) = 0$.

Putting these values in general form: $y''(t) + a_1 y'(t) + a_2 y(t) = b$

$$(0) + a_1(k) + (0)kt = b \Rightarrow a_1 k = b \Rightarrow k = \frac{b}{a_1}$$

Thus $\boxed{y_p = kt = \frac{b}{a_1}(t)}$ (while $a_1 \neq 0$) or $y_p = \frac{b}{a_1} t$

Example 4. We find the y_p of the equation: $y''(t) + y'(t) = -10$.

Here the coefficients are: $a_2 = 0$, $a_1 = 1$ and $b = -10$. Thus $y_p = \frac{b}{a_1} t = \frac{-10}{1} t = -10t$.

C-3 ~~Example 5.~~ If $y(t) = kt^2$, while $a_1 = a_2 = 0$, the resultant differential equation will be as: $y''(t) = b$.

$y'(t) = 2kt$, $y''(t) = 2k$. Putting in general form: $y''(t) + a_1 y'(t) + a_2 y(t) + b$

$2k + 0(2kt) + 0(k^2 t) = b. \Rightarrow 2k = b \Rightarrow k = \frac{b}{2}$. Thus $\boxed{y_p = kt^2 = \frac{b}{2} t^2.}$

Example 6. We find the y_p of the equation $y''(t) = -10$.

The coefficient are $a_1 = a_2 = 0$ and $b = -10$.

The formula $y_p = \frac{b}{2} t^2$ will be applied: $y_p = \frac{-10}{2} t^2 = -5t^2$.

Complementary Function:-

There are three conditions for Complementary Soln.

(1):- Distinct real Roots when $(a_1)^2 > 4a_2$

(235) Honey Advanced ...

(ii):- Repeated real Roots when $(a_1)^2 = 4a_2$

(iii):- Complex real Roots when $(a_1)^2 < 4a_2$

(1):- Distinct Real Roots:- when $(a_1)^2 > 4a_2$

then $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$

and where $r_1, r_2 = \dfrac{-a_1 \pm \sqrt{(a_1)^2 - 4a_2}}{2}$

(2):- Repeated Real Roots:-

when $(a_1)^2 = 4a_2$ then

$$y_c = A_3 e^{rt} + A_4 t e^{rt}$$

where $r = -a_1/2$ $\left(\because r_1 = r_2 = r = \dfrac{-a_1}{2}\right)$

(3):- Complex Real Roots:-

when $(a_1)^2 < 4a_2$ then

$$y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$$

where $h = \dfrac{-a_1}{2}$ and $v = \sqrt{\dfrac{4a_2 - (a_1)^2}{2}}$

Example No4:- Solve the differential equation

$$y''(t) + y'(t) - 2y = -10$$

Solution:- $y''(t) + y'(t) - 2y = -10$

Here $a_1 = 1$, $a_2 = -2$, and $b = -10$

The particular integral is

$$Y_p = \frac{b}{a_2} \implies Y_p = \frac{-10}{-2} = 5$$

Check for Rule:-

$$(a_1)^2 > 4a_2$$

$$(1)^2 > 4(-2) \implies 1 > -8$$

So it is distinct real root.

The characteristic roots are

$$r_1, r_2 = -a_1 \pm \sqrt{(a_1)^2 - 4a_2} / 2$$

$$r_1, r_2 = \frac{-1 \pm \sqrt{(1)^2 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$r_1, r_2 = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$$

$$r_1, r_2 = \frac{-1+3}{2} = 2/2 = 1$$

$$r_1, r_2 = \frac{-1-3}{2} = -4/2 = -2$$

$$r_1 = 1, \quad r_2 = -2$$

The complementary integral is

$$Y_c = A_1 e^{rt} + A_2 e^{rt}$$

$$Y_c = A_1 e^{t} + A_2 e^{-2t}$$

The general solution is

$$y(t) = y_c + y_p$$

$$y(t) = A_1 e^{t} + A_2 e^{-2t} + 5$$

To find the constants A_1, A_2 we put $t=0$

$$y(0) = A_1 e^{0} + A_2 e^{-2(0)} + 5$$

$$y(0) = A_1 + A_2 e^{0} + 5 \qquad (\because e^{0} = 1)$$

$$y(0) = A_1 + A_2 + 5$$

Let $y(0) = 12$ and $y'(0) = -2$

$$12 = A_1 + A_2 + 5 \Rightarrow A_1 + A_2 = 12 - 5$$

$$A_1 + A_2 = 7 \longrightarrow (A)$$

Now differentiating w.r.to "t" of G.S and putting $t=0$

$$y'(t) = A_1 e^{t} . 1 + A_2 e^{-2t} (-2)$$

$$y'(t) = A_1 e^{t} - 2 A_2 e^{-2t}$$

putting $t=0$

$$y'(0) = A_1 e^{0} - 2 A_2 e^{-2(0)} \qquad (\because e^{0} = 1)$$

$$y'(0) = A_1 e^{0} - 2 A_2 e^{0}$$

$$y'(0) = A_1 - 2 A_2 \qquad (\because y'(0) = -2)$$

$$-2 = A_1 - 2 A_2$$

$$A_1 - 2 A_2 = -2 \longrightarrow (B)$$

Now subtracting eqn A & B

$$\begin{aligned} \cancel{A_1} + A_2 &= 7 \\ -\cancel{A_1} \mp 2A_2 &= \mp 2 \\ \hline 3A_2 &= 9 \end{aligned} \Rightarrow A_2 = 9/3 = 3$$

$$\boxed{A_2 = 3}$$

putting the value of A_2 in eqn A

$$A_1 + 3 = 7 \Longrightarrow A_1 = 7-3$$

$$\boxed{A_1 = 4}$$

Therefore, the definite solution is

$$\boxed{\therefore y(t) = 4e^{t} + 3e^{-2t} + 5}$$

verification:- For verification putting $t=0$

$$y(0) = 4e^{0} + 3e^{-2(0)} + 5$$

$$\left(\begin{aligned} \because y(0) &= 12 \\ e^{0} &= 1 \end{aligned}\right)$$

$$12 = 4(1) + 3e^{0} + 5$$

$$12 = 4 + 3(1) + 5$$

$$12 = 4 + 3 + 5$$

$$12 = 12 \quad \text{Ans.}$$

Example NO 5:- Solve the differential equation.

$$y''(t) + 6y'(t) + 9y = 27$$

Solution:- $y''(t)+6y'(t)+9y=27$

Here $a_1=6$, $a_2=9$, and $b=27$

The particular integral is

$$y_p=\frac{b}{a_2}=27/9=3 \Rightarrow \boxed{y_p=3}$$

Check for Rule:-

$$(a_1)^2=4a_2$$

$$(6)^2=4(9)$$

$$36=36$$

So it is repeated real root.

The characteristic roots are

$$r=\frac{-a_1}{2} \Rightarrow r=\frac{-6}{2}=-3$$

The Complementary integral is

$$y_c=A_3e^{rt}+A_4te^{rt}$$

$$\boxed{y_c=A_3e^{-3t}+A_4te^{-3t}}$$

The general Solution is

$y(t)=y_c+y_p$

$y(t)=A_3e^{-3t}+A_4te^{-3t}+3$

To find the general solution putting $t=0$

$y(0)=A_3e^{-3(0)}+A_4(0)e^{-3(0)}+3$ $\left(\because y'(0)=5,\ y(0)=5\right)$

$y(0)=A_3e^0+0+3$

$y(0) = A_3 + 3 \qquad (\because y(0) = 5)$

$5 = A_3 + 3$

$A_3 = 5 - 3 \qquad \boxed{A_3 = 2}$

Now differentiating w.r. to "t" putting $t = 0$

$\dot{y}(t) = A_3 e^{-3t}(-3) + (-3) A_4 t e^{-3t} + A_4 e^{-3t}(1)$

$\dot{y}(t) = -3A_3 e^{-3t} - 3A_4 t e^{-3t} + A_4 e^{-3t}$

Now

$\dot{y}(0) = -3A_3 e^{-3(0)} - 3A_4 e^{-3(0)} + A_4 e^{-3(0)}$

$\dot{y}(0) = -3A_3 e^{0} - 3A_4 e^{0}(0) + A_4 e^{0}$

$\dot{y}(0) = (-3A_3 - 3A_4(0) + A_4) e^{0} \qquad (\because e^{0} = 1)$

$\dot{y}(0) = -3A_3 + A_4$

$-5 = -3(2) + A_4 \qquad \left(\because \begin{array}{c} \dot{y}(0) = -5 \\ A_3 = 2 \end{array}\right)$

$-5 = -6 + A_4$

$A_4 = 6 - 5 \qquad \boxed{A_4 = 1}$

Therefore, the definite solution is

$$\boxed{\therefore y(t) = 2e^{-3t} + te^{-3t} + 3}$$

Verification:- $y(0) = 2e^{-3(0)} + (0)e^{-3(0)} + 3$

$5 = 2e^{0} + 0 + 3 \qquad (\because e^{0} = 1)$

$5 = 2(1) + 3 \Rightarrow 5 = 5$ Ans

$5 = 5$