

Q: NO: What is equations reducible to the linear form.

ANS If the differentiable equations $dy/dt = h(y, t)$ happens or becomes in specific form which is non-linear form

$$dy/dt + Ry = Ty^m \rightarrow (1)$$

Where R and T are two functions of "t" and m is only or any number other than 0 and 1 then the equation referred to as a Bernoulli equation. The reduction procedure is relatively simple.

first of all we divide the equation (1) by y^m to get

$$\frac{1}{y^m} \frac{dy}{dt} + R \frac{y}{y^m} = \frac{T y^m}{y^m}$$

$$y^{-m} \frac{dy}{dt} + R \cdot y \cdot y^{-m} = T$$
$$y^{-m} \frac{dy}{dt} + R y^{1-m} = T \rightarrow (2)$$

$$\text{Let } z = y^{1-m}$$

$$\text{So that } \frac{dz}{dt} = \frac{dz}{dy} \frac{dy}{dt} \quad \text{(chain rule)} \quad \text{--- (1)}$$

$$\text{Now } \frac{dz}{dy} = (1-m)y^{1-m-1} \Rightarrow \frac{dz}{dy} = (1-m)y^{-m}$$

Put in (1)

$$\frac{dz}{dt} = (1-m)y^{-m} \frac{dy}{dt} \quad \div \text{ by } (1-m)$$

$$\frac{1}{1-m} \frac{dz}{dt} = \frac{dy}{dt} y^{-m}$$

Thus the equation (2) can be written as

$$\frac{1}{1-m} \frac{dz}{dt} + RZ = T \quad \therefore z = y^{-m}$$

Now multiplying by $(1-m)$ on both sides:

$$(1-m) \frac{1}{1-m} \frac{dz}{dt} + (1-m)RZ = T(1-m)$$

$$\boxed{\frac{dz}{dt} + (1-m)RZ = (1-m)T}$$

Example 1:- Solve the Bernoulli equation: $\frac{dy}{dt} + ty = 3ty^2$.

Here $n = 2$, $z = y^{1-n} = y^{1-2} = y^{-1}$, $R = t$ and $F = 3t$.

The linear version of Bernoulli's equation is as: $\frac{dz}{dt} + (1-n)Rz = (1-n)F$ or $\frac{dz}{dt} + (1-n)Rz = (1-n)F$

Putting their values: $\frac{dz}{dt} + (1-2)tz = (1-2)3t$

$$\frac{dz}{dt} + tz - 2tz = 3t - 6t \Rightarrow \frac{dz}{dt} - tz = -3t$$

Thus $\frac{dz}{dt} - tz = -3t$ is the linear differential equation of the form: $\frac{dy}{dt} + uy = w$.

Reference above solved equation: $y = z$, $u = -t$ and $w = -3t$.

Applying the general formula: $z(t) = e^{\int u dt} \left(A + \int -3t e^{\int u dt} dt \right)$

$$= e^{\int -t dt} \left(A + \int -3t e^{\int -t dt} dt \right) = e^{-1/2 t^2} \left(A + \int -3t e^{-(1/2)t^2} dt \right) = e^{1/2 t^2} \left(A - \int 3t e^{-1/2 t^2} dt \right)$$

Finding $\int -3t e^{-1/2 t^2} dt$. Let $u = -\frac{1}{2} t^2$, $\frac{du}{dt} = -t \Rightarrow dt = \frac{du}{-t}$.

By following Integration by substitution: $-\int 3t e^{-1/2 t^2} dt$

$$\int -3t e^u \frac{du}{-t} = -3 \int e^u du = -3 e^u = -3 e^{-1/2 t^2}$$

Thus $z(t) = e^{1/2 t^2} (A + 3e^{-1/2 t^2}) \Rightarrow z(t) = A e^{1/2 t^2} + 3$.

As $y^{-1} = z \Rightarrow y = z^{-1}$. Then putting the value $y = z^{-1} = (A e^{1/2 t^2} + 3)^{-1}$.

Example 2 :- We solve the following Bernoulli equation. $\frac{dy}{dt} - y = ty^2$.

Here $n = 2$, so $z = y^{1-n} = y^{1-2} = y^{-1}$, $R = -1$, $F = t$ and $(1-n) = -1$.

Substituting the value:

$$\frac{dz}{dt} + (1-n)Rz = (1-n)F \Rightarrow \frac{dz}{dt} + (1-2)(-1)(z) = (1-2)t \Rightarrow \frac{dz}{dt} + z = -t$$

Thus $\frac{dz}{dt} + z = -t$ is the linear differential equation of the form $\frac{dy}{dt} + uy = w$.

Reference above solved equation: $-y = z$, $u = 1$ and $w = -t$. Applying the formula:

$$z(t) = e^{\int u dt} \left(A + \int -t e^{\int u dt} dt \right) = e^t (A + \int -t e^t dt)$$

$$z(t) = e^t (A - t e^t + e^t) = A e^t - t + 1. \text{ Since } z(t) = y(t)^{-1}. \text{ Thus } y(t) = (A e^t - t + 1)^{-1}.$$