CHAPTER 8

THE GAMMA FUNCTION (FACTORIAL FUNCTION)

The gamma function appears occasionally in physical problems such as the normalization of Coulomb wave functions and the computation of probabilities in statistical mechanics. In general, however, it has less direct physical application and interpretation than, say, the Legendre and Bessel functions of Chapters 11 and 12. Rather, its importance stems from its usefulness in developing other functions that have direct physical application. The gamma function, therefore, is included here.

8.1 DEFINITIONS, SIMPLE PROPERTIES

At least three different, convenient definitions of the gamma function are in common use. Our first task is to state these definitions, to develop some simple, direct consequences, and to show the equivalence of the three forms.

Infinite Limit (Euler)

The first definition, named after Euler, is

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \qquad z \neq 0, -1, -2, -3, \ldots. \tag{8.1}$$

This definition of $\Gamma(z)$ is useful in developing the Weierstrass infinite-product form of $\Gamma(z)$, Eq. (8.16), and in obtaining the derivative of $\ln\Gamma(z)$ (Section 8.2). Here and elsewhere in this chapter z may be either real or complex. Replacing z with $z+1$, we have

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)(z+3)\cdots(z+n+1)} n^{z+1} \\ &= \lim_{n\to\infty} \frac{nz}{z+n+1} \cdot \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z} \\ &= z\Gamma(z). \end{aligned} \tag{8.2}$$

This is the basic functional relation for the gamma function. It should be noted that it is a **difference** equation. It has been shown that the gamma function is one of a general class of functions that do not satisfy any differential equation with rational coefficients. Specifically, the gamma function is one of the very few functions of mathematical physics that does not satisfy either the hypergeometric differential equation (Section 13.4) or the confluent hypergeometric equation (Section 13.5).

Also, from the definition,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)} n = 1. \tag{8.3}$$

Now, application of Eq. (8.2) gives

$$\begin{aligned}&\Gamma(2) = 1, \\ &\Gamma(3) = 2\Gamma(2) = 2, \dots \\ &\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!. \end{aligned} \tag{8.4}$$

Definite Integral (Euler)

A second definition, also frequently called the Euler integral, is

$$\boxed{\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \Re(z) > 0.} \tag{8.5}$$

The restriction on z is necessary to avoid divergence of the integral. When the gamma function does appear in physical problems, it is often in this form or some variation, such as

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt, \qquad \Re(z) > 0. \tag{8.6}$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \qquad \Re(z) > 0. \tag{8.7}$$

When $z = \frac{1}{2}$, Eq. (8.6) is just the Gauss error integral, and we have the interesting result

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \tag{8.8}$$

Generalizations of Eq. (8.6), the Gaussian integrals, are considered in Exercise 8.1.11. This definite integral form of $\Gamma(z)$, Eq. (8.5), leads to the beta function, Section 8.4.

To show the equivalence of these two definitions, Eqs. (8.1) and (8.5), consider the function of two variables

$$F(z,n)=\int_0^n\left(1-\frac{t}{n}\right)^n t^{z-1}\,dt,\qquad \Re(z)>0, \tag{8.9}$$

with n a positive integer.[1] Since

$$\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n\equiv e^{-t}, \tag{8.10}$$

from the definition of the exponential

$$\lim_{n\to\infty}F(z,n)=F(z,\infty)=\int_0^\infty e^{-t}t^{z-1}\,dt\equiv\Gamma(z) \tag{8.11}$$

by Eq. (8.5).

Returning to $F(z,n)$, we evaluate it in successive integrations by parts. For convenience let $u=t/n$. Then

$$F(z,n)=n^z\int_0^1(1-u)^n u^{z-1}\,du. \tag{8.12}$$

Integrating by parts, we obtain

$$\frac{F(z,n)}{n^z}=(1-u)^n\frac{u^z}{z}\bigg|_0^1+\frac{n}{z}\int_0^1(1-u)^{n-1}u^z\,du. \tag{8.13}$$

Repeating this with the integrated part vanishing at both endpoints each time, we finally get

$$\begin{aligned}F(z,n)&=n^z\frac{n(n-1)\cdots1}{z(z+1)\cdots(z+n-1)}\int_0^1 u^{z+n-1}\,du\\&=\frac{1\cdot2\cdot3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^z.\end{aligned} \tag{8.14}$$

This is identical with the expression on the right side of Eq. (8.1). Hence

$$\lim_{n\to\infty}F(z,n)=F(z,\infty)\equiv\Gamma(z), \tag{8.15}$$

by Eq. (8.1), completing the proof.

Infinite Product (Weierstrass)

The third definition (Weierstrass' form) is

$$\boxed{\frac{1}{\Gamma(z)}\equiv ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},} \tag{8.16}$$

[1] The form of $F(z,n)$ is suggested by the beta function (compare Eq. (8.60)).

where γ is the Euler–Mascheroni constant,

$$\gamma = 0.5772156619\ldots. \tag{8.17}$$

This infinite-product form may be used to develop the reflection identity, Eq. (8.23), and applied in the exercises, such as Exercise 8.1.17. This form can be derived from the original definition (Eq. (8.1)) by rewriting it as

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n)} n^z = \lim_{n\to\infty} \frac{1}{z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)^{-1} n^z. \tag{8.18}$$

Inverting Eq. (8.18) and using

$$n^{-z} = e^{(-\ln n)z}, \tag{8.19}$$

we obtain

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} e^{(-\ln n)z} \prod_{m=1}^{n}\left(1+\frac{z}{m}\right). \tag{8.20}$$

Multiplying and dividing by

$$\exp\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)z\right] = \prod_{m=1}^{n} e^{z/m}, \tag{8.21}$$

we get

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z\left\{\lim_{n\to\infty} \exp\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)z\right]\right\}\\ &\quad\times\left[\lim_{n\to\infty}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)e^{-z/m}\right].\end{aligned} \tag{8.22}$$

As shown in Section 5.2, the parenthesis in the exponent approaches a limit, namely γ, the Euler–Mascheroni constant. Hence Eq. (8.16) follows.

It was shown in Section 5.11 that the Weierstrass infinite-product definition of $\Gamma(z)$ led directly to an important identity,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}. \tag{8.23}$$

Alternatively, we can start from the product of Euler integrals,

$$\begin{aligned}\Gamma(z+1)\Gamma(1-z) &= \int_0^\infty s^z e^{-s} ds \int_0^\infty t^{-z} e^{-t}\, dt\\ &= \int_0^\infty v^z \frac{dv}{(v+1)^2}\int_0^\infty e^{-u} u\, du = \frac{\pi z}{\sin \pi z},\end{aligned}$$

transforming from the variables s, t to $u = s+t$, $v = s/t$, as suggested by combining the exponentials and the powers in the integrands. The Jacobian is

$$J = -\begin{vmatrix} 1 & 1 \\ \frac{1}{t} & -\frac{s}{t^2}\end{vmatrix} = \frac{s+t}{t^2} = \frac{(v+1)^2}{u},$$

where $(v+1)t = u$. The integral $\int_0^\infty e^{-u} u\,du = 1$, while that over v may be derived by contour integration, giving $\frac{\pi z}{\sin \pi z}$.

This identity may also be derived by contour integration (Example 7.1.6 and Exercises 7.1.18 and 7.1.19) and the beta function, Section 8.4. Setting $z = \frac{1}{2}$ in Eq. (8.23), we obtain

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{8.24a}$$

(taking the positive square root), in agreement with Eq. (8.8).

Similarly one can establish **Legendre's duplication formula**,

$$\Gamma(1+z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\,\Gamma(2z+1). \tag{8.24b}$$

The Weierstrass definition shows immediately that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \ldots$ and that $[\Gamma(z)]^{-1}$ has no poles in the finite complex plane, which means that $\Gamma(z)$ has no zeros. This behavior may also be seen in Eq. (8.23), in which we note that $\pi/(\sin \pi z)$ is never equal to zero.

Actually the infinite-product definition of $\Gamma(z)$ may be derived from the Weierstrass factorization theorem with the specification that $[\Gamma(z)]^{-1}$ have simple zeros at $z = 0, -1, -2, -3, \ldots$. The Euler–Mascheroni constant is fixed by requiring $\Gamma(1) = 1$. See also the products expansions of entire functions in Section 7.1.

In probability theory the gamma distribution (probability density) is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \le 0. \end{cases} \tag{8.24c}$$

The constant $[\beta^\alpha \Gamma(\alpha)]^{-1}$ is chosen so that the total (integrated) probability will be unity. For $x \to E$, kinetic energy, $\alpha \to \frac{3}{2}$, and $\beta \to kT$, Eq. (8.24c) yields the classical Maxwell–Boltzmann statistics.

Factorial Notation

So far this discussion has been presented in terms of the classical notation. As pointed out by Jeffreys and others, the -1 of the $z-1$ exponent in our second definition (Eq. (8.5)) is a continual nuisance. Accordingly, Eq. (8.5) is sometimes rewritten as

$$\int_0^\infty e^{-t} t^z\,dt \equiv z!, \qquad \Re(z) > -1, \tag{8.25}$$

to **define** a factorial function $z!$. Occasionally we may still encounter Gauss' notation, $\prod(z)$, for the factorial function:

$$\prod(z) = z! = \Gamma(z+1). \tag{8.26}$$

The Γ notation is due to Legendre. The factorial function of Eq. (8.25) is related to the gamma function by

$$\Gamma(z) = (z-1)! \qquad \text{or} \qquad \Gamma(z+1) = z!. \tag{8.27}$$

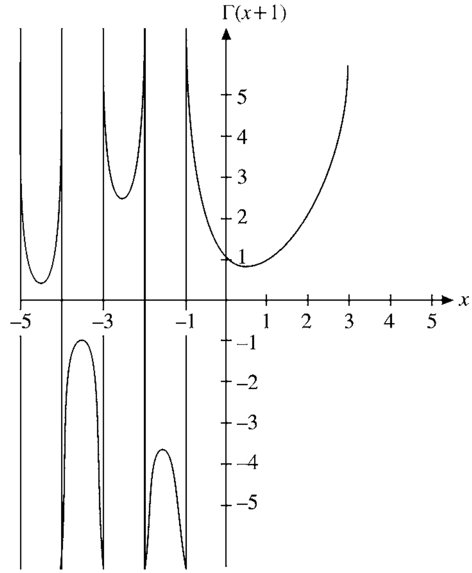

FIGURE 8.1 The factorial function — extension to negative arguments.

If $z = n$, a positive integer (Eq. (8.4)) shows that

$$z! = n! = 1 \cdot 2 \cdot 3 \cdots n, \tag{8.28}$$

the familiar factorial. However, it should be noted that since $z!$ is now defined by Eq. (8.25) (or equivalently by Eq. (8.27)) the factorial function is no longer limited to positive integral values of the argument (Fig. 8.1). The difference relation (Eq. (8.2)) becomes

$$(z-1)! = \frac{z!}{z}. \tag{8.29}$$

This shows immediately that

$$0! = 1 \tag{8.30}$$

and

$$n! = \pm\infty \qquad \text{for } n, \text{ a } \textbf{negative} \text{ integer.} \tag{8.31}$$

In terms of the factorial, Eq. (8.23) becomes

$$z!(-z)! = \frac{\pi z}{\sin \pi z}. \tag{8.32}$$

By restricting ourselves to the real values of the argument, we find that $\Gamma(x+1)$ defines the curves shown in Figs. 8.1 and 8.2. The minimum of the curve is

$$\Gamma(x+1) = x! = (0.46163\ldots)! = 0.88560\ldots. \tag{8.33a}$$

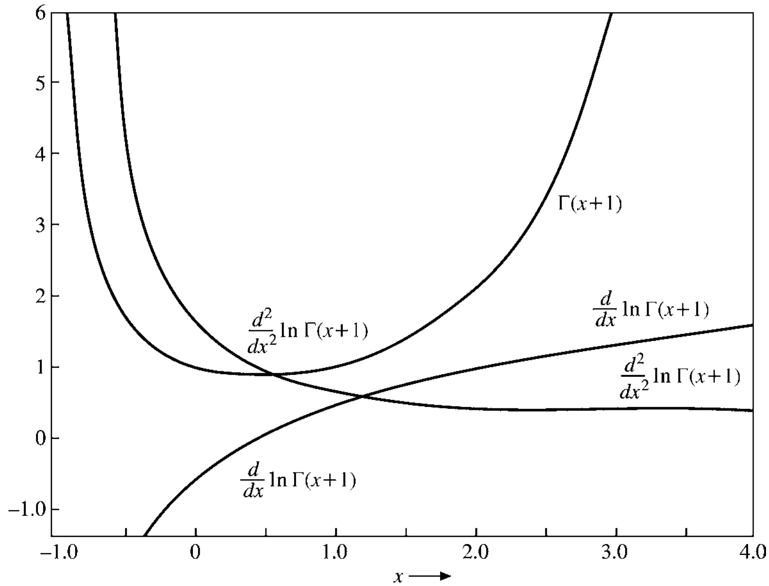

FIGURE 8.2 The factorial function and the first two derivatives of $\ln(\Gamma(x+1))$.

Double Factorial Notation

In many problems of mathematical physics, particularly in connection with Legendre polynomials (Chapter 12), we encounter products of the odd positive integers and products of the even positive integers. For convenience these are given special labels as double factorials:

$$\begin{aligned} 1\cdot 3\cdot 5\cdots(2n+1) &= (2n+1)!! \\ 2\cdot 4\cdot 6\cdots(2n) &= (2n)!!. \end{aligned} \tag{8.33b}$$

Clearly, these are related to the regular factorial functions by

$$(2n)!! = 2^n n! \qquad \text{and} \qquad (2n+1)!! = \frac{(2n+1)!}{2^n n!}. \tag{8.33c}$$

We also define $(-1)!! = 1$, a special case that does not follow from Eq. (8.33c).

Integral Representation

An integral representation that is useful in developing asymptotic series for the Bessel functions is

$$\int_C e^{-z} z^{\nu}\, dz = \left(e^{2\pi i\nu} - 1\right)\Gamma(\nu+1), \tag{8.34}$$

where C is the contour shown in Fig. 8.3. This contour integral representation is only useful when ν is not an integer, $z = 0$ then being a **branch point**. Equation (8.34) may be

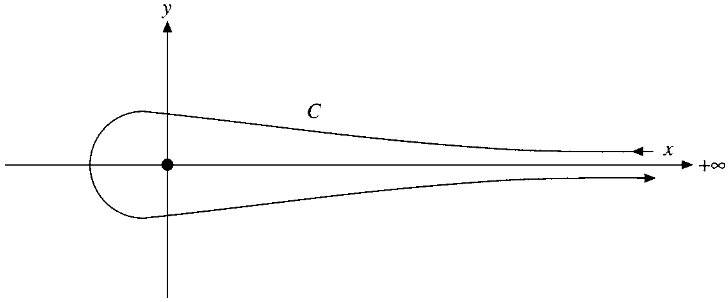

FIGURE 8.3 Factorial function contour.

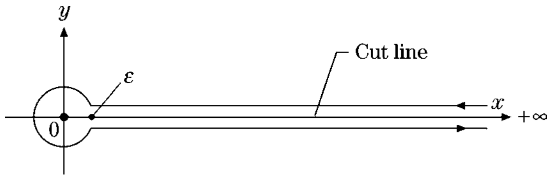

FIGURE 8.4 The contour of Fig. 8.3 deformed.

readily verified for $\nu > -1$ by deforming the contour as shown in Fig. 8.4. The integral from ∞ into the origin yields $-(\nu!)$, placing the phase of z at 0. The integral out to ∞ (in the fourth quadrant) then yields $e^{2\pi i\nu}\nu!$, the phase of z having increased to 2π. Since the circle around the origin contributes nothing when $\nu > -1$, Eq. (8.34) follows.

It is often convenient to cast this result into a more symmetrical form:

$$\int_C e^{-z}(-z)^\nu\,dz = 2i\Gamma(\nu+1)\sin(\nu\pi). \tag{8.35}$$

This analysis establishes Eqs. (8.34) and (8.35) for $\nu > -1$. It is relatively simple to extend the range to include all nonintegral ν. First, we note that the integral exists for $\nu < -1$ as long as we stay away from the origin. Second, integrating by parts we find that Eq. (8.35) yields the familiar difference relation (Eq. (8.29)). If we take the difference relation to define the factorial function of $\nu < -1$, then Eqs. (8.34) and (8.35) are verified for all ν (except negative integers).

Exercises

8.1.1 Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral (Eq. (8.5)),

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt.$$

8.1.2 In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)},$$

in which s is a positive integer. Rewrite this expression in terms of factorials.

8.1.3 Show that, as $s-n\to$ negative integer,

$$\frac{(s-n)!}{(2s-2n)!}\to\frac{(-1)^{n-s}(2n-2s)!}{(n-s)!}.$$

Here s and n are integers with $s<n$. This result can be used to avoid negative factorials, such as in the series representations of the spherical Neumann functions and the Legendre functions of the second kind.

8.1.4 Show that $\Gamma(z)$ may be written

$$\Gamma(z)=2\int_0^\infty e^{-t^2}t^{2z-1}\,dt,\qquad \Re(z)>0,$$

$$\Gamma(z)=\int_0^1\left[\ln\left(\frac{1}{t}\right)\right]^{z-1}dt,\qquad \Re(z)>0.$$

8.1.5 In a Maxwellian distribution the fraction of particles with speed between v and $v+dv$ is

$$\frac{dN}{N}=4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{mv^2}{2kT}\right)v^2\,dv,$$

N being the total number of particles. The average or expectation value of v^n is defined as $\langle v^n\rangle=N^{-1}\int v^n\,dN$. Show that

$$\langle v^n\rangle=\left(\frac{2kT}{m}\right)^{n/2}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma(3/2)}.$$

8.1.6 By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k\ln x\,dx=\frac{1}{(k+1)^2},\qquad k>-1.$$

8.1.7 Show that

$$\int_0^\infty e^{-x^4}\,dx=\Gamma\left(\frac{5}{4}\right).$$

8.1.8 Show that

$$\lim_{x\to 0}\frac{(ax-1)!}{(x-1)!}=\frac{1}{a}.$$

8.1.9 Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

8.1.10 Show that the equation $x!=k$, $k\neq 0$, has an infinite number of real roots.

8.1.11 Show that

(a) $$\int_0^\infty x^{2s+1}\exp(-ax^2)\,dx = \frac{s!}{2a^{s+1}}.$$

(b) $$\int_0^\infty x^{2s}\exp(-ax^2)\,dx = \frac{(s-\frac{1}{2})!}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}.$$

These Gaussian integrals are of major importance in statistical mechanics.

8.1.12 (a) Develop recurrence relations for $(2n)!!$ and for $(2n+1)!!$.
(b) Use these recurrence relations to calculate (or to define) $0!!$ and $(-1)!!$.

ANS. $0!! = 1, \quad (-1)!! = 1.$

8.1.13 For s a nonnegative integer, show that

$$(-2s-1)!! = \frac{(-1)^s}{(2s-1)!!} = \frac{(-1)^s 2^s s!}{(2s)!}.$$

8.1.14 Express the coefficient of the nth term of the expansion of $(1+x)^{1/2}$

(a) in terms of factorials of integers,
(b) in terms of the double factorial (!!) functions.

ANS. $$a_n = (-1)^{n+1}\frac{(2n-3)!}{2^{2n-2}n!(n-2)!} = (-1)^{n+1}\frac{(2n-3)!!}{(2n)!!}, \qquad n = 2, 3, \ldots.$$

8.1.15 Express the coefficient of the nth term of the expansion of $(1+x)^{-1/2}$

(a) in terms of the factorials of integers,
(b) in terms of the double factorial (!!) functions.

ANS. $$a_n = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n\frac{(2n-1)!!}{(2n)!!}, \qquad n = 1, 2, 3, \ldots.$$

8.1.16 The Legendre polynomial may be written as

$$\begin{aligned}P_n(\cos\theta) = 2\frac{(2n-1)!!}{(2n)!!}\bigg\{&\cos n\theta + \frac{1}{1}\cdot\frac{n}{2n-1}\cos(n-2)\theta \\ &+ \frac{1\cdot 3}{1\cdot 2}\frac{n(n-1)}{(2n-1)(2n-3)}\cos(n-4)\theta \\ &+ \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos(n-6)\theta + \cdots\bigg\}.\end{aligned}$$

Let $n = 2s+1$. Then

$$P_n(\cos\theta) = P_{2s+1}(\cos\theta) = \sum_{m=0}^{s} a_m \cos(2m+1)\theta.$$

Find a_m in terms of factorials and double factorials.

8.1.17 (a) Show that

$$\Gamma\left(\tfrac{1}{2}-n\right)\Gamma\left(\tfrac{1}{2}+n\right)=(-1)^n\pi,$$

where n is an integer.

(b) Express $\Gamma(\frac{1}{2}+n)$ and $\Gamma(\frac{1}{2}-n)$ separately in terms of $\pi^{1/2}$ and a !! function.

$$\text{ANS. } \Gamma(\tfrac{1}{2}+n)=\frac{(2n-1)!!}{2^n}\pi^{1/2}.$$

8.1.18 From one of the definitions of the factorial or gamma function, show that

$$\left|(ix)!\right|^2=\frac{\pi x}{\sinh \pi x}.$$

8.1.19 Prove that

$$\left|\Gamma(\alpha+i\beta)\right|=\left|\Gamma(\alpha)\right|\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}.$$

This equation has been useful in calculations of beta decay theory.

8.1.20 Show that

$$\left|(n+ib)!\right|=\left(\frac{\pi b}{\sinh \pi b}\right)^{1/2}\prod_{s=1}^{n}\left(s^2+b^2\right)^{1/2}$$

for n, a positive integer.

8.1.21 Show that

$$|x!|\geq\left|(x+iy)!\right|$$

for all x. The variables x and y are real.

8.1.22 Show that

$$\left|\Gamma\left(\tfrac{1}{2}+iy\right)\right|^2=\frac{\pi}{\cosh \pi y}.$$

8.1.23 The probability density associated with the normal distribution of statistics is given by

$$f(x)=\frac{1}{\sigma(2\pi)^{1/2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],$$

with $(-\infty,\infty)$ for the range of x. Show that

(a) the mean value of x, $\langle x\rangle$ is equal to μ,
(b) the standard deviation $(\langle x^2\rangle-\langle x\rangle^2)^{1/2}$ is given by σ.

8.1.24 From the gamma distribution

$$f(x)=\begin{cases}\dfrac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}, & x>0,\\ 0, & x\leq 0,\end{cases}$$

show that

(a) $\langle x\rangle$ (mean) $=\alpha\beta$, (b) σ^2 (variance) $\equiv\langle x^2\rangle-\langle x\rangle^2=\alpha\beta^2$.

8.1.25 The wave function of a particle scattered by a Coulomb potential is $\psi(r,\theta)$. At the origin the wave function becomes

$$\psi(0) = e^{-\pi\gamma/2}\Gamma(1+i\gamma),$$

where $\gamma = Z_1 Z_2 e^2/\hbar v$. Show that

$$|\psi(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma}-1}.$$

8.1.26 Derive the contour integral representation of Eq. (8.34),

$$2i\nu!\sin\nu\pi = \int_C e^{-z}(-z)^\nu\,dz.$$

8.1.27 Write a function subprogram *FACT*(N) (fixed-point independent variable) that will calculate $N!$. Include provision for rejection and appropriate error message if N is negative.
Note. For small integer N, direct multiplication is simplest. For large N, Eq. (8.55), Stirling's series would be appropriate.

8.1.28 (a) Write a function subprogram to calculate the double factorial ratio $(2N-1)!!/(2N)!!$. Include provision for $N = 0$ and for rejection and an error message if N is negative. Calculate and tabulate this ratio for $N = 1(1)100$.
(b) Check your function subprogram calculation of 199!!/200!! against the value obtained from Stirling's series (Section 8.3).

$$\text{ANS. } \frac{199!!}{200!!} = 0.056348.$$

8.1.29 Using either the FORTRAN-supplied GAMMA or a library-supplied subroutine for $x!$ or $\Gamma(x)$, determine the value of x for which $\Gamma(x)$ is a minimum $(1 \le x \le 2)$ and this minimum value of $\Gamma(x)$. Notice that although the minimum value of $\Gamma(x)$ may be obtained to about six significant figures (single precision), the corresponding value of x is much less accurate. Why this relatively low accuracy?

8.1.30 The factorial function expressed in integral form can be evaluated by the Gauss–Laguerre quadrature. For a 10-point formula the resultant $x!$ is theoretically exact for x an integer, 0 up through 19. What happens if x is not an integer? Use the Gauss–Laguerre quadrature to evaluate $x!$, $x = 0.0(0.1)2.0$. Tabulate the absolute error as a function of x.

Check value. $x!_{\text{exact}} - x!_{\text{quadrature}} = 0.00034 \quad \text{for} \quad x = 1.3.$

8.2 DIGAMMA AND POLYGAMMA FUNCTIONS

Digamma Functions

As may be noted from the three definitions in Section 8.1, it is inconvenient to deal with the derivatives of the gamma or factorial function directly. Instead, it is customary to take the natural logarithm of the factorial function (Eq. (8.1)), convert the product to a sum, and then differentiate; that is,

$$\Gamma(z+1) = z\Gamma(z) = \lim_{n\to\infty} \frac{n!}{(z+1)(z+2)\cdots(z+n)} n^z \tag{8.36}$$

and

$$\begin{aligned}\ln\Gamma(z+1) = \lim_{n\to\infty} \big[&\ln(n!) + z\ln n - \ln(z+1) \\ &- \ln(z+2) - \cdots - \ln(z+n)\big],\end{aligned} \tag{8.37}$$

in which the logarithm of the limit is equal to the limit of the logarithm. Differentiating with respect to z, we obtain

$$\frac{d}{dz}\ln\Gamma(z+1) \equiv \psi(z+1) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n}\right), \tag{8.38}$$

which defines $\psi(z+1)$, the digamma function. From the definition of the Euler–Mascheroni constant,[2] Eq. (8.38) may be rewritten as

$$\begin{aligned}\psi(z+1) &= -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) \\ &= -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}.\end{aligned} \tag{8.39}$$

One application of Eq. (8.39) is in the derivation of the series form of the Neumann function (Section 11.3). Clearly,

$$\psi(1) = -\gamma = -0.577\,215\,664\,901\ldots.^3 \tag{8.40}$$

Another, perhaps more useful, expression for $\psi(z)$ is derived in Section 8.3.

Polygamma Function

The digamma function may be differentiated repeatedly, giving rise to the polygamma function:

$$\begin{aligned}\psi^{(m)}(z+1) &\equiv \frac{d^{m+1}}{dz^{m+1}}\ln(z!) \\ &= (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}, \qquad m = 1, 2, 3, \ldots.\end{aligned} \tag{8.41}$$

[2]Compare Sections 5.2 and 5.9. We add and substract $\sum_{s=1}^{n} s^{-1}$.

[3]γ has been computed to 1271 places by D. E. Knuth, *Math. Comput.* **16**: 275 (1962), and to 3566 decimal places by D. W. Sweeney, *ibid.* **17**: 170 (1963). It may be of interest that the fraction 228/395 gives γ accurate to six places.

A plot of $\psi(x+1)$ and $\psi'(x+1)$ is included in Fig. 8.2. Since the series in Eq. (8.41) defines the Riemann zeta function[4] (with $z=0$),

$$\zeta(m) \equiv \sum_{n=1}^{\infty} \frac{1}{n^m}, \tag{8.42}$$

we have

$$\psi^{(m)}(1) = (-1)^{m+1} m!\zeta(m+1), \qquad m = 1, 2, 3, \ldots. \tag{8.43}$$

The values of the polygamma functions of positive integral argument, $\psi^{(m)}(n+1)$, may be calculated by using Exercise 8.2.6.

In terms of the perhaps more common Γ notation,

$$\frac{d^{n+1}}{dz^{n+1}} \ln\Gamma(z) = \frac{d^n}{dz^n}\psi(z) = \psi^{(n)}(z). \tag{8.44a}$$

Maclaurin Expansion, Computation

It is now possible to write a Maclaurin expansion for $\ln\Gamma(z+1)$:

$$\ln\Gamma(z+1) = \sum_{n=1}^{\infty} \frac{z^n}{n!}\psi^{(n-1)}(1) = -\gamma z + \sum_{n=2}^{\infty}(-1)^n \frac{z^n}{n}\zeta(n) \tag{8.44b}$$

convergent for $|z| < 1$; for $z = x$, the range is $-1 < x \le 1$. Alternate forms of this series appear in Exercise 5.9.14. Equation (8.44b) is a possible means of computing $\Gamma(z+1)$ for real or complex z, but Stirling's series (Section 8.3) is usually better, and in addition, an excellent table of values of the gamma function for complex arguments based on the use of Stirling's series and the recurrence relation (Eq. (8.29)) is now available.[5]

Series Summation

The digamma and polygamma functions may also be used in summing series. If the general term of the series has the form of a rational fraction (with the highest power of the index in the numerator at least two less than the highest power of the index in the denominator), it may be transformed by the method of partial fractions (compare Section 15.8). The infinite series may then be expressed as a finite sum of digamma and polygamma functions. The usefulness of this method depends on the availability of tables of digamma and polygamma functions. Such tables and examples of series summation are given in AMS-55, Chapter 6 (see Additional Readings for the reference).

[4]See Section 5.9. For $z \neq 0$ this series may be used to define a generalized zeta function.

[5]*Table of the Gamma Function for Complex Arguments*, Applied Mathematics Series No. 34. Washington, DC: National Bureau of Standards (1954).

Example 8.2.1 CATALAN'S CONSTANT

Catalan's constant, Exercise 5.2.22, or $\beta(2)$ of Section 5.9 is given by

$$K = \beta(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}. \tag{8.44c}$$

Grouping the positive and negative terms separately and starting with unit index (to match the form of $\psi^{(1)}$, Eq. (8.41)), we obtain

$$K = 1 + \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} - \frac{1}{9} - \sum_{n=1}^{\infty} \frac{1}{(4n+3)^2}.$$

Now, quoting Eq. (8.41), we get

$$K = \tfrac{8}{9} + \tfrac{1}{16}\psi^{(1)}\big(1 + \tfrac{1}{4}\big) - \tfrac{1}{16}\psi^{(1)}\big(1 + \tfrac{3}{4}\big). \tag{8.44d}$$

Using the values of $\psi^{(1)}$ from Table 6.1 of AMS-55 (see Additional Readings for the reference), we obtain

$$K = 0.91596559\ldots.$$

Compare this calculation of Catalan's constant with the calculations of Chapter 5, either direct summation or a modification using Riemann zeta function values. ■

Exercises

8.2.1 Verify that the following two forms of the digamma function,

$$\psi(x+1) = \sum_{r=1}^{x} \frac{1}{r} - \gamma$$

and

$$\psi(x+1) = \sum_{r=1}^{\infty} \frac{x}{r(r+x)} - \gamma,$$

are equal to each other (for x a positive integer).

8.2.2 Show that $\psi(z+1)$ has the series expansion

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

8.2.3 For a power-series expansion of $\ln(z!)$, AMS-55 (see Additional Readings for reference) lists

$$\ln(z!) = -\ln(1+z) + z(1-\gamma) + \sum_{n=2}^{\infty} (-1)^n \frac{[\zeta(n) - 1] z^n}{n}.$$

(a) Show that this agrees with Eq. (8.44b) for $|z| < 1$.
(b) What is the range of convergence of this new expression?

8.2.4 Show that

$$\frac{1}{2}\ln\left(\frac{\pi z}{\sin \pi z}\right) = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \qquad |z| < 1.$$

Hint. Try Eq. (8.32).

8.2.5 Write out a Weierstrass infinite-product definition of $\ln(z!)$. Without differentiating, show that this leads directly to the Maclaurin expansion of $\ln(z!)$, Eq. (8.44b).

8.2.6 Derive the difference relation for the polygamma function

$$\psi^{(m)}(z+2) = \psi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}}, \qquad m = 0, 1, 2, \ldots.$$

8.2.7 Show that if

$$\Gamma(x+iy) = u + iv,$$

then

$$\Gamma(x-iy) = u - iv.$$

This is a special case of the Schwarz reflection principle, Section 6.5.

8.2.8 The Pochhammer symbol $(a)_n$ is defined as

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1$$

(for integral n).

(a) Express $(a)_n$ in terms of factorials.
(b) Find $(d/da)(a)_n$ in terms of $(a)_n$ and digamma functions.

ANS. $\dfrac{d}{da}(a)_n = (a)_n\big[\psi(a+n) - \psi(a)\big]$.

(c) Show that

$$(a)_{n+k} = (a+n)_k \cdot (a)_n.$$

8.2.9 Verify the following special values of the ψ form of the di- and polygamma functions:

$$\psi(1) = -\gamma, \qquad \psi^{(1)}(1) = \zeta(2), \qquad \psi^{(2)}(1) = -2\zeta(3).$$

8.2.10 Derive the polygamma function recurrence relation

$$\psi^{(m)}(1+z) = \psi^{(m)}(z) + (-1)^m m!/z^{m+1}, \qquad m = 0, 1, 2, \ldots.$$

8.2.11 Verify

(a) $\displaystyle\int_0^\infty e^{-r} \ln r \, dr = -\gamma.$

(b) $\displaystyle\int_0^\infty re^{-r}\ln r\,dr = 1-\gamma.$

(c) $\displaystyle\int_0^\infty r^n e^{-r}\ln r\,dr = (n-1)! + n\int_0^\infty r^{n-1}e^{-r}\ln r\,dr, \qquad n = 1, 2, 3, \ldots.$

Hint. These may be verified by integration by parts, three parts, or differentiating the integral form of $n!$ with respect to n.

8.2.12 Dirac relativistic wave functions for hydrogen involve factors such as $[2(1-\alpha^2Z^2)^{1/2}]!$ where α, the fine structure constant, is $\frac{1}{137}$ and Z is the atomic number. Expand $[2(1-\alpha^2Z^2)^{1/2}]!$ in a series of powers of α^2Z^2.

8.2.13 The quantum mechanical description of a particle in a Coulomb field requires a knowledge of the phase of the complex factorial function. Determine the phase of $(1+ib)!$ for small b.

8.2.14 The total energy radiated by a blackbody is given by

$$u = \frac{8\pi k^4T^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x-1}dx.$$

Show that the integral in this expression is equal to $3!\zeta(4)$.
$[\zeta(4) = \pi^4/90 = 1.0823\ldots]$ The final result is the Stefan–Boltzmann law.

8.2.15 As a generalization of the result in Exercise 8.2.14, show that

$$\int_0^\infty \frac{x^s\,dx}{e^x-1} = s!\zeta(s+1), \qquad \Re(s) > 0.$$

8.2.16 The neutrino energy density (Fermi distribution) in the early history of the universe is given by

$$\rho_\nu = \frac{4\pi}{h^3}\int_0^\infty \frac{x^3}{\exp(x/kT)+1}dx.$$

Show that

$$\rho_\nu = \frac{7\pi^5}{30h^3}(kT)^4.$$

8.2.17 Prove that

$$\int_0^\infty \frac{x^s\,dx}{e^x+1} = s!\big(1-2^{-s}\big)\zeta(s+1), \qquad \Re(s) > 0.$$

Exercises 8.2.15 and 8.2.17 actually constitute Mellin integral transforms (compare Section 15.1).

8.2.18 Prove that

$$\psi^{(n)}(z) = (-1)^{n+1}\int_0^\infty \frac{t^ne^{-zt}}{1-e^{-t}}dt, \qquad \Re(z) > 0.$$

8.2.19 Using di- and polygamma functions, sum the series

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Note. You can use Exercise 8.2.6 to calculate the needed digamma functions.

8.2.20 Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{(b-a)}\bigl\{\psi(1+b) - \psi(1+a)\bigr\},$$

where $a \neq b$ and neither a nor b is a negative integer. It is of some interest to compare this summation with the corresponding integral,

$$\int_1^{\infty} \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\bigl\{\ln(1+b) - \ln(1+a)\bigr\}.$$

The relation between $\psi(x)$ and $\ln x$ is made explicit in Eq. (8.51) in the next section.

8.2.21 Verify the contour integral representation of $\zeta(s)$,

$$\zeta(s) = -\frac{(-s)!}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz.$$

The contour C is the same as that for Eq. (8.35). The points $z = \pm 2n\pi i,\ n = 1, 2, 3, \ldots,$ are all excluded.

8.2.22 Show that $\zeta(s)$ is analytic in the entire finite complex plane except at $s = 1$, where it has a simple pole with a residue of $+1$.
Hint. The contour integral representation will be useful.

8.2.23 Using the complex variable capability of FORTRAN calculate $\Re(1+ib)!$, $\Im(1+ib)!$, $|(1+ib)!|$ and phase $(1+ib)!$ for $b = 0.0(0.1)1.0$. Plot the phase of $(1+ib)!$ versus b.
Hint. Exercise 8.2.3 offers a convenient approach. You will need to calculate $\zeta(n)$.

8.3 Stirling's Series

For computation of $\ln(z!)$ for very large z (statistical mechanics) and for numerical computations at nonintegral values of z, a series expansion of $\ln(z!)$ in negative powers of z is desirable. Perhaps the most elegant way of deriving such an expansion is by the method of steepest descents (Section 7.3). The following method, starting with a numerical integration formula, does not require knowledge of contour integration and is particularly direct.

Derivation from Euler–Maclaurin Integration Formula

The Euler–Maclaurin formula for evaluating a definite integral[6] is

$$\int_0^n f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + \tfrac{1}{2}f(n)$$
$$- b_2\big[f'(n) - f'(0)\big] - b_4\big[f'''(n) - f'''(0)\big] - \cdots, \tag{8.45}$$

in which the b_{2n} are related to the Bernoulli numbers B_{2n} (compare Section 5.9) by

$$(2n)!b_{2n} = B_{2n}, \tag{8.46}$$

$$\begin{aligned} B_0 &= 1, & B_6 &= \tfrac{1}{42}, \\ B_2 &= \tfrac{1}{6}, & B_8 &= -\tfrac{1}{30}, \\ B_4 &= -\tfrac{1}{30}, & B_{10} &= \tfrac{5}{66}, \quad \text{and so on.} \end{aligned} \tag{8.47}$$

By applying Eq. (8.45) to the definite integral

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z} \tag{8.48}$$

(for z not on the negative real axis), we obtain

$$\frac{1}{z} = \frac{1}{2z^2} + \psi^{(1)}(z+1) - \frac{2!b_2}{z^3} - \frac{4!b_4}{z^5} - \cdots. \tag{8.49}$$

This is the reason for using Eq. (8.48). The Euler–Maclaurin evaluation yields $\psi^{(1)}(z+1)$, which is $d^2 \ln\Gamma(z+1)/dz^2$.

Using Eq. (8.46) and solving for $\psi^{(1)}(z+1)$, we have

$$\begin{aligned} \psi^{(1)}(z+1) = \frac{d}{dz}\psi(z+1) &= \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \cdots \\ &= \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^{\infty}\frac{B_{2n}}{z^{2n+1}}. \end{aligned} \tag{8.50}$$

Since the Bernoulli numbers diverge strongly, this series does not converge. It is a semi-convergent, or asymptotic, series, useful if one retains a small enough number of terms (compare Section 5.10).

Integrating once, we get the digamma function

$$\begin{aligned} \psi(z+1) &= C_1 + \ln z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \cdots \\ &= C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}. \end{aligned} \tag{8.51}$$

Integrating Eq. (8.51) with respect to z from $z-1$ to z and then letting z approach infinity, C_1, the constant of integration, may be shown to vanish. This gives us a second expression for the digamma function, often more useful than Eq. (8.38) or (8.44b).

[6]This is obtained by repeated integration by parts, Section 5.9.

Stirling's Series

The indefinite integral of the digamma function (Eq. (8.51)) is

$$\ln\Gamma(z+1) = C_2 + \left(z+\frac{1}{2}\right)\ln z - z + \frac{B_2}{2z} + \cdots + \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \cdots, \tag{8.52}$$

in which C_2 is another constant of integration. To fix C_2 we find it convenient to use the doubling, or Legendre duplication, formula derived in Section 8.4,

$$\Gamma(z+1)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{-2z}\pi^{1/2}\Gamma(2z+1). \tag{8.53}$$

This may be proved directly when z is a positive integer by writing $\Gamma(2z+1)$ as a product of even terms times a product of odd terms and extracting a factor of 2 from each term (Exercise 8.3.5). Substituting Eq. (8.52) into the logarithm of the doubling formula, we find that C_2 is

$$C_2 = \tfrac{1}{2}\ln 2\pi, \tag{8.54}$$

giving

$$\boxed{\ln\Gamma(z+1) = \frac{1}{2}\ln 2\pi + \left(z+\frac{1}{2}\right)\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots.} \tag{8.55}$$

This is Stirling's series, an asymptotic expansion. The absolute value of the error is less than the absolute value of the first term omitted.

The constants of integration C_1 and C_2 may also be evaluated by comparison with the first term of the series expansion obtained by the method of "steepest descent." This is carried out in Section 7.3.

To help convey a feeling of the remarkable precision of Stirling's series for $\Gamma(s+1)$, the ratio of the first term of Stirling's approximation to $\Gamma(s+1)$ is plotted in Fig. 8.5. A tabulation gives the ratio of the first term in the expansion to $\Gamma(s+1)$ and the ratio of the first two terms in the expansion to $\Gamma(s+1)$ (Table 8.1). The derivation of these forms is Exercise 8.3.1.

Exercises

8.3.1 Rewrite Stirling's series to give $\Gamma(z+1)$ instead of $\ln\Gamma(z+1)$.

$$\text{ANS. } \Gamma(z+1) = \sqrt{2\pi}\,z^{z+1/2}e^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51{,}840z^3} + \cdots\right).$$

8.3.2 Use Stirling's formula to estimate 52!, the number of possible rearrangements of cards in a standard deck of playing cards.

8.3.3 By integrating Eq. (8.51) from $z-1$ to z and then letting $z\to\infty$, evaluate the constant C_1 in the asymptotic series for the digamma function $\psi(z)$.

8.3.4 Show that the constant C_2 in Stirling's formula equals $\frac{1}{2}\ln 2\pi$ by using the logarithm of the doubling formula.

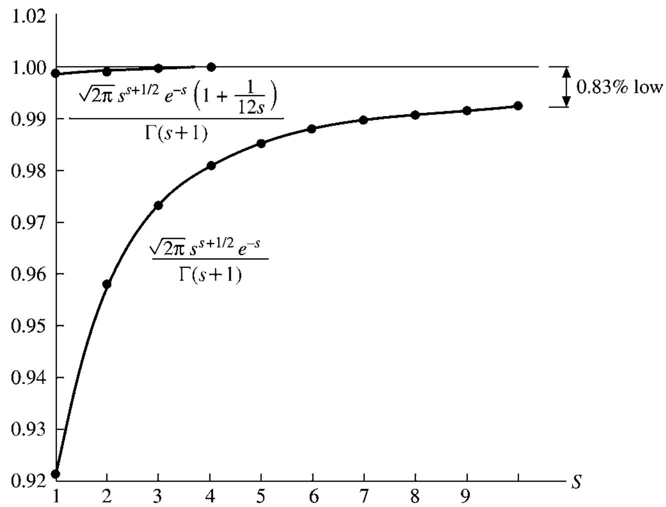

FIGURE 8.5 Accuracy of Stirling's formula.

Table 8.1

s	$\frac{1}{\Gamma(s+1)}\sqrt{2\pi}s^{s+1/2}e^{-s}$	$\frac{1}{\Gamma(s+1)}\sqrt{2\pi}s^{s+1/2}e^{-s}\left(1+\frac{1}{12s}\right)$
1	0.92213	0.99898
2	0.95950	0.99949
3	0.97270	0.99972
4	0.97942	0.99983
5	0.98349	0.99988
6	0.98621	0.99992
7	0.98817	0.99994
8	0.98964	0.99995
9	0.99078	0.99996
10	0.99170	0.99998

8.3.5 By direct expansion, verify the doubling formula for $z = n + \frac{1}{2}$; n is an integer.

8.3.6 Without using Stirling's series show that

$$\text{(a) } \ln(n!) < \int_1^{n+1} \ln x\, dx, \qquad \text{(b) } \ln(n!) > \int_1^{n} \ln x\, dx; \; n \text{ is an integer} \geq 2.$$

Notice that the arithmetic mean of these two integrals gives a good approximation for Stirling's series.

8.3.7 Test for convergence

$$\sum_{p=0}^{\infty}\left[\frac{(p-\frac{1}{2})!}{p!}\right]^2 \times \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}.$$