1.1 Sets, Real Numbers, and Numerical Expressions

OBJECTIVES

1. Identify certain sets of numbers
2. Apply the properties of equality
3. Simplify numerical expressions

In arithmetic, we use symbols such as 6, $\frac{2}{3}$, 0.27, and π to represent numbers. The symbols $+$, $-$, $\cdot$, and $\div$ commonly indicate the basic operations of addition, subtraction, multiplication, and division, respectively. Thus we can form specific **numerical expressions**. For example, we can write the indicated sum of six and eight as $6 + 8$.

In algebra, the concept of a variable provides the basis for generalizing arithmetic ideas. For example, by using x and y to represent any numbers, we can use the expression $x + y$ to represent the indicated sum of any two numbers. The x and y in such an expression are called **variables**, and the phrase $x + y$ is called an **algebraic expression**.

We can extend to algebra many of the notational agreements we make in arithmetic, with a few modifications. The following chart summarizes the notational agreements that pertain to the four basic operations.

Operation	Arithmetic	Algebra	Vocabulary
Addition	$4 + 6$	$x + y$	The *sum* of x and y
Subtraction	$14 - 10$	$a - b$	The *difference* of a and b
Multiplication	$7 \cdot 5$ or 7×5	$a \cdot b$, $a(b)$, $(a)b$, $(a)(b)$, or ab	The *product* of a and b
Division	$8 \div 4$, $\frac{8}{4}$, or $4\overline{)8}$	$x \div y$, $\frac{x}{y}$, or $y\overline{)x}$	The *quotient* of x and y

Note the different ways to indicate a product, including the use of parentheses. The ab form is the simplest and probably the most widely used form. Expressions such as abc, $6xy$, and $14xyz$ all indicate multiplication. We also call your attention to the various forms that indicate division; in algebra, we usually use the fractional form $\frac{x}{y}$ although the other forms do serve a purpose at times.

Use of Sets

We can use some of the basic vocabulary and symbolism associated with the concept of sets in the study of algebra. A **set** is a collection of objects, and the objects are called **elements** or **members** of the set. In arithmetic and algebra the elements of a set are usually numbers.

The use of set braces, { }, to enclose the elements (or a description of the elements) and the use of capital letters to name sets provide a convenient way to communicate about sets. For example, we can represent a set A, which consists of the vowels of the alphabet, in any of the following ways:

$A = \{\text{vowels of the alphabet}\}$ Word description

$A = \{a, e, i, o, u\}$ List or roster description

$A = \{x \mid x \text{ is a vowel}\}$ Set builder notation

We can modify the listing approach if the number of elements is quite large. For example, all of the letters of the alphabet can be listed as

$\{a, b, c, \ldots, z\}$

We simply begin by writing enough elements to establish a pattern; then the three dots indicate that the set continues in that pattern. The final entry indicates the last element of the pattern. If we write

$\{1, 2, 3, \ldots\}$

the set begins with the counting numbers 1, 2, and 3. The three dots indicate that it continues in a like manner forever; there is no last element. A set that consists of no elements is called the **null set** (written $\varnothing$).

Set builder notation combines the use of braces and the concept of a variable. For example, $\{x|x$ is a vowel$\}$ is read "the set of all x such that x is a vowel." Note that the vertical line is read "such that." We can use set builder notation to describe the set $\{1, 2, 3, \ldots\}$ as $\{x|x > 0$ and x is a whole number$\}$.

We use the symbol $\in$ to denote set membership. Thus if $A = \{a, e, i, o, u\}$, we can write $e \in A$, which we read as "e is an element of A." The slash symbol, /, is commonly used in mathematics as a negation symbol. For example, $m \notin A$ is read as "m is not an element of A."

Two sets are said to be *equal* if they contain exactly the same elements. For example,

$\{1, 2, 3\} = \{2, 1, 3\}$

because both sets contain the same elements; the order in which the elements are written doesn't matter. The slash mark through the equality symbol denotes "is not equal to." Thus if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, we can write $A \neq B$, which we read as "set A is not equal to set B."

Real Numbers

We refer to most of the algebra that we will study in this text as the **algebra of real numbers**. This simply means that the variables represent real numbers. Therefore, it is necessary for us to be familiar with the various terms that are used to classify different types of real numbers.

$\{1, 2, 3, 4, \ldots\}$	Natural numbers, counting numbers, positive integers
$\{0, 1, 2, 3, \ldots\}$	Whole numbers, nonnegative integers
$\{\ldots -3, -2, -1\}$	Negative integers
$\{\ldots -3, -2, -1, 0\}$	Nonpositive integers
$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Integers

We define a **rational number** as follows:

Definition 1.1 Rational Numbers

A rational number is any number that can be written in the form $\frac{a}{b}$, where a and b are integers, and b does not equal zero.

We can easily recognize that each of the following numbers fits the definition of a rational number.

$$\frac{-3}{4} \quad \frac{2}{3} \quad \frac{15}{4} \quad \text{and} \quad \frac{1}{-5}$$

However, numbers such as -4, 0, 0.3, and $6\frac{1}{2}$ are also rational numbers. All of these numbers could be written in the form $\frac{a}{b}$ as follows.

-4 can be written as $\frac{-4}{1}$ or $\frac{4}{-1}$

0 can be written as $\frac{0}{1} = \frac{0}{2} = \frac{0}{3} = \ldots$

0.3 can be written as $\frac{3}{10}$

$6\frac{1}{2}$ can be written as $\frac{13}{2}$

We can also define a rational number in terms of decimal representation. We classify decimals as terminating, repeating, or nonrepeating.

Type	Definition	Examples	Rational numbers
Terminating	A terminating decimal ends.	0.3, 0.46, 0.6234, 1.25	Yes
Repeating	A repeating decimal has a block of digits that repeats indefinitely.	0.66666 . . . 0.141414 . . . 0.694694694 . . . 0.23171717 . . .	Yes
Nonrepeating	A nonrepeating decimal does not have a block of digits that repeats indefinitely and does not terminate.	3.1415926535 . . . 1.414213562 . . . 0.276314583 . . .	No

A repeating decimal has a block of digits that can be any number of digits and may or may not begin immediately after the decimal point. A small horizontal bar (overbar) is commonly used to indicate the repeat block. Thus 0.6666 . . . is written as $0.\overline{6}$, and 0.2317171717 . . . is written as $0.23\overline{17}$.

In terms of decimals, we define a rational number as a number that has a terminating or a repeating decimal representation. The following examples illustrate some rational numbers written in $\frac{a}{b}$ form and in decimal form.

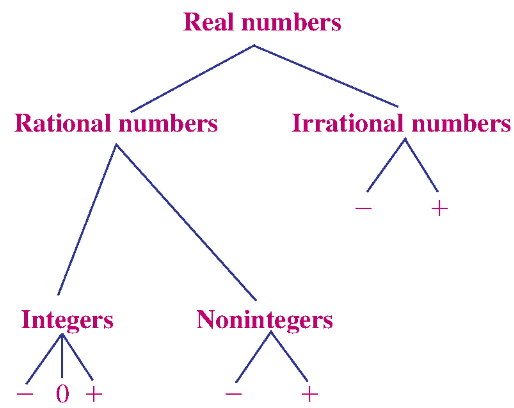
$\frac{3}{4} = 0.75$ $\quad \frac{3}{11} = 0.\overline{27}$ $\quad \frac{1}{8} = 0.125$ $\quad \frac{1}{7} = 0.\overline{142857}$ $\quad \frac{1}{3} = 0.\overline{3}$

We define an **irrational number** as a number that *cannot* be expressed in $\frac{a}{b}$ form, where a and b are integers, and b is not zero. Furthermore, an irrational number has a nonrepeating and nonterminating decimal representation. Some examples of irrational numbers and a partial decimal representation for each follow.

$\sqrt{2} = 1.414213562373095 \ldots$ $\quad \sqrt{3} = 1.73205080756887 \ldots$

$\pi = 3.14159265358979 \ldots$

The set of **real numbers** is composed of the rational numbers along with the irrational numbers. Every real number is either a rational number or an irrational number. The following tree diagram summarizes the various classifications of the real number system.



We can trace any real number down through the diagram as follows:

7 is real, rational, an integer, and positive

$-\frac{2}{3}$ is real, rational, noninteger, and negative

$\sqrt{7}$ is real, irrational, and positive

0.38 is real, rational, noninteger, and positive

Remark: We usually refer to the set of nonnegative integers, $\{0, 1, 2, 3, \ldots\}$, as the set of **whole numbers**, and we refer to the set of positive integers, $\{1, 2, 3, \ldots\}$, as the set of **natural numbers**. The set of whole numbers differs from the set of natural numbers by the inclusion of the number zero.

The concept of subset is convenient to discuss at this time. A set A is a **subset** of a set B if and only if every element of A is also an element of B. This is written as $A \subseteq B$ and read as "A is a subset of B." For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 5, 9\}$, then $A \subseteq B$ because every element of A is also an element of B. The slash mark denotes negation, so if $A = \{1, 2, 5\}$ and $B = \{2, 4, 7\}$, we can say that A is not a subset of B by writing $A \nsubseteq B$. Figure 1.1 represents the subset relationships for the set of real numbers. Refer to Figure 1.1 as you study the following statements, which use subset vocabulary and subset symbolism.

1. The set of whole numbers is a subset of the set of integers.

$$\{0, 1, 2, 3, \ldots\} \subseteq \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

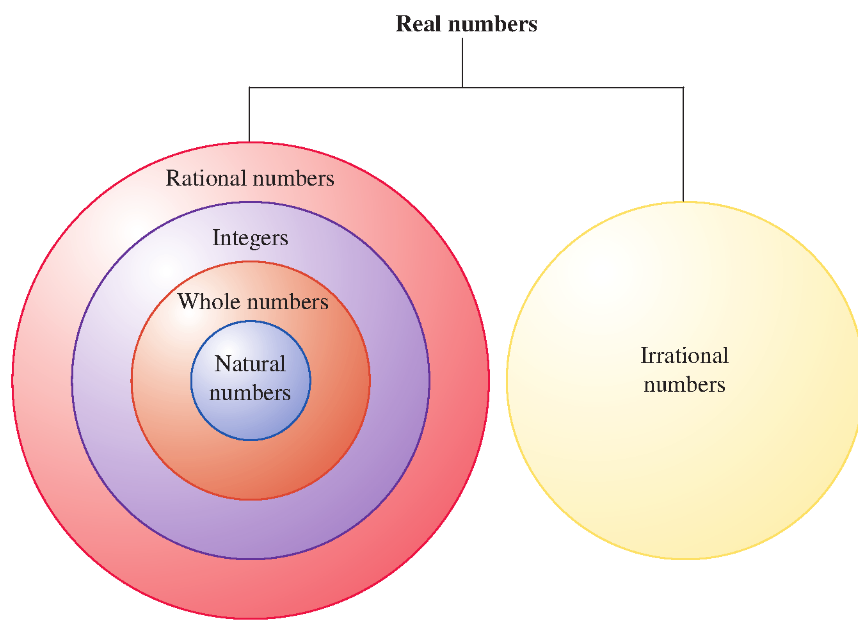


Figure 1.1

2. The set of integers is a subset of the set of rational numbers.

$\{\ldots, -2, -1, 0, 1, 2, \ldots\} \subseteq \{x|x \text{ is a rational number}\}$

3. The set of rational numbers is a subset of the set of real numbers.

$\{x|x \text{ is a rational number}\} \subseteq \{y|y \text{ is a real number}\}$

Properties of Equality

The relation *equality* plays an important role in mathematics—especially when we are manipulating real numbers and algebraic expressions that represent real numbers. An **equality** is a statement in which two symbols, or groups of symbols, are names for the same number. The symbol $=$ is used to express an equality. Thus we can write

$6 + 1 = 7 \qquad 18 - 2 = 16 \qquad 36 \div 4 = 9$

(The symbol $\neq$ denotes *is not equal to*.) The following four basic properties of equality are self-evident, but we do need to keep them in mind. (We will expand this list in Chapter 2 when we work with solutions of equations.)

Properties of equality	Definition: For real numbers *a*, *b*, and *c*	Examples
Reflexive property	$a = a$	$14 = 14$, $x = x$, $a + b = a + b$
Symmetric property	If $a = b$, then $b = a$.	If $3 + 1 = 4$, then $4 = 3 + 1$. If $x = 10$, then $10 = x$.
Transitive property	If $a = b$ and $b = c$, then $a = c$.	If $x = 7$ and $7 = y$, then $x = y$. If $x + 5 = y$ and $y = 8$, then $x + 5 = 8$.
Substitution property	If $a = b$, then a may be replaced by b, or b may be replaced by a, without changing the meaning of the statement.	If $x + y = 4$ and $x = 2$, then we can replace x in the first equation with the value 2, which will yield $2 + y = 4$.

Simplifying Numerical Expressions

Let's conclude this section by *simplifying some numerical expressions* that involve whole numbers. When simplifying numerical expressions, we perform the operations in the following order. Be sure that you agree with the result in each example.

1. Perform the operations inside the symbols of inclusion (parentheses, brackets, and braces) and above and below each fraction bar. Start with the innermost inclusion symbol.

2. Perform all multiplications and divisions in the order in which they appear from left to right.

3. Perform all additions and subtractions in the order in which they appear from left to right.

Classroom Example
Simplify $25 + 55 \div 11 \cdot 4$.

EXAMPLE 1 Simplify $20 + 60 \div 10 \cdot 2$.

Solution

First do the division.

$20 + 60 \div 10 \cdot 2 = 20 + 6 \cdot 2$

Next do the multiplication.

$$20 + 6 \cdot 2 = 20 + 12$$

Then do the addition.

$$20 + 12 = 32$$

Thus $20 + 60 \div 10 \cdot 2$ simplifies to 32.

Classroom Example
Simplify $4 \cdot 9 \div 3 \cdot 6 \div 8$.

EXAMPLE 2 Simplify $7 \cdot 4 \div 2 \cdot 3 \cdot 2 \div 4$.

Solution

The multiplications and divisions are to be done from left to right in the order in which they appear.

$$\begin{aligned} 7 \cdot 4 \div 2 \cdot 3 \cdot 2 \div 4 &= 28 \div 2 \cdot 3 \cdot 2 \div 4 \\ &= 14 \cdot 3 \cdot 2 \div 4 \\ &= 42 \cdot 2 \div 4 \\ &= 84 \div 4 \\ &= 21 \end{aligned}$$

Thus $7 \cdot 4 \div 2 \cdot 3 \cdot 2 \div 4$ simplifies to 21.

Classroom Example
Simplify $3 \cdot 7 + 16 \div 4 - 3 \cdot 8 + 6 \div 2$.

EXAMPLE 3 Simplify $5 \cdot 3 + 4 \div 2 - 2 \cdot 6 - 28 \div 7$.

Solution

First we do the multiplications and divisions in the order in which they appear. Then we do the additions and subtractions in the order in which they appear. Our work may take on the following format.

$$5 \cdot 3 + 4 \div 2 - 2 \cdot 6 - 28 \div 7 = 15 + 2 - 12 - 4 = 1$$

Classroom Example
Simplify $(7 + 2)(3 + 8)$.

EXAMPLE 4 Simplify $(4 + 6)(7 + 8)$.

Solution

We use the parentheses to indicate the *product* of the quantities $4 + 6$ and $7 + 8$. We perform the additions inside the parentheses first and then multiply.

$$(4 + 6)(7 + 8) = (10)(15) = 150$$

Classroom Example
Simplify $(2 \cdot 5 + 3 \cdot 6) \cdot (7 \cdot 4 - 8 \cdot 3)$.

EXAMPLE 5 Simplify $(3 \cdot 2 + 4 \cdot 5)(6 \cdot 8 - 5 \cdot 7)$.

Solution

First we do the multiplications inside the parentheses.

$$(3 \cdot 2 + 4 \cdot 5)(6 \cdot 8 - 5 \cdot 7) = (6 + 20)(48 - 35)$$

Then we do the addition and subtraction inside the parentheses.

$$(6 + 20)(48 - 35) = (26)(13)$$

Then we find the final product.

$$(26)(13) = 338$$

Classroom Example
Simplify $3 + 9[2(5 + 4)]$.

EXAMPLE 6 Simplify $6 + 7[3(4 + 6)]$.

Solution

We use brackets for the same purposes as parentheses. In such a problem we need to simplify *from the inside out*; that is, we perform the operations in the innermost parentheses first. We thus obtain

$$\begin{aligned} 6 + 7[3(4 + 6)] &= 6 + 7[3(10)] \\ &= 6 + 7[30] \\ &= 6 + 210 \\ &= 216 \end{aligned}$$

Classroom Example
Simplify $\dfrac{7 \cdot 6 - 3 \cdot 3}{2 \cdot 6 \div 3 - 1}$.

EXAMPLE 7 Simplify $\dfrac{6 \cdot 8 \div 4 - 2}{5 \cdot 4 - 9 \cdot 2}$.

Solution

First we perform the operations above and below the fraction bar. Then we find the final quotient.

$$\frac{6 \cdot 8 \div 4 - 2}{5 \cdot 4 - 9 \cdot 2} = \frac{48 \div 4 - 2}{20 - 18} = \frac{12 - 2}{2} = \frac{10}{2} = 5$$

Remark: With parentheses we could write the problem in Example 7 as $(6 \cdot 8 \div 4 - 2) \div (5 \cdot 4 - 9 \cdot 2)$.

Concept Quiz 1.1

For Problems 1–10, answer true or false.

1. The expression ab indicates the sum of a and b.
2. The set $\{1, 2, 3 \ldots\}$ contains infinitely many elements.
3. The sets $A = \{1, 2, 4, 6\}$ and $B = \{6, 4, 1, 2\}$ are equal sets.
4. Every irrational number is also classified as a real number.
5. To evaluate $24 \div 6 \cdot 2$, the first operation to be performed is to multiply 6 times 2.
6. To evaluate $6 + 8 \cdot 3$, the first operation to be performed is to multiply 8 times 3.
7. The number 0.15 is real, irrational, and positive.
8. If $4 = x + 3$, then $x + 3 = 4$ is an example of the symmetric property of equality.
9. The numerical expression $6 \cdot 2 + 3 \cdot 5 - 6$ simplifies to 21.
10. The number represented by $0.\overline{12}$ is a rational number.

Problem Set 1.1

For Problems 1–10, identify each statement as true or false. **(Objective 1)**

1. Every irrational number is a real number.
2. Every rational number is a real number.
3. If a number is real, then it is irrational.
4. Every real number is a rational number.
5. All integers are rational numbers.
6. Some irrational numbers are also rational numbers.
7. Zero is a positive integer.

8. Zero is a rational number.

9. All whole numbers are integers.

10. Zero is a negative integer.

For Problems 11–18, from the list 0, 14, $\frac{2}{3}$, π, $\sqrt{7}$, $-\frac{11}{14}$, 2.34, -19, $\frac{55}{8}$, $-\sqrt{17}$, $3.2\overline{1}$, and -2.6, identify each of the following. **(Objective 1)**

11. The whole numbers

12. The natural numbers

13. The rational numbers

14. The integers

15. The nonnegative integers

16. The irrational numbers

17. The real numbers

18. The nonpositive integers

For Problems 19–28, use the following set designations.

$N = \{x|x \text{ is a natural number}\}$

$Q = \{x|x \text{ is a rational number}\}$

$W = \{x|x \text{ is a whole number}\}$

$H = \{x|x \text{ is an irrational number}\}$

$I = \{x|x \text{ is an integer}\}$

$R = \{x|x \text{ is a real number}\}$

Place $\subseteq$ or $\nsubseteq$ in each blank to make a true statement. **(Objective 1)**

19. R ______ N

20. N ______ R

21. I ______ Q

22. N ______ I

23. Q ______ H

24. H ______ Q

25. N ______ W

26. W ______ I

27. I ______ N

28. I ______ W

For Problems 29–32, classify the real number by tracing through the diagram in the text (see page 5). **(Objective 1)**

29. -8

30. 0.9

31. $-\sqrt{2}$

32. $\frac{5}{6}$

For Problems 33–42, list the elements of each set. For example, the elements of $\{x|x$ is a natural number less than 4$\}$ can be listed as $\{1, 2, 3\}$. **(Objective 1)**

33. $\{x|x$ is a natural number less than 3$\}$

34. $\{x|x$ is a natural number greater than 3$\}$

35. $\{n|n$ is a whole number less than 6$\}$

36. $\{y|y$ is an integer greater than $-4\}$

37. $\{y|y$ is an integer less than 3$\}$

38. $\{n|n$ is a positive integer greater than $-7\}$

39. $\{x|x$ is a whole number less than 0$\}$

40. $\{x|x$ is a negative integer greater than $-3\}$

41. $\{n|n$ is a nonnegative integer less than 5$\}$

42. $\{n|n$ is a nonpositive integer greater than 3$\}$

For Problems 43–50, replace each question mark to make the given statement an application of the indicated property of equality. For example, $16 = ?$ becomes $16 = 16$ because of the reflexive property of equality. **(Objective 2)**

43. If $y = x$ and $x = -6$, then $y = ?$ (Transitive property of equality)

44. $5x + 7 = ?$ (Reflexive property of equality)

45. If $n = 2$ and $3n + 4 = 10$, then $3(?) + 4 = 10$ (Substitution property of equality)

46. If $y = x$ and $x = z + 2$, then $y = ?$ (Transitive property of equality)

47. If $4 = 3x + 1$, then $? = 4$ (Symmetric property of equality)

48. If $t = 4$ and $s + t = 9$, then $s + ? = 9$ (Substitution property of equality)

49. $5x = ?$ (Reflexive property of equality)

50. If $5 = n + 3$, then $n + 3 = ?$ (Symmetric property of equality)

For Problems 51–74, simplify each of the numerical expressions. **(Objective 3)**

51. $16 + 9 - 4 - 2 + 8 - 1$

52. $18 + 17 - 9 - 2 + 14 - 11$

53. $9 \div 3 \cdot 4 \div 2 \cdot 14$

54. $21 \div 7 \cdot 5 \cdot 2 \div 6$

55. $7 + 8 \cdot 2$

56. $21 - 4 \cdot 3 + 2$

57. $9 \cdot 7 - 4 \cdot 5 - 3 \cdot 2 + 4 \cdot 7$

58. $6 \cdot 3 + 5 \cdot 4 - 2 \cdot 8 + 3 \cdot 2$

59. $(17 - 12)(13 - 9)(7 - 4)$

60. $(14 - 12)(13 - 8)(9 - 6)$

61. $13 + (7 - 2)(5 - 1)$

62. $48 - (14 - 11)(10 - 6)$

63. $(5 \cdot 9 - 3 \cdot 4)(6 \cdot 9 - 2 \cdot 7)$

64. $(3 \cdot 4 + 2 \cdot 1)(5 \cdot 2 + 6 \cdot 7)$

65. $7[3(6 - 2)] - 64$

66. $12 + 5[3(7 - 4)]$

67. $[3 + 2(4 \cdot 1 - 2)][18 - (2 \cdot 4 - 7 \cdot 1)]$

68. $3[4(6 + 7)] + 2[3(4 - 2)]$

69. $14 + 4\left(\frac{8 - 2}{12 - 9}\right) - 2\left(\frac{9 - 1}{19 - 15}\right)$

70. $12 + 2\left(\frac{12 - 2}{7 - 2}\right) - 3\left(\frac{12 - 9}{17 - 14}\right)$

71. $[7 + 2 \cdot 3 \cdot 5 - 5] \div 8$

72. $[27 - (4 \cdot 2 + 5 \cdot 2)][(5 \cdot 6 - 4) - 20]$

73. $\frac{3 \cdot 8 - 4 \cdot 3}{5 \cdot 7 - 34} + 19$

74. $\frac{4 \cdot 9 - 3 \cdot 5 - 3}{18 - 12}$

75. You must of course be able to do calculations like those in Problems 51–74 both with and without a calculator. Furthermore, different types of calculators handle the priority-of-operations issue in different ways. Be sure you can do Problems 51–74 with *your* calculator.

Thoughts Into Words

76. Explain in your own words the difference between the reflexive property of equality and the symmetric property of equality.

77. Your friend keeps getting an answer of 30 when simplifying $7 + 8(2)$. What mistake is he making and how would you help him?

78. Do you think $3\sqrt{2}$ is a rational or an irrational number? Defend your answer.

79. Explain why every integer is a rational number but not every rational number is an integer.

80. Explain the difference between $1.\overline{3}$ and 1.3.

Answers to the Concept Quiz

1. False **2.** True **3.** True **4.** True **5.** False **6.** True **7.** False **8.** True **9.** True **10.** True

1.2 Operations with Real Numbers

OBJECTIVES

1. Review the real number line
2. Find the absolute value of a number
3. Add real numbers
4. Subtract real numbers
5. Multiply real numbers
6. Divide real numbers
7. Simplify numerical expressions
8. Use real numbers to represent problems

Before we review the four basic operations with real numbers, let's briefly discuss some concepts and terminology we commonly use with this material. It is often helpful to have a geometric representation of the set of real numbers as indicated in Figure 1.2. Such a representation, called the **real number line**, indicates a one-to-one correspondence between the set of real numbers and the points on a line. In other words, to each real number there corresponds one and only one point on the line, and to each point on the line there corresponds one