**The Fundamental Identities**

**(1) The reciprocal identities:**

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

**(2) The tangent and cotangent identities:**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

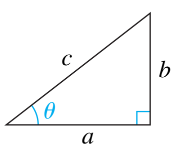
**(3) The Pythagorean identities:**

$$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$$

PROOFS

(1) The reciprocal identities were established earlier in this section.

Figure 10




(2) To prove the tangent identity, we refer to the right triangle in Figure 10 and use definitions of trigonometric functions as follows:

$$\tan\theta = \frac{b}{a} = \frac{b/c}{a/c} = \frac{\sin\theta}{\cos\theta}$$

To verify the cotangent identity, we use a reciprocal identity and the tangent identity:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sin\theta/\cos\theta} = \frac{\cos\theta}{\sin\theta}$$

(3) The Pythagorean identities are so named because of the first step in the following proof. Referring to Figure 10, we obtain

$$b^2 + a^2 = c^2 \qquad \text{Pythagorean theorem}$$

$$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \left(\frac{c}{c}\right)^2 \qquad \text{divide by } c^2$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1 \qquad \text{definitions of } \sin\theta \text{ and } \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1. \qquad \text{equivalent notation}$$

We may use this identity to verify the second Pythagorean identity as follows:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \text{divide by } \cos^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \text{equivalent equation}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \left(\frac{\cos\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2 \qquad \text{law of exponents}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{tangent and reciprocal identities}$$

To prove the third Pythagorean identity, $1 + \cot^2\theta = \csc^2\theta$, we could divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

We can use the fundamental identities to express each trigonometric function in terms of any other trigonometric function. Two illustrations are given in the next example.

### EXAMPLE 4 Using fundamental identities

Let $\theta$ be an acute angle.

**(a)** Express $\sin\theta$ in terms of $\cos\theta$.

**(b)** Express $\tan\theta$ in terms of $\sin\theta$.

**SOLUTION**

**(a)** We may proceed as follows:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{Pythagorean identity}$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{isolate } \sin^2\theta$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \qquad \text{take the square root}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \qquad \sin\theta > 0 \text{ for acute angles}$$

Later in this section (Example 12) we will consider a simplification involving a *non*-acute angle $\theta$.

**(b)** If we begin with the fundamental identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

then all that remains is to express $\cos\theta$ in terms of $\sin\theta$. We can do this by solving $\sin^2\theta + \cos^2\theta = 1$ for $\cos\theta$, obtaining

$$\cos\theta = \sqrt{1 - \sin^2\theta} \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}.$$

*(continued)*

Hence,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}.$$

Fundamental identities are often used to simplify expressions involving trigonometric functions, as illustrated in the next example.

**EXAMPLE 5** **Showing that an equation is an identity**

Show that the following equation is an identity by transforming the left-hand side into the right-hand side:

$$(\sec\theta + \tan\theta)(1 - \sin\theta) = \cos\theta$$

**SOLUTION** We begin with the left-hand side and proceed as follows:

$$\begin{aligned}
(\sec\theta + \tan\theta)(1 - \sin\theta) &= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)(1 - \sin\theta) && \text{reciprocal and tangent identities}\\
&= \left(\frac{1 + \sin\theta}{\cos\theta}\right)(1 - \sin\theta) && \text{add fractions}\\
&= \frac{1 - \sin^2\theta}{\cos\theta} && \text{multiply}\\
&= \frac{\cos^2\theta}{\cos\theta} && \sin^2\theta + \cos^2\theta = 1\\
&= \cos\theta && \text{cancel } \cos\theta
\end{aligned}$$

There are other ways to simplify the expression on the left-hand side in Example 5. We could first multiply the two factors and then simplify and combine terms. The method we employed—changing all expressions to expressions that involve only sines and cosines—is often useful. However, that technique does not always lead to the shortest possible simplification.

Hereafter, we shall use the phrase *verify an identity* instead of *show that an equation is an identity*. When verifying an identity, we often use fundamental identities and algebraic manipulations to simplify expressions, as we did in the preceding example. As with the fundamental identities, we understand that an identity that contains fractions is valid for all values of the variables such that no denominator is zero.

**EXAMPLE 6 Verifying an identity**

Verify the following identity by transforming the left-hand side into the right-hand side:

$$\frac{\tan\theta + \cos\theta}{\sin\theta} = \sec\theta + \cot\theta$$

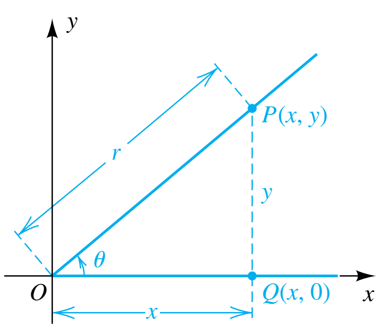
**SOLUTION** We may transform the left-hand side into the right-hand side as follows:

$$\begin{aligned}
\frac{\tan\theta + \cos\theta}{\sin\theta} &= \frac{\tan\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} && \text{divide numerator by } \sin\theta \\
&= \frac{\left(\dfrac{\sin\theta}{\cos\theta}\right)}{\sin\theta} + \cot\theta && \text{tangent and cotangent identities} \\
&= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} + \cot\theta && \text{rule for quotients} \\
&= \frac{1}{\cos\theta} + \cot\theta && \text{cancel } \sin\theta \\
&= \sec\theta + \cot\theta && \text{reciprocal identity}
\end{aligned}$$

In Section 7.1 we will verify many other identities using methods similar to those used in Examples 5 and 6.

Since many applied problems involve angles that are not acute, it is necessary to extend the definition of the trigonometric functions. We make this extension by using the standard position of an angle $\theta$ on a rectangular coordinate system. If $\theta$ is acute, we have the situation illustrated in Figure 11, where we have chosen a point $P(x, y)$ on the terminal side of $\theta$ and where $d(O, P) = r = \sqrt{x^2 + y^2}$. Referring to triangle $OQP$, we have

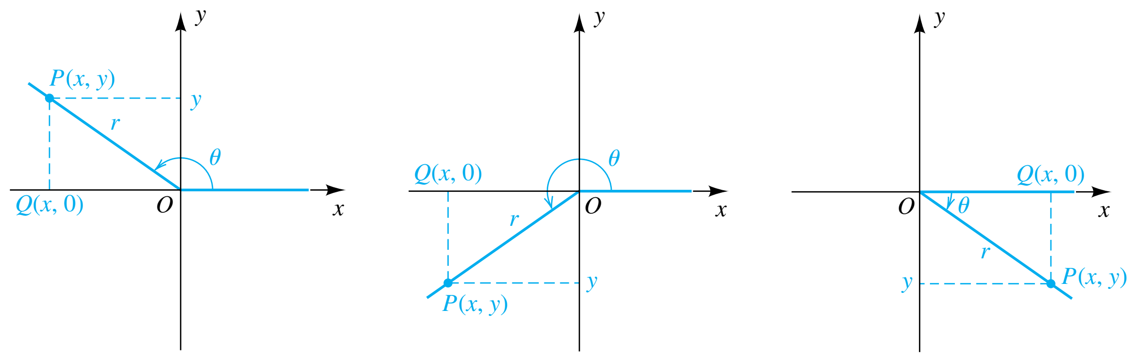
**Figure 11**




$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}, \quad \text{and} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.$$

We now wish to consider angles of the types illustrated in Figure 12 on the next page (or any *other* angle, either positive, negative, or zero). Note that in Figure 12 the value of $x$ or $y$ may be negative. In each case, side $QP$ (opp in Figure 12) has length $|y|$, side $OQ$ (adj in Figure 12) has length $|x|$, and the hypotenuse $OP$ has length $r$. We shall define the six trigonometric functions so that their values agree with those given previously whenever the angle is acute. It is understood that if a zero denominator occurs, then the corresponding function value is undefined.

Figure 12




**Definition of the Trigonometric Functions of Any Angle**

Let $\theta$ be an angle in standard position on a rectangular coordinate system, and let $P(x, y)$ be any point other than the origin $O$ on the terminal side of $\theta$.

If $d(O, P) = r = \sqrt{x^2 + y^2}$, then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (\text{if } y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (\text{if } x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (\text{if } y \neq 0).$$

We can show, using similar triangles, that the formulas in this definition do not depend on the point $P(x, y)$ that is chosen on the terminal side of $\theta$. The fundamental identities, which were established for acute angles, are also true for trigonometric functions of any angle.

The domains of the sine and cosine functions consist of all angles $\theta$. However, $\tan\theta$ and $\sec\theta$ are undefined if $x = 0$ (that is, if the terminal side of $\theta$ is on the $y$-axis). Thus, the domains of the tangent and the secant functions consist of all angles *except* those of radian measure $(\pi/2) + \pi n$ for any integer $n$. Some special cases are $\pm\pi/2$, $\pm 3\pi/2$, and $\pm 5\pi/2$. The corresponding degree measures are $\pm 90°$, $\pm 270°$, and $\pm 450°$.

The domains of the cotangent and cosecant functions consist of all angles except those that have $y = 0$ (that is, all angles except those having terminal sides on the $x$-axis). These are the angles of radian measure $\pi n$ (or degree measure $180° \cdot n$) for any integer $n$.

Our discussion of domains is summarized in the following table, where $n$ denotes any integer.

| Function | | Domain |
|---|---|---|
| sine, | cosine | every angle $\theta$ |
| tangent, | secant | every angle $\theta$ except $\theta = \frac{\pi}{2} + \pi n = 90° + 180° \cdot n$ |
| cotangent, | cosecant | every angle $\theta$ except $\theta = \pi n = 180° \cdot n$ |

For any point $P(x, y)$ in the preceding definition, $|x| \leq r$ and $|y| \leq r$ or, equivalently, $|x/r| \leq 1$ and $|y/r| \leq 1$. Thus,

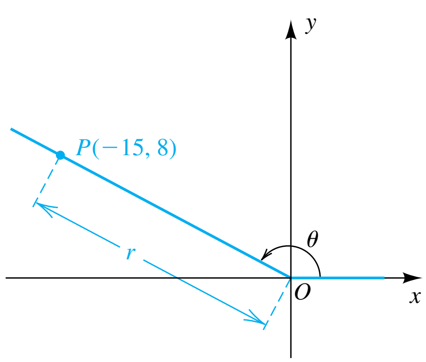
$$|\sin\theta| \leq 1, \quad |\cos\theta| \leq 1, \quad |\csc\theta| \geq 1, \quad \text{and} \quad |\sec\theta| \geq 1$$

for every $\theta$ in the domains of these functions.

**EXAMPLE 7** **Finding trigonometric function values of an angle in standard position**

If $\theta$ is an angle in standard position on a rectangular coordinate system and if $P(-15, 8)$ is on the terminal side of $\theta$, find the values of the six trigonometric functions of $\theta$.

Figure 13




**SOLUTION** The point $P(-15, 8)$ is shown in Figure 13. Applying the definition of the trigonometric functions of any angle with $x = -15$, $y = 8$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + 8^2} = \sqrt{289} = 17,$$

we obtain the following:

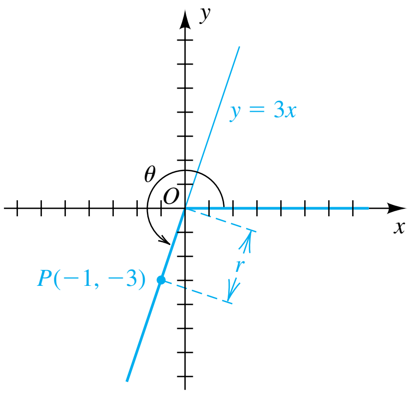
$$\sin\theta = \frac{y}{r} = \frac{8}{17} \qquad \cos\theta = \frac{x}{r} = -\frac{15}{17} \qquad \tan\theta = \frac{y}{x} = -\frac{8}{15}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{8} \qquad \sec\theta = \frac{r}{x} = -\frac{17}{15} \qquad \cot\theta = \frac{x}{y} = -\frac{15}{8}$$

**EXAMPLE 8** **Finding trigonometric function values of an angle in standard position**

An angle $\theta$ is in standard position, and its terminal side lies in quadrant III on the line $y = 3x$. Find the values of the trigonometric functions of $\theta$.

Figure 14




**SOLUTION** The graph of $y = 3x$ is sketched in Figure 14, together with the initial and terminal sides of $\theta$. Since the terminal side of $\theta$ is in quadrant III, we begin by choosing a convenient negative value of $x$, say $x = -1$. Substituting in $y = 3x$ gives us $y = 3(-1) = -3$, and hence $P(-1, -3)$ is on the terminal side. Applying the definition of the trigonometric functions of any angle with

$$x = -1, \quad y = -3, \quad \text{and} \quad r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$$

gives us

$$\sin\theta = -\frac{3}{\sqrt{10}} \qquad \cos\theta = -\frac{1}{\sqrt{10}} \qquad \tan\theta = \frac{-3}{-1} = 3$$

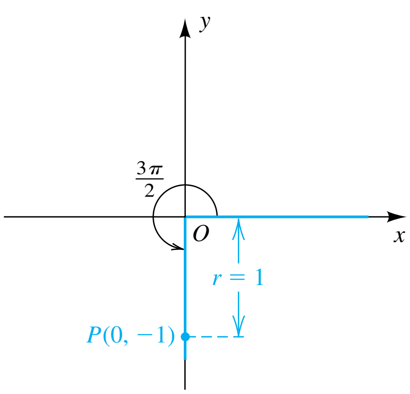
$$\csc\theta = -\frac{\sqrt{10}}{3} \qquad \sec\theta = -\frac{\sqrt{10}}{1} \qquad \cot\theta = \frac{-1}{-3} = \frac{1}{3}.$$

The definition of the trigonometric functions of any angle may be applied if $\theta$ is a quadrantal angle. The procedure is illustrated by the next example.

**EXAMPLE 9 Finding trigonometric function values of a quadrantal angle**

If $\theta = 3\pi/2$, find the values of the trigonometric functions of $\theta$.

Figure 15




**SOLUTION** Note that $3\pi/2 = 270°$. If $\theta$ is placed in standard position, the terminal side of $\theta$ coincides with the negative $y$-axis, as shown in Figure 15. To apply the definition of the trigonometric functions of any angle, we may choose *any* point $P$ on the terminal side of $\theta$. For simplicity, we use $P(0, -1)$. In this case, $x = 0$, $y = -1$, $r = 1$, and hence

$$\sin\frac{3\pi}{2} = \frac{-1}{1} = -1 \qquad \cos\frac{3\pi}{2} = \frac{0}{1} = 0$$

$$\csc\frac{3\pi}{2} = \frac{1}{-1} = -1 \qquad \cot\frac{3\pi}{2} = \frac{0}{-1} = 0.$$

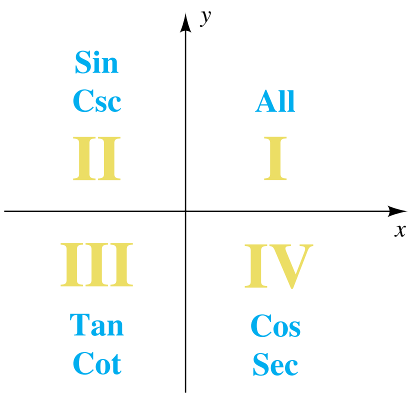
The tangent and secant functions are undefined, since the meaningless expressions $\tan\theta = (-1)/0$ and $\sec\theta = 1/0$ occur when we substitute in the appropriate formulas.

Let us determine the signs associated with values of the trigonometric functions. If $\theta$ is in quadrant II and $P(x, y)$ is a point on the terminal side, then $x$ is negative and $y$ is positive. Hence, $\sin\theta = y/r$ and $\csc\theta = r/y$ are positive, and the other four trigonometric functions, which all involve $x$, are negative. Checking the remaining quadrants in a similar fashion, we obtain the following table.

**Signs of the Trigonometric Functions**

| Quadrant containing $\theta$ | Positive functions | Negative functions |
|---|---|---|
| I | all | none |
| II | sin, csc | cos, sec, tan, cot |
| III | tan, cot | sin, csc, cos, sec |
| IV | cos, sec | sin, csc, tan, cot |

**Figure 16**
Positive trigonometric functions




*A mnemonic device for remembering the quadrants in which the trigonometric functions are positive is "A Smart Trig Class," which corresponds to All Sin Tan Cos.*

The diagram in Figure 16 may be useful for remembering quadrants in which trigonometric functions are *positive*. If a function is not listed (such as cos in quadrant II), then that function is negative. We finish this section with three examples that require using the information in the preceding table.

### EXAMPLE 10 Finding the quadrant containing an angle

Find the quadrant containing $\theta$ if both $\cos\theta > 0$ and $\sin\theta < 0$.

**SOLUTION** Referring to the table of signs or Figure 16, we see that $\cos\theta > 0$ (cosine is positive) if $\theta$ is in quadrant I or IV and that $\sin\theta < 0$ (sine is negative) if $\theta$ is in quadrant III or IV. Hence, for both conditions to be satisfied, $\theta$ must be in quadrant IV.

### EXAMPLE 11 Finding values of trigonometric functions from prescribed conditions

If $\sin\theta = \frac{3}{5}$ and $\tan\theta < 0$, use fundamental identities to find the values of the other five trigonometric functions.

**SOLUTION** Since $\sin\theta = \frac{3}{5} > 0$ (positive) and $\tan\theta < 0$ (negative), $\theta$ is in quadrant II. Using the relationship $\sin^2\theta + \cos^2\theta = 1$ and the fact that $\cos\theta$ is negative in quadrant II, we have

$$\cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1 - \left(\tfrac{3}{5}\right)^2} = -\sqrt{\tfrac{16}{25}} = -\tfrac{4}{5}.$$

Next we use the tangent identity to obtain

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -\frac{3}{4}.$$

Finally, using the reciprocal identities gives us

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-4/5} = -\frac{5}{4}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-3/4} = -\frac{4}{3}.$$

**EXAMPLE 12** **Using fundamental identities**

Rewrite $\sqrt{\cos^2\theta + \sin^2\theta + \cot^2\theta}$ in nonradical form without using absolute values for $\pi < \theta < 2\pi$.

**SOLUTION**

$$\begin{aligned}\sqrt{\cos^2\theta + \sin^2\theta + \cot^2\theta} &= \sqrt{1 + \cot^2\theta} && \cos^2\theta + \sin^2\theta = 1\\ &= \sqrt{\csc^2\theta} && 1 + \cot^2\theta = \csc^2\theta\\ &= |\csc\theta| && \sqrt{x^2} = |x|\end{aligned}$$

Since $\pi < \theta < 2\pi$, we know that $\theta$ is in quadrant III or IV. Thus, $\csc\theta$ is *negative*, and by the definition of absolute value, we have

$$|\csc\theta| = -\csc\theta.$$

## 6.2 Exercises

**Exer. 1–2: Use common sense to match the variables and the values. (The triangles are drawn to scale, and the angles are measured in radians.)**

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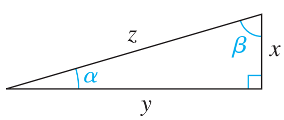


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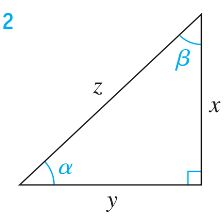
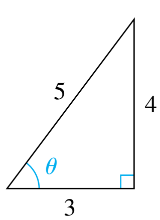



| | | | |
|---|---|---|---|
| (a) $\alpha$ | (A) 7 | (a) $\alpha$ | (A) 23.35 |
| (b) $\beta$ | (B) 0.28 | (b) $\beta$ | (B) 16 |
| (c) $x$ | (C) 24 | (c) $x$ | (C) 17 |
| (d) $y$ | (D) 1.29 | (d) $y$ | (D) 0.82 |
| (e) $z$ | (E) 25 | (e) $z$ | (E) 0.76 |

**Exer. 3–10: Find the values of the six trigonometric functions for the angle $\theta$.**

3
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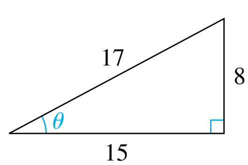


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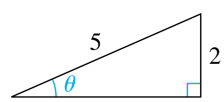


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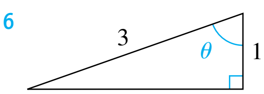


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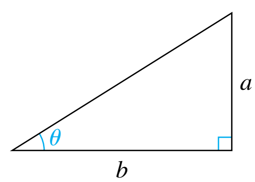


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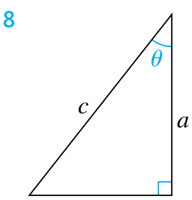


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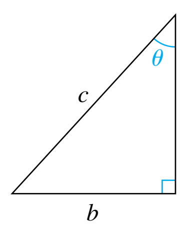


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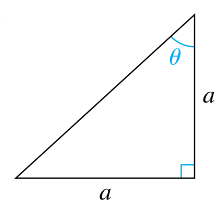
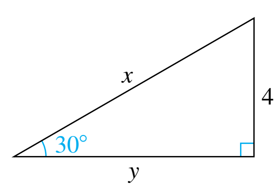



**Exer. 11–16: Find the exact values of $x$ and $y$.**

11
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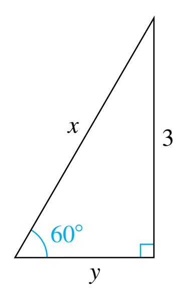

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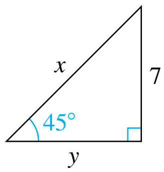


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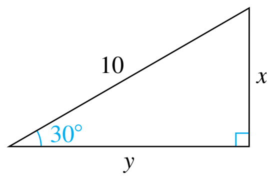


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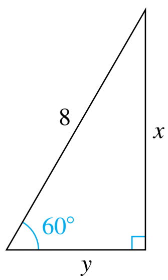


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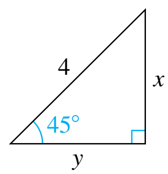



**Exer. 17–22: Find the exact values of the trigonometric functions for the acute angle $\theta$.**

17 $\sin\theta = \frac{3}{5}$

18 $\cos\theta = \frac{8}{17}$

19 $\tan\theta = \frac{5}{12}$

20 $\cot\theta = \frac{7}{24}$

21 $\sec\theta = \frac{6}{5}$

22 $\csc\theta = 4$

23 **Height of a tree** A forester, 200 feet from the base of a redwood tree, observes that the angle between the ground and the top of the tree is 60°. Estimate the height of the tree.

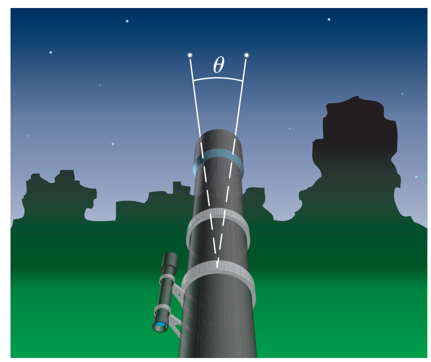
24 **Distance to Mt. Fuji** The peak of Mt. Fuji in Japan is approximately 12,400 feet high. A trigonometry student, several miles away, notes that the angle between level ground and the peak is 30°. Estimate the distance from the student to the point on level ground directly beneath the peak.

25 **Stonehenge blocks** Stonehenge in Salisbury Plains, England, was constructed using solid stone blocks weighing over 99,000 pounds each. Lifting a single stone required 550 people, who pulled the stone up a ramp inclined at an angle of 9°. Approximate the distance that a stone was moved in order to raise it to a height of 30 feet.

26 **Advertising sign height** Added in 1990 and removed in 1997, the highest advertising sign in the world was a large letter I situated at the top of the 73-story First Interstate World Center building in Los Angeles. At a distance of 200 feet from a point directly below the sign, the angle between the ground and the top of the sign was 78.87°. Approximate the height of the top of the sign.

27 **Telescope resolution** Two stars that are very close may appear to be one. The ability of a telescope to separate their images is called its resolution. The smaller the resolution, the better a telescope's ability to separate images in the sky. In a refracting telescope, resolution $\theta$ (see the figure) can be improved by using a lens with a larger diameter $D$. The relationship between $\theta$ in degrees and $D$ in meters is given by $\sin\theta = 1.22\lambda/D$, where $\lambda$ is the wavelength of light in meters. The largest refracting telescope in the world is at the University of Chicago. At a wavelength of $\lambda = 550 \times 10^{-9}$ meter, its resolution is 0.000 037 69°. Approximate the diameter of the lens.

**Exercise 27**




28 **Moon phases** The phases of the moon can be described using the phase angle $\theta$, determined by the sun, the moon, and Earth, as shown in the figure. Because the moon orbits Earth, $\theta$ changes during the course of a month. The area of the region $A$ of the moon, which appears illuminated to an observer on Earth, is given by $A = \frac{1}{2}\pi R^2(1 + \cos\theta)$, where $R = 1080$ mi is the radius of the moon. Approximate $A$ for the following positions of the moon:

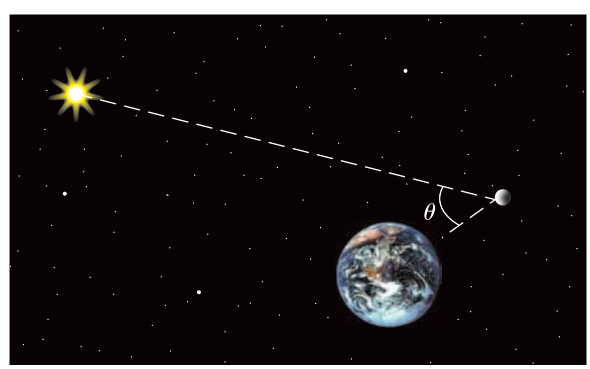
(a) $\theta = 0°$ (full moon)

(b) $\theta = 180°$ (new moon)

(c) $\theta = 90°$ (first quarter)

(d) $\theta = 103°$

**Exercise 28**



**Exer. 29–34: Approximate to four decimal places, when appropriate.**

29 (a) $\sin 42°$ (b) $\cos 77°$
(c) $\csc 123°$ (d) $\sec(-190°)$

30 (a) $\tan 282°$ (b) $\cot(-81°)$
(c) $\sec 202°$ (d) $\sin 97°$

31 (a) $\cot(\pi/13)$ (b) $\csc 1.32$
(c) $\cos(-8.54)$ (d) $\tan(3\pi/7)$

32 (a) $\sin(-0.11)$ (b) $\sec \frac{31}{27}$
(c) $\tan\left(-\frac{3}{13}\right)$ (d) $\cos 2.4\pi$

33 (a) $\sin 30°$ (b) $\sin 30$
(c) $\cos \pi°$ (d) $\cos \pi$

34 (a) $\sin 45°$ (b) $\sin 45$
(c) $\cos(3\pi/2)°$ (d) $\cos(3\pi/2)$

**Exer. 35–38: Use the Pythagorean identities to write the expression as an integer.**

35 (a) $\tan^2 4\beta - \sec^2 4\beta$ (b) $4\tan^2\beta - 4\sec^2\beta$

36 (a) $\csc^2 3\alpha - \cot^2 3\alpha$ (b) $3\csc^2\alpha - 3\cot^2\alpha$

37 (a) $5\sin^2\theta + 5\cos^2\theta$
(b) $5\sin^2(\theta/4) + 5\cos^2(\theta/4)$

38 (a) $7\sec^2\gamma - 7\tan^2\gamma$
(b) $7\sec^2(\gamma/3) - 7\tan^2(\gamma/3)$

**Exer. 39–42: Simplify the expression.**

39 $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

40 $\dfrac{\cot^2\alpha - 4}{\cot^2\alpha - \cot\alpha - 6}$

41 $\dfrac{2 - \tan\theta}{2\csc\theta - \sec\theta}$

42 $\dfrac{\csc\theta + 1}{(1/\sin^2\theta) + \csc\theta}$

**Exer. 43–48: Use fundamental identities to write the first expression in terms of the second, for any acute angle $\theta$.**

43 $\cot\theta$, $\sin\theta$

44 $\tan\theta$, $\cos\theta$

45 $\sec\theta$, $\sin\theta$

46 $\csc\theta$, $\cos\theta$

47 $\sin\theta$, $\sec\theta$

48 $\cos\theta$, $\cot\theta$

**Exer. 49–70: Verify the identity by transforming the left-hand side into the right-hand side.**

49 $\cos\theta\sec\theta = 1$

50 $\tan\theta\cot\theta = 1$

51 $\sin\theta\sec\theta = \tan\theta$

52 $\sin\theta\cot\theta = \cos\theta$

53 $\dfrac{\csc\theta}{\sec\theta} = \cot\theta$

54 $\cot\theta\sec\theta = \csc\theta$

55 $(1 + \cos 2\theta)(1 - \cos 2\theta) = \sin^2 2\theta$

56 $\cos^2 2\theta - \sin^2 2\theta = 2\cos^2 2\theta - 1$

57 $\cos^2\theta(\sec^2\theta - 1) = \sin^2\theta$

58 $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$

59 $\dfrac{\sin(\theta/2)}{\csc(\theta/2)} + \dfrac{\cos(\theta/2)}{\sec(\theta/2)} = 1$

60 $1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$

61 $(1 + \sin\theta)(1 - \sin\theta) = \dfrac{1}{\sec^2\theta}$

62 $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

63 $\sec\theta - \cos\theta = \tan\theta\sin\theta$

64 $\dfrac{\sin\theta + \cos\theta}{\cos\theta} = 1 + \tan\theta$

65 $(\cot\theta + \csc\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$

66 $\cot\theta + \tan\theta = \csc\theta\sec\theta$

67 $\sec^2 3\theta\csc^2 3\theta = \sec^2 3\theta + \csc^2 3\theta$

68 $\dfrac{1 + \cos^2 3\theta}{\sin^2 3\theta} = 2\csc^2 3\theta - 1$

69 $\log\csc\theta = -\log\sin\theta$

70 $\log\tan\theta = \log\sin\theta - \log\cos\theta$

**Exer. 71–74: Find the exact values of the six trigonometric functions of $\theta$ if $\theta$ is in standard position and $P$ is on the terminal side.**

71 $P(4, -3)$

72 $P(-8, -15)$

73 $P(-2, -5)$

74 $P(-1, 2)$