6

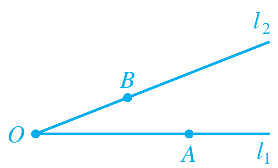
The Trigonometric Functions

Trigonometry was invented over 2000 years ago by the Greeks, who needed precise methods for measuring angles and sides of triangles. In fact, the word *trigonometry* was derived from the two Greek words *trigonon* (triangle) and *metria* (measurement). This chapter begins with a discussion of angles and how they are measured. We next introduce the trigonometric functions by using ratios of sides of a right triangle. After extending the domains of the trigonometric functions to arbitrary angles and real numbers, we consider their graphs and graphing techniques that make use of amplitudes, periods, and phase shifts. The chapter concludes with a section on applied problems.

6.1 Angles

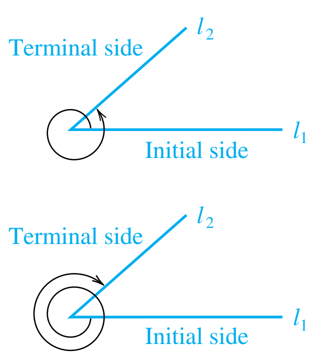
In geometry an **angle** is defined as the set of points determined by two rays, or half-lines, l_1 and l_2, having the same endpoint O. If A and B are points on l_1 and l_2, as in Figure 1, we refer to **angle *AOB*** (denoted $\angle AOB$). An angle may also be considered as two finite line segments with a common endpoint.

Figure 1



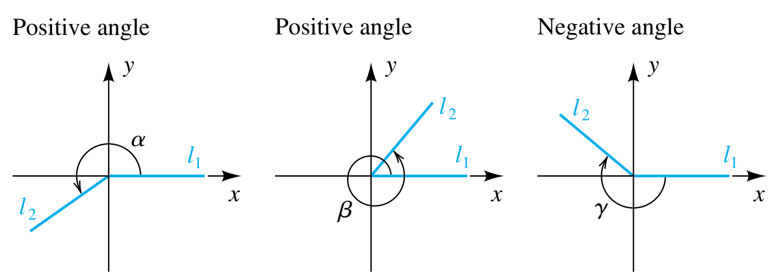
In trigonometry we often interpret angles as rotations of rays. Start with a fixed ray l_1, having endpoint O, and rotate it about O, in a plane, to a position specified by ray l_2. We call l_1 the **initial side,** l_2 the **terminal side,** and O the **vertex** of $\angle AOB$. The amount or direction of rotation is not restricted in any way. We might let l_1 make several revolutions in either direction about O before coming to position l_2, as illustrated by the curved arrows in Figure 2. Thus, many different angles have the same initial and terminal sides. Any two such angles are called **coterminal angles.** A **straight angle** is an angle whose sides lie on the same straight line but extend in opposite directions from its vertex.

Figure 2
Coterminal angles



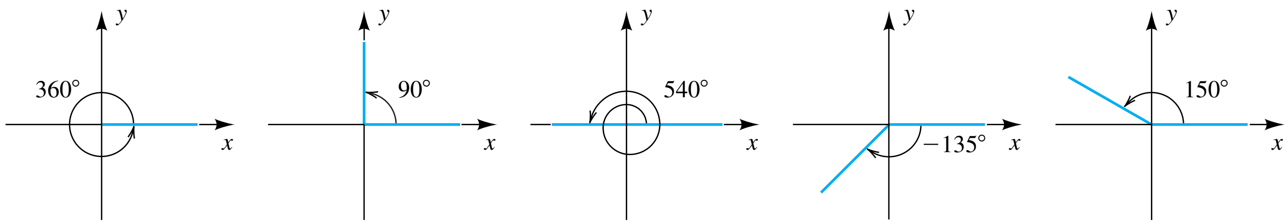
If we introduce a rectangular coordinate system, then the **standard position** of an angle is obtained by taking the vertex at the origin and letting the initial side l_1 coincide with the positive x-axis. If l_1 is rotated in a *counterclockwise* direction to the terminal position l_2, then the angle is considered **positive.** If l_1 is rotated in a *clockwise* direction, the angle is **negative.** We often denote angles by lowercase Greek letters such as α (*alpha*), β (*beta*), γ (*gamma*), θ (*theta*), ϕ (*phi*), and so on. Figure 3 contains sketches of two positive angles, α and β, and a negative angle, γ. If the terminal side of an angle in standard position is in a certain quadrant, we say that the *angle* is in that quadrant. In Figure 3, α is in quadrant III, β is in quadrant I, and γ is in quadrant II. An angle is called a **quadrantal angle** if its terminal side lies on a coordinate axis.

Figure 3 Standard position of an angle



One unit of measurement for angles is the **degree.** The angle in standard position obtained by one complete revolution in the counterclockwise direction has measure 360 degrees, written 360°. Thus, an angle of measure 1 degree (1°) is obtained by $\frac{1}{360}$ of one complete counterclockwise revolution. In Figure 4, several angles measured in degrees are shown in standard position on rectangular coordinate systems. Note that the first three are quadrantal angles.

Figure 4



Throughout our work, a notation such as $\theta = 60°$ specifies an angle θ whose measure is 60°. We also refer to *an angle of* 60° or *a* 60° *angle,* instead of using the more precise (but cumbersome) phrase *an angle having measure* 60°.

EXAMPLE 1 **Finding coterminal angles**

If $\theta = 60°$ is in standard position, find two positive angles and two negative angles that are coterminal with θ.

SOLUTION The angle θ is shown in standard position in the first sketch in Figure 5. To find positive coterminal angles, we may add 360° or 720° (or any other positive integer multiple of 360°) to θ, obtaining

$$60° + 360° = 420° \qquad \text{and} \qquad 60° + 720° = 780°.$$

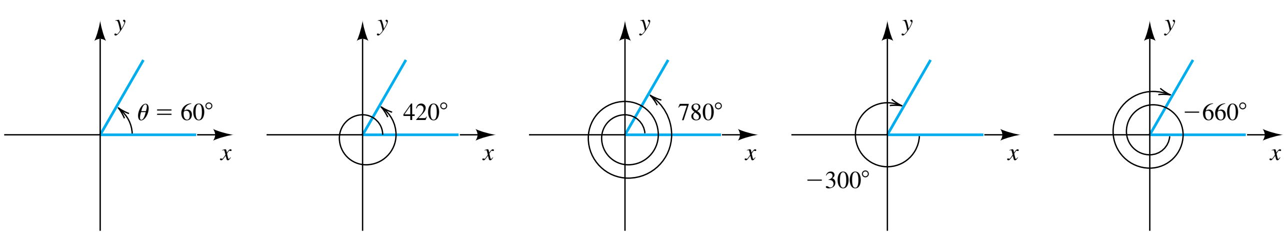
These coterminal angles are also shown in Figure 5.

To find negative coterminal angles, we may add −360° or −720° (or any other negative integer multiple of 360°), obtaining

$$60° + (-360°) = -300° \qquad \text{and} \qquad 60° + (-720°) = -660°,$$

as shown in the last two sketches in Figure 5.

Figure 5



A **right angle** is half of a straight angle and has measure 90°. The following chart contains definitions of other special types of angles.

Terminology	Definition	Illustrations
acute angle θ	$0° < \theta < 90°$	12°; 37°
obtuse angle θ	$90° < \theta < 180°$	95°; 157°
complementary angles α, β	$\alpha + \beta = 90°$	20°, 70°; 7°, 83°
supplementary angles α, β	$\alpha + \beta = 180°$	115°, 65°; 18°, 162°

If smaller measurements than the degree are required, we can use tenths, hundredths, or thousandths of degrees. Alternatively, we can divide the degree into 60 equal parts, called **minutes** (denoted by $'$), and each minute into 60 equal parts, called **seconds** (denoted by $''$). Thus, $1° = 60'$, and $1' = 60''$. The notation $\theta = 73°56'18''$ refers to an angle θ that has measure 73 degrees, 56 minutes, 18 seconds.

EXAMPLE 2 Finding complementary angles

Find the angle that is complementary to θ:

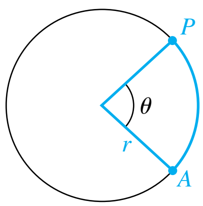
(a) $\theta = 25°43'37''$ **(b)** $\theta = 73.26°$

SOLUTION We wish to find $90° - \theta$. It is convenient to write 90° as an equivalent measure, $89°59'60''$.

(a)
$$\begin{aligned} 90° &= 89°59'60'' \\ \underline{\quad\theta\quad} &= \underline{25°43'37''} \\ 90° - \theta &= 64°16'23'' \end{aligned}$$

(b)
$$\begin{aligned} 90° &= 90.00° \\ \underline{\quad\theta\quad} &= \underline{73.26°} \\ 90° - \theta &= 16.74° \end{aligned}$$

Figure 6
Central angle θ



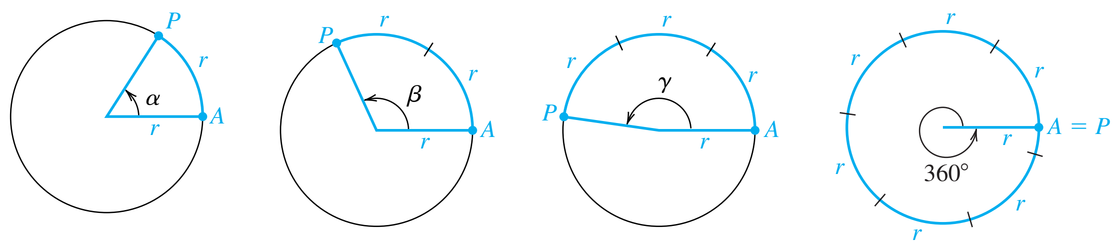
Degree measure for angles is used in applied areas such as surveying, navigation, and the design of mechanical equipment. In scientific applications that require calculus, it is customary to employ *radian measure.* To define an angle of radian measure 1, we consider a circle of any radius r. A **central angle** of a circle is an angle whose vertex is at the center of the circle. If θ is the central angle shown in Figure 6, we say that the **arc AP** (denoted $\overset{\frown}{AP}$) of the circle **subtends** θ or that θ **is subtended by** $\overset{\frown}{AP}$. If the length of $\overset{\frown}{AP}$ is equal to the radius r of the circle, then θ has a measure of one radian, as in the next definition.

Definition of Radian Measure	
	One radian is the measure of the central angle of a circle subtended by an arc equal in length to the radius of the circle.

If we consider a circle of radius r, then an angle α whose measure is 1 radian intercepts an arc AP of length r, as illustrated in Figure 7(a). The angle β in Figure 7(b) has radian measure 2, since it is subtended by an arc of length $2r$. Similarly, γ in (c) of the figure has radian measure 3, since it is subtended by an arc of length $3r$.

Figure 7

(a) $\alpha = 1$ radian **(b)** $\beta = 2$ radians **(c)** $\gamma = 3$ radians **(d)** $360° = 2\pi \approx 6.28$ radians



To find the radian measure corresponding to 360°, we must find the number of times that a circular arc of length r can be laid off along the circumference (see Figure 7(d)). This number is not an integer or even a rational number. Since the circumference of the circle is $2\pi r$, the number of times r units can be laid off is 2π. Thus, an angle of measure 2π radians corresponds to the degree measure 360°, and we write $360° = 2\pi$ radians. This result gives us the following relationships.

Relationships Between Degrees and Radians

(1) $180° = \pi$ radians

(2) $1° = \dfrac{\pi}{180}$ radian ≈ 0.0175 radian

(3) 1 radian $= \left(\dfrac{180°}{\pi}\right) \approx 57.2958°$

When radian measure of an angle is used, no units will be indicated. Thus, if an angle has radian measure 5, we write $\theta = 5$ instead of $\theta = 5$ *radians.* There should be no confusion as to whether radian or degree measure is being used, since if θ has *degree* measure 5°, we write $\theta = 5°$, and *not* $\theta = 5$.

The next chart illustrates how to change from one angular measure to another.

Changing Angular Measures

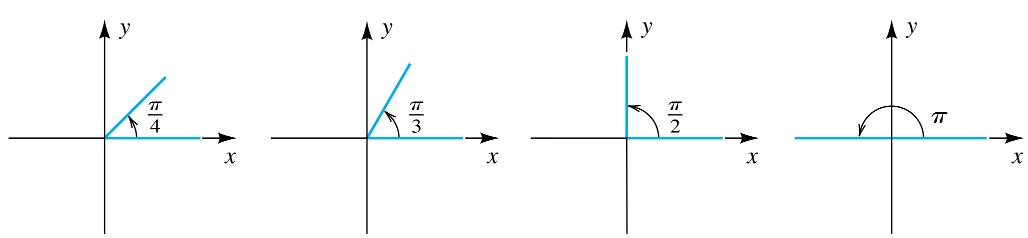
To change	Multiply by	Illustrations
degrees to radians	$\dfrac{\pi}{180^\circ}$	$150^\circ = 150^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{5\pi}{6}$ $225^\circ = 225^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{5\pi}{4}$
radians to degrees	$\dfrac{180^\circ}{\pi}$	$\dfrac{7\pi}{4} = \dfrac{7\pi}{4}\left(\dfrac{180^\circ}{\pi}\right) = 315^\circ$ $\dfrac{\pi}{3} = \dfrac{\pi}{3}\left(\dfrac{180^\circ}{\pi}\right) = 60^\circ$

We may use the techniques illustrated in the preceding chart to obtain the following table, which displays the corresponding radian and degree measures of special angles.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°

Several of these special angles, in radian measure, are shown in standard position in Figure 8.

Figure 8



EXAMPLE 3 **Changing radians to degrees, minutes, and seconds**

If $\theta = 3$, approximate θ in terms of degrees, minutes, and seconds.

SOLUTION

$$\begin{aligned}
3 \text{ radians} &= 3\left(\frac{180^\circ}{\pi}\right) && \text{multiply by } \frac{180^\circ}{\pi} \\
&\approx 171.8873^\circ && \text{approximate} \\
&= 171^\circ + (0.8873)(60') && 1^\circ = 60' \\
&= 171^\circ + 53.238' && \text{multiply} \\
&= 171^\circ + 53' + (0.238)(60'') && 1' = 60'' \\
&= 171^\circ 53' + 14.28'' && \text{multiply} \\
&\approx 171^\circ 53' 14'' && \text{approximate}
\end{aligned}$$

EXAMPLE 4 **Expressing minutes and seconds as decimal degrees**

Express $19^\circ 47' 23''$ as a decimal, to the nearest ten-thousandth of a degree.

SOLUTION Since $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$,

$$\begin{aligned}
19^\circ 47' 23'' &= 19^\circ + \left(\tfrac{47}{60}\right)^\circ + \left(\tfrac{23}{3600}\right)^\circ \\
&\approx 19^\circ + 0.7833^\circ + 0.0064^\circ \\
&= 19.7897^\circ.
\end{aligned}$$

The next result specifies the relationship between the length of a circular arc and the central angle that it subtends.

Formula for the Length of a Circular Arc	If an arc of length s on a circle of radius r subtends a central angle of radian measure θ, then $$s = r\theta.$$

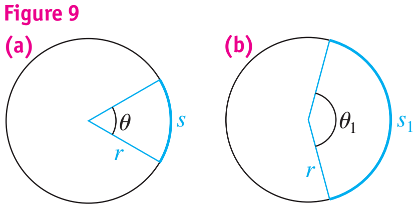
A mnemonic device for remembering $s = r\theta$ is SRO (Standing Room Only).

PROOF A typical arc of length s and the corresponding central angle θ are shown in Figure 9(a). Figure 9(b) shows an arc of length s_1 and central angle θ_1. If radian measure is used, then, from plane geometry, the ratio of the lengths of the arcs is the same as the ratio of the angular measures; that is,

$$\frac{s}{s_1} = \frac{\theta}{\theta_1}, \qquad \text{or} \qquad s = \frac{\theta}{\theta_1} s_1.$$

(continued)

Figure 9



If we consider the special case in which θ_1 has radian measure 1, then, from the definition of radian, $s_1 = r$ and the last equation becomes

$$s = \frac{\theta}{1} \cdot r = r\theta.$$

Notice that if $\theta = 2\pi$, then the formula for the length of a circular arc becomes $s = r(2\pi)$, which is simply the formula for the circumference of a circle, $C = 2\pi r$.

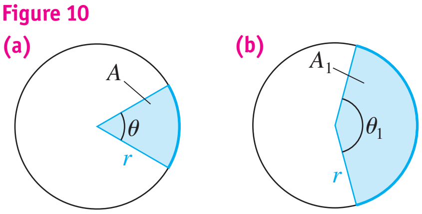
The next formula is proved in a similar manner.

Formula for the Area of a Circular Sector

If θ is the radian measure of a central angle of a circle of radius r and if A is the area of the circular sector determined by θ, then

$$A = \tfrac{1}{2}r^2\theta.$$

Figure 10



PROOF If A and A_1 are the areas of the sectors in Figures 10(a) and 10(b), respectively, then, from plane geometry,

$$\frac{A}{A_1} = \frac{\theta}{\theta_1}, \quad \text{or} \quad A = \frac{\theta}{\theta_1}A_1.$$

If we consider the special case $\theta_1 = 2\pi$, then $A_1 = \pi r^2$ and

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta.$$

When using the preceding formulas, it is important to remember to use the radian measure of θ rather than the degree measure, as illustrated in the next example.

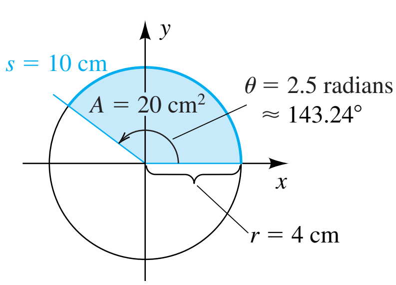
EXAMPLE 5 **Using the circular arc and sector formulas**

In Figure 11, a central angle θ is subtended by an arc 10 centimeters long on a circle of radius 4 centimeters.

(a) Approximate the measure of θ in degrees.

(b) Find the area of the circular sector determined by θ.

Figure 11



SOLUTION We proceed as follows:

(a) $s = r\theta$ — length of a circular arc formula

$\theta = \dfrac{s}{r}$ — solve for θ

$= \frac{10}{4} = 2.5$ — let $s = 10$, $r = 4$

This is the *radian* measure of θ. Changing to degrees, we have

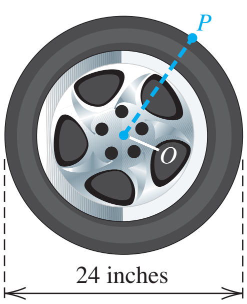
$$\theta = 2.5\left(\frac{180°}{\pi}\right) = \frac{450°}{\pi} \approx 143.24°.$$

(b) $A = \frac{1}{2}r^2\theta$ area of a circular sector formula

$= \frac{1}{2}(4)^2(2.5)$ let $r = 4$, $\theta = 2.5$ radians

$= 20\ \text{cm}^2$ multiply

Figure 12



The **angular speed** of a wheel that is rotating at a constant rate is the angle generated in one unit of time by a line segment from the center of the wheel to a point P on the circumference (see Figure 12). The **linear speed** of a point P on the circumference is the distance that P travels per unit of time. By dividing both sides of the formula for a circular arc by time t, we obtain a relationship for linear speed and angular speed; that is,

$$\frac{s}{t} = \frac{r\theta}{t}, \quad \text{or, equivalently,} \quad \underbrace{\frac{s}{t}}_{\text{linear speed}} = r \cdot \underbrace{\frac{\theta}{t}}_{\text{angular speed}}.$$

EXAMPLE 6 **Finding angular and linear speeds**

Suppose that the wheel in Figure 12 is rotating at a rate of 800 rpm (revolutions per minute).

(a) Find the angular speed of the wheel.

(b) Find the linear speed (in in./min and mi/hr) of a point P on the circumference of the wheel.

SOLUTION

(a) Let O denote the center of the wheel, and let P be a point on the circumference. Because the number of revolutions per minute is 800 and because each revolution generates an angle of 2π radians, the angle generated by the line segment OP in one minute has radian measure $(800)(2\pi)$; that is,

$$\text{angular speed} = \frac{800 \text{ revolutions}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \text{ revolution}} = 1600\pi \text{ radians per minute.}$$

Note that the diameter of the wheel is irrelevant in finding the angular speed.

(b) linear speed $=$ radius $\cdot$ angular speed

$= (12 \text{ in.})(1600\pi \text{ rad/min})$

$= 19{,}200\pi \text{ in./min}$

Converting in./min to mi/hr, we get

$$\frac{19{,}200\pi \text{ in.}}{1 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 57.1 \text{ mi/hr}.$$

Unlike the angular speed, the linear speed *is* dependent on the diameter of the wheel.

6.1 Exercises

Exer. 1–4: If the given angle is in standard position, find two positive coterminal angles and two negative coterminal angles.

1 (a) $120°$ (b) $135°$ (c) $-30°$

2 (a) $240°$ (b) $315°$ (c) $-150°$

3 (a) $620°$ (b) $\dfrac{5\pi}{6}$ (c) $-\dfrac{\pi}{4}$

4 (a) $570°$ (b) $\dfrac{2\pi}{3}$ (c) $-\dfrac{5\pi}{4}$

Exer. 5–6: Find the angle that is complementary to θ.

5 (a) $\theta = 5°17'34''$ (b) $\theta = 32.5°$

6 (a) $\theta = 63°4'15''$ (b) $\theta = 82.73°$

Exer. 7–8: Find the angle that is supplementary to θ.

7 (a) $\theta = 48°51'37''$ (b) $\theta = 136.42°$

8 (a) $\theta = 152°12'4''$ (b) $\theta = 15.9°$

Exer. 9–12: Find the exact radian measure of the angle.

9 (a) $150°$ (b) $-60°$ (c) $225°$

10 (a) $120°$ (b) $-135°$ (c) $210°$

11 (a) $450°$ (b) $72°$ (c) $100°$

12 (a) $630°$ (b) $54°$ (c) $95°$

Exer. 13–16: Find the exact degree measure of the angle.

13 (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{11\pi}{6}$ (c) $\dfrac{3\pi}{4}$

14 (a) $\dfrac{5\pi}{6}$ (b) $\dfrac{4\pi}{3}$ (c) $\dfrac{11\pi}{4}$

15 (a) $-\dfrac{7\pi}{2}$ (b) 7π (c) $\dfrac{\pi}{9}$

16 (a) $-\dfrac{5\pi}{2}$ (b) 9π (c) $\dfrac{\pi}{16}$

Exer. 17–20: Express θ in terms of degrees, minutes, and seconds, to the nearest second.

17 $\theta = 2$

18 $\theta = 1.5$

19 $\theta = 5$

20 $\theta = 4$

Exer. 21–24: Express the angle as a decimal, to the nearest ten-thousandth of a degree.

21 $37°41'$

22 $83°17'$

23 $115°26'27''$

24 $258°39'52''$

Exer. 25–28: Express the angle in terms of degrees, minutes, and seconds, to the nearest second.

25 $63.169°$

26 $12.864°$

27 $310.6215°$

28 $81.7238°$

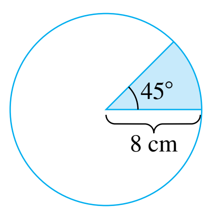
Exer. 29–30: If a circular arc of the given length s subtends the central angle θ on a circle, find the radius of the circle.

29 $s = 10$ cm, $\theta = 4$

30 $s = 3$ km, $\theta = 20°$

Exer. 31–32: (a) Find the length of the arc of the colored sector in the figure. (b) Find the area of the sector.

31



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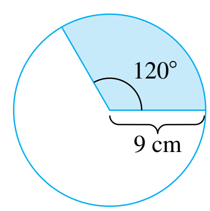


Exer. 33–34: (a) Find the radian and degree measures of the central angle θ subtended by the given arc of length s on a circle of radius r. (b) Find the area of the sector determined by θ.

33 $s = 7$ cm, $r = 4$ cm

34 $s = 3$ ft, $r = 20$ in.

Exer. 35–36: (a) Find the length of the arc that subtends the given central angle θ on a circle of diameter d. (b) Find the area of the sector determined by θ.

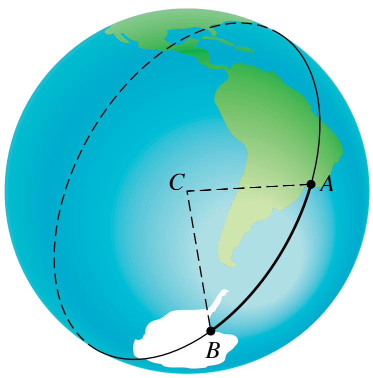
35 $\theta = 50°$, $d = 16$ m

36 $\theta = 2.2$, $d = 120$ cm

37 **Measuring distances on Earth** The distance between two points A and B on Earth is measured along a circle having center C at the center of Earth and radius equal to the distance from C to the surface (see the figure). If the diameter of Earth is approximately 8000 miles, approximate the distance between A and B if angle ACB has the indicated measure:

(a) $60°$ (b) $45°$ (c) $30°$ (d) $10°$ (e) $1°$

Exercise 37



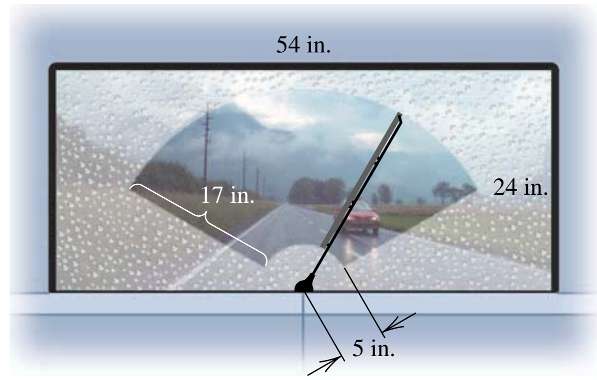
38 **Nautical miles** Refer to Exercise 37. If angle ACB has measure $1'$, then the distance between A and B is a nautical mile. Approximate the number of land (statute) miles in a nautical mile.

39 **Measuring angles using distance** Refer to Exercise 37. If two points A and B are 500 miles apart, express angle ACB in radians and in degrees.

40 A hexagon is inscribed in a circle. If the difference between the area of the circle and the area of the hexagon is 24 m^2, use the formula for the area of a sector to approximate the radius r of the circle.

41 **Window area** A rectangular window measures 54 inches by 24 inches. There is a 17-inch wiper blade attached by a 5-inch arm at the center of the base of the window, as shown in the figure. If the arm rotates 120°, approximate the percentage of the window's area that is wiped by the blade.

Exercise 41



42 **A tornado's core** A simple model of the core of a tornado is a right circular cylinder that rotates about its axis. If a tornado has a core diameter of 200 feet and maximum wind speed of 180 mi/hr (or 264 ft/sec) at the perimeter of the core, approximate the number of revolutions the core makes each minute.

43 **Earth's rotation** Earth rotates about its axis once every 23 hours, 56 minutes, and 4 seconds. Approximate the number of radians Earth rotates in one second.

44 **Earth's rotation** Refer to Exercise 43. The equatorial radius of Earth is approximately 3963.3 miles. Find the linear speed of a point on the equator as a result of Earth's rotation.

Exer. 45–46: A wheel of the given radius is rotating at the indicated rate.
(a) Find the angular speed (in radians per minute).
(b) Find the linear speed of a point on the circumference (in ft/min).

45 radius 5 in., 40 rpm **46** radius 9 in., 2400 rpm

47 **Rotation of compact discs (CDs)** The drive motor of a particular CD player is controlled to rotate at a speed of 200 rpm when reading a track 5.7 centimeters from the center of the CD. The speed of the drive motor must vary so that the reading of the data occurs at a constant rate.

(a) Find the angular speed (in radians per minute) of the drive motor when it is reading a track 5.7 centimeters from the center of the CD.

(b) Find the linear speed (in cm/sec) of a point on the CD that is 5.7 centimeters from the center of the CD.

(c) Find the angular speed (in rpm) of the drive motor when it is reading a track 3 centimeters from the center of the CD.

(d) Find a function S that gives the drive motor speed in rpm for any radius r in centimeters, where $2.3 \le r \le 5.9$. What type of variation exists between the drive motor speed and the radius of the track being read? Check your answer by graphing S and finding the speeds for $r = 3$ and $r = 5.7$.

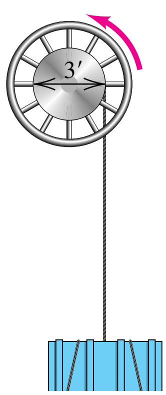
48 **Tire revolutions** A typical tire for a compact car is 22 inches in diameter. If the car is traveling at a speed of 60 mi/hr, find the number of revolutions the tire makes per minute.

49 **Cargo winch** A large winch of diameter 3 feet is used to hoist cargo, as shown in the figure.

(a) Find the distance the cargo is lifted if the winch rotates through an angle of radian measure $7\pi/4$.

(b) Find the angle (in radians) through which the winch must rotate in order to lift the cargo d feet.

Exercise 49

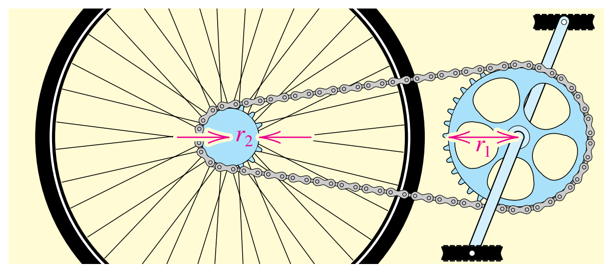


50 **Pendulum's swing** A pendulum in a grandfather clock is 4 feet long and swings back and forth along a 6-inch arc. Approximate the angle (in degrees) through which the pendulum passes during one swing.

51 **Pizza values** A vender sells two sizes of pizza by the slice. The *small* slice is $\frac{1}{6}$ of a circular 18-inch-diameter pizza, and it sells for \$2.00. The *large* slice is $\frac{1}{8}$ of a circular 26-inch-diameter pizza, and it sells for \$3.00. Which slice provides more pizza per dollar?

52 **Bicycle mechanics** The sprocket assembly for a bicycle is shown in the figure. If the sprocket of radius r_1 rotates through an angle of θ_1 radians, find the corresponding angle of rotation for the sprocket of radius r_2.

Exercise 52



53 **Bicycle mechanics** Refer to Exercise 52. An expert cyclist can attain a speed of 40 mi/hr. If the sprocket assembly has $r_1 = 5$ in., $r_2 = 2$ in., and the wheel has a diameter of 28 inches, approximately how many revolutions per minute of the front sprocket wheel will produce a speed of 40 mi/hr? (*Hint:* First change 40 mi/hr to in./sec.)

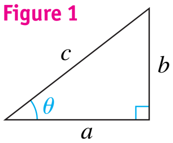
54 **Magnetic pole drift** The geographic and magnetic north poles have different locations. Currently, the magnetic north pole is drifting westward through 0.0017 radian per year, where the angle of drift has its vertex at the center of Earth. If this movement continues, approximately how many years will it take for the magnetic north pole to drift a total of 5°?

6.2 Trigonometric Functions of Angles

We shall introduce the trigonometric functions in the manner in which they originated historically—as ratios of sides of a right triangle. A triangle is a **right triangle** if one of its angles is a right angle. If θ is any acute angle, we may consider a right triangle having θ as one of its angles, as in Figure 1, where the symbol ⌜ specifies the 90° angle. Six ratios can be obtained using the lengths a, b, and c of the sides of the triangle:

$$\frac{b}{c}, \quad \frac{a}{c}, \quad \frac{b}{a}, \quad \frac{a}{b}, \quad \frac{c}{a}, \quad \frac{c}{b}$$

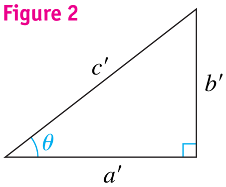
Figure 1



We can show that these ratios depend only on θ, and not on the size of the triangle, as indicated in Figure 2. Since the two triangles have equal angles, they are similar, and therefore ratios of corresponding sides are proportional. For example,

$$\frac{b}{c} = \frac{b'}{c'}, \quad \frac{a}{c} = \frac{a'}{c'}, \quad \frac{b}{a} = \frac{b'}{a'}.$$

Figure 2



Thus, for each θ, the six ratios are uniquely determined and hence are functions of θ. They are called the **trigonometric functions*** and are designated as the **sine, cosine, tangent, cotangent, secant,** and **cosecant** functions, abbreviated **sin, cos, tan, cot, sec,** and **csc,** respectively. The symbol $\sin(\theta)$, or $\sin\theta$, is used for the ratio b/c, which the sine function associates with θ. Values of the other five functions are denoted in similar fashion. To summarize, if θ is the acute angle of the right triangle in Figure 1, then, by definition,

$$\sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c} \qquad \tan\theta = \frac{b}{a}$$

$$\csc\theta = \frac{c}{b} \qquad \sec\theta = \frac{c}{a} \qquad \cot\theta = \frac{a}{b}.$$

*We will refer to these six trigonometric functions as **the** trigonometric functions. Here are some other, less common trigonometric functions that we will not use in this text:*

$$\text{vers }\theta = 1 - \cos\theta$$

$$\text{covers }\theta = 1 - \sin\theta$$

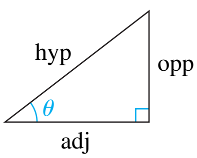
$$\text{exsec }\theta = \sec\theta - 1$$

$$\text{hav }\theta = \tfrac{1}{2}\text{ vers }\theta$$

The domain of each of the six trigonometric functions is the set of all acute angles. Later in this section we will extend the domains to larger sets of angles, and in the next section, to real numbers.

If θ is the angle in Figure 1, we refer to the sides of the triangle of lengths a, b, and c as the **adjacent side, opposite side,** and **hypotenuse,** respectively. We shall use **adj, opp,** and **hyp** to denote the lengths of the sides. We may then represent the triangle as in Figure 3. With this notation, the trigonometric functions may be expressed as follows.

Figure 3



Definition of the Trigonometric Functions of an Acute Angle of a Right Triangle

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The formulas in the preceding definition can be applied to any right triangle without attaching the labels a, b, c to the sides. Since the lengths of the sides of a triangle are positive real numbers, *the values of the six trigonometric functions are positive for every acute angle* θ. Moreover, the hypotenuse is always greater than the adjacent or opposite side, and hence $\sin\theta < 1$, $\cos\theta < 1$, $\csc\theta > 1$, and $\sec\theta > 1$ for every acute angle θ.

A mnemonic device for remembering the top row in the definition is

SOH CAH TOA,

where SOH is an abbreviation for <u>S</u>in θ = <u>O</u>pp/<u>H</u>yp, *and so forth.*

Note that since

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \text{and} \quad \csc\theta = \frac{\text{hyp}}{\text{opp}},$$

$\sin\theta$ and $\csc\theta$ are reciprocals of each other, giving us the two identities in the left-hand column of the next box. Similarly, $\cos\theta$ and $\sec\theta$ are reciprocals of each other, as are $\tan\theta$ and $\cot\theta$.

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

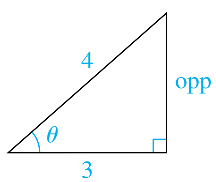
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Several other important identities involving the trigonometric functions will be discussed at the end of this section.

EXAMPLE 1 Finding trigonometric function values

If θ is an acute angle and $\cos\theta = \frac{3}{4}$, find the values of the trigonometric functions of θ.

Figure 4



SOLUTION We begin by sketching a right triangle having an acute angle θ with adj $= 3$ and hyp $= 4$, as shown in Figure 4, and proceed as follows:

$$3^2 + (\text{opp})^2 = 4^2 \qquad \text{Pythagorean theorem}$$
$$(\text{opp})^2 = 16 - 9 = 7 \qquad \text{isolate } (\text{opp})^2$$
$$\text{opp} = \sqrt{7} \qquad \text{take the square root}$$

Applying the definition of the trigonometric functions of an acute angle of a right triangle, we obtain the following:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{7}}{4} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{4} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4}{\sqrt{7}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{\sqrt{7}}$$

In Example 1 we could have rationalized the denominators for $\csc\theta$ and $\cot\theta$, writing

$$\csc\theta = \frac{4\sqrt{7}}{7} \quad \text{and} \quad \cot\theta = \frac{3\sqrt{7}}{7}.$$

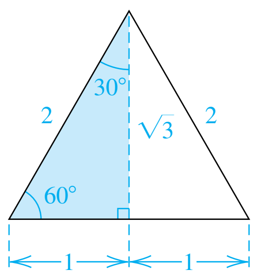
However, in most examples and exercises we will leave expressions in unrationalized form. An exception to this practice is the special trigonometric function values corresponding to 60°, 30°, and 45°, which are obtained in the following example.

EXAMPLE 2 **Finding trigonometric function values of 60°, 30°, and 45°**

Find the values of the trigonometric functions that correspond to θ:

(a) $\theta = 60°$ **(b)** $\theta = 30°$ **(c)** $\theta = 45°$

Figure 5

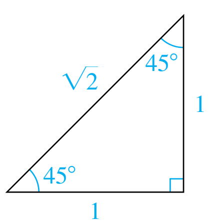


SOLUTION Consider an equilateral triangle with sides of length 2. The median from one vertex to the opposite side bisects the angle at that vertex, as illustrated by the dashes in Figure 5. By the Pythagorean theorem, the side opposite 60° in the shaded right triangle has length $\sqrt{3}$. Using the formulas for the trigonometric functions of an acute angle of a right triangle, we obtain the values corresponding to 60° and 30° as follows:

(a)
$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$
$$\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec 60° = \frac{2}{1} = 2 \qquad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

(b)
$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\csc 30° = \frac{2}{1} = 2 \qquad \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Figure 6



(c) To find the values for $\theta = 45°$, we may consider an isosceles right triangle whose two equal sides have length 1, as illustrated in Figure 6. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{2}$. Hence, the values corresponding to 45° are as follows:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos 45° \qquad \tan 45° = \frac{1}{1} = 1$$
$$\csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2} = \sec 45° \qquad \cot 45° = \frac{1}{1} = 1$$

For reference, we list the values found in Example 2, together with the radian measures of the angles, in the following table. Two reasons for stressing these values are that they are exact and that they occur frequently in work involving trigonometry. Because of the importance of these special values, it is a good idea either to memorize the table or to learn to find the values quickly by using triangles, as in Example 2.

Special Values of the Trigonometric Functions

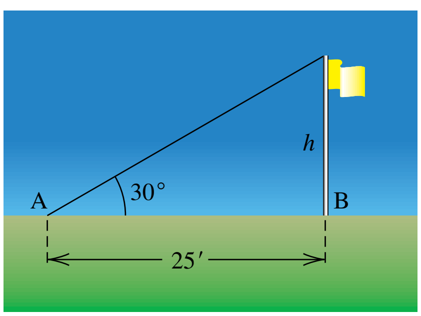
θ (radians)	θ (degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

The next example illustrates a practical use for trigonometric functions of acute angles. Additional applications involving right triangles will be considered in Section 6.7.

EXAMPLE 3 **Finding the height of a flagpole**

A surveyor observes that at a point A, located on level ground a distance 25.0 feet from the base B of a flagpole, the angle between the ground and the top of the pole is 30°. Approximate the height h of the pole to the nearest tenth of a foot.

Figure 7



SOLUTION Referring to Figure 7, we see that we want to relate the opposite side and the adjacent side, h and 25, respectively, to the 30° angle. This suggests that we use a trigonometric function involving those two sides—namely, tan or cot. It is usually easier to solve the problem if we select the function for which the variable is in the numerator. Hence, we have

$$\tan 30^\circ = \frac{h}{25} \quad \text{or, equivalently,} \quad h = 25\tan 30^\circ.$$

We use the value of tan 30° from Example 2 to find h:

$$h = 25\left(\frac{\sqrt{3}}{3}\right) \approx 14.4 \text{ ft}$$

It is possible to approximate, to any degree of accuracy, the values of the trigonometric functions for any acute angle. Calculators have keys labeled SIN, COS, and TAN that can be used to approximate values of these functions. The values of csc, sec, and cot may then be found by means of the reciprocal key. *Before using a calculator to find function values that correspond to the radian measure of an acute angle, be sure that the calculator is in radian mode. For values corresponding to degree measure, select degree mode.*

Figure 8
In degree mode

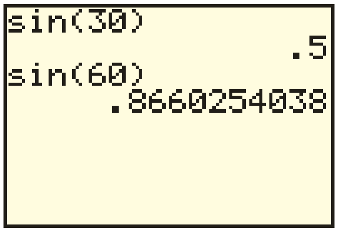
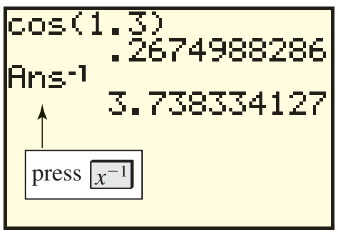


Figure 9
In radian mode



As an illustration (see Figure 8), to find sin 30° on a typical calculator, we place the calculator in degree mode and use the $\boxed{\text{SIN}}$ key to obtain $\sin 30° = 0.5$, which is the exact value. Using the same procedure for 60°, we obtain a decimal approximation to $\sqrt{3}/2$, such as

$$\sin 60° \approx 0.8660.$$

Most calculators give eight- to ten-decimal-place accuracy for such function values; throughout the text, however, we will usually round off values to four decimal places.

To find a value such as cos 1.3 (see Figure 9), where 1.3 is the radian measure of an acute angle, we place the calculator in radian mode and use the $\boxed{\text{COS}}$ key, obtaining

$$\cos 1.3 \approx 0.2675.$$

For sec 1.3, we could find cos 1.3 and then use the reciprocal key, usually labeled $\boxed{1/x}$ or $\boxed{x^{-1}}$ (as shown in Figure 9), to obtain

$$\sec 1.3 = \frac{1}{\cos 1.3} \approx 3.7383.$$

The formulas listed in the box on the next page are, without doubt, the most important identities in trigonometry, because they can be used to simplify and unify many different aspects of the subject. Since the formulas are part of the foundation for work in trigonometry, they are called the *fundamental identities*.

Three of the fundamental identities involve squares, such as $(\sin \theta)^2$ and $(\cos \theta)^2$. In general, if n is an integer different from -1, then a power such as $(\cos \theta)^n$ is written $\cos^n \theta$. The symbols $\sin^{-1} \theta$ and $\cos^{-1} \theta$ are reserved for inverse trigonometric functions, which we will discuss in Section 6.4 and treat thoroughly in the next chapter. With this agreement on notation, we have, for example,

$$\cos^2 \theta = (\cos \theta)^2 = (\cos \theta)(\cos \theta)$$

$$\tan^3 \theta = (\tan \theta)^3 = (\tan \theta)(\tan \theta)(\tan \theta)$$

$$\sec^4 \theta = (\sec \theta)^4 = (\sec \theta)(\sec \theta)(\sec \theta)(\sec \theta).$$

Let us next list all the fundamental identities and then discuss the proofs. These identities are true for every acute angle θ, and θ may take on various forms. For example, using the first Pythagorean identity with $\theta = 4\alpha$, we know that

$$\sin^2 4\alpha + \cos^2 4\alpha = 1.$$

We shall see later that these identities are also true for other angles and for real numbers.