10.7 Distinguishable Permutations and Combinations

Certain problems involve finding different arrangements of objects, some of which are indistinguishable. For example, suppose we are given five disks of the same size, of which three are black, one is white, and one is red. Let us find the number of ways they can be arranged in a row so that different color arrangements are obtained. If the disks were all different colors, then the number of arrangements would be 5!, or 120. However, since some of the disks have the same appearance, we cannot obtain 120 different arrangements. To clarify this point, let us write

B B B W R

for the arrangement having black disks in the first three positions in the row, the white disk in the fourth position, and the red disk in the fifth position. The first three disks can be arranged in 3!, or 6, different ways, but these arrangements cannot be distinguished from one another because the first three disks look alike. We say that those 3! permutations are **nondistinguishable.** Similarly, given any other arrangement, say

B R B W B,

there are 3! different ways of arranging the three black disks, but again each such arrangement is nondistinguishable from the others. Let us call two arrangements of objects **distinguishable permutations** if one arrangement cannot be obtained from the other by rearranging like objects. Thus, B B B W R and B R B W B are distinguishable permutations of the five disks. Let k denote the number of distinguishable permutations. Since to each such arrangement there correspond 3! *nondistinguishable* permutations, we must have $3!\,k = 5!$, the number of permutations of five *different* objects. Hence, $k = 5!/3! = 5 \cdot 4 = 20$. By the same type of reasoning we can obtain the following extension of this discussion.

First Theorem on Distinguishable Permutations

If r objects in a collection of n objects are alike and if the remaining objects are different from each other and from the r objects, then the number of distinguishable permutations of the n objects is

$$\frac{n!}{r!}.$$

We can generalize this theorem to the case in which there are several subcollections of nondistinguishable objects. For example, consider eight disks, of which four are black, three are white, and one is red. In this case, with each arrangement, such as

B W B W B W B R,

there are 4! arrangements of the black disks and 3! arrangements of the white disks that have no effect on the color arrangement. Hence, $4!\,3!$ possible arrangements of the disks will not produce distinguishable permutations. If we let k denote the number of *distinguishable* permutations, then $4!\,3!k = 8!$, since 8! is the number of permutations we would obtain if the disks were all different. Thus, the number of distinguishable permutations is

$$k = \frac{8!}{4!\,3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{3!} \cdot \frac{4!}{4!} = 280.$$

The following general result can be proved.

Second Theorem on Distinguishable Permutations

If, in a collection of n objects, n_1 are alike of one kind, n_2 are alike of another kind, . . . , n_k are alike of a further kind, and

$$n = n_1 + n_2 + \cdots + n_k,$$

then the number of distinguishable permutations of the n objects is

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_k!}.$$

EXAMPLE 1 **Finding a number of distinguishable permutations**

Find the number of distinguishable permutations of the letters in the word *Mississippi*.

SOLUTION In this example we are given a collection of eleven objects in which four are of one kind (the letter s), four are of another kind (i), two are of a third kind (p), and one is of a fourth kind (M). Hence, by the preceding theorem, we have $11 = 4 + 4 + 2 + 1$ and the number of distinguishable permutations is

$$\frac{11!}{4!\,4!\,2!\,1!} = 34{,}650.$$

When we work with permutations, our concern is with the orderings or arrangements of elements. Let us now ignore the order or arrangement of elements and consider the following question: Given a set containing n distinct elements, in how many ways can a subset of r elements be chosen with $r \leq n$? Before answering, let us state a definition.

Definition of Combination

Let S be a set of n elements and let $1 \leq r \leq n$. A **combination** of r elements of S is a subset of S that contains r distinct elements.

If S contains n elements, we also use the phrase **combination of *n* elements taken *r* at a time.** The symbol $C(n, r)$ will denote the number of combinations of r elements that can be obtained from a set of n elements.

Theorem on the Number of Combinations

The number of combinations of r elements that can be obtained from a set of n elements is

$$C(n, r) = \frac{n!}{(n-r)!\,r!}, \quad 1 \le r \le n.$$

The formula for $C(n, r)$ is identical to the formula for the binomial coefficient $\binom{n}{r}$ in Section 10.5.

PROOF If S contains n elements, then, to find $C(n, r)$, we must find the total number of subsets of the form

$$\{x_1, x_2, \ldots, x_r\}$$

such that the x_i are *different* elements of S. Since the r elements $x_1, x_2, \ldots, x_r$ can be arranged in $r!$ different ways, each such subset produces $r!$ different r-tuples. Thus, the total number of different r-tuples is $r!\,C(n, r)$. However, in the previous section we found that the total number of r-tuples is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Hence,
$$r!\,C(n, r) = \frac{n!}{(n-r)!}.$$

Dividing both sides of the last equation by $r!$ gives us the formula for $C(n, r)$.

From the proof, note that

$$P(n, r) = r!\,C(n, r),$$

which means that there are *more permutations than combinations* when we choose a subset of r elements from a set of n elements. To remember this relationship, consider a presidency, say Bush-Quayle. There is only one group or combination of these two people, but when a president–vice-president ordering is associated with these two people, there are two permutations, and Bush-Quayle is clearly different from Quayle-Bush.

As you read the examples and work the exercises, keep the following in mind.

If the order of selection *is* important, use a *permutation.*

If the order of selection *is not* important, use a *combination.*

EXAMPLE 2 **Choosing a baseball squad**

A little league baseball squad has six outfielders, seven infielders, five pitchers, and two catchers. Each outfielder can play any of the three outfield positions, and each infielder can play any of the four infield positions. In how many ways can a team of nine players be chosen?

Remember—if the order of selection can be ignored, use a combination.

SOLUTION The number of ways of choosing three outfielders from the six candidates is

$$C(6, 3) = \frac{6!}{(6-3)!\,3!} = \frac{6!}{3!\,3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3 \cdot 2 \cdot 1 \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20.$$

The number of ways of choosing the four infielders is

$$C(7, 4) = \frac{7!}{(7-4)!\,4!} = \frac{7!}{3!\,4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 1 \cdot 4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

There are five ways of choosing a pitcher and two choices for the catcher. It follows from the fundamental counting principle that the total number of ways to choose a team is

$$20 \cdot 35 \cdot 5 \cdot 2 = 7000.$$

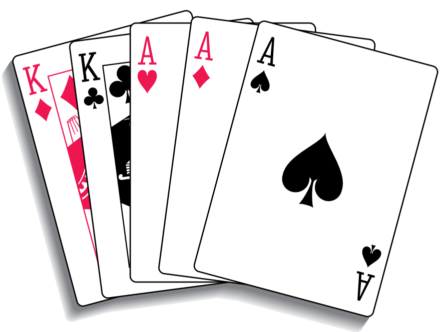
EXAMPLE 3 Being dealt a full house

In one type of poker, a five-card hand is dealt from a standard 52-card deck.

(a) How many hands are possible?

(b) A *full house* is a hand that consists of three cards of one denomination and two cards of another denomination. (The 13 denominations are 2's, 3's, 4's, 5's, 6's, 7's, 8's, 9's, 10's, J's, Q's, K's, and A's.) How many hands are full houses?

Figure 1



SOLUTION

(a) The order in which the five cards are dealt is not important, so we use a combination:

$$C(52, 5) = \frac{52!}{(52-5)!\,5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47! \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

(b) We first determine how many ways we can be dealt a specific full house—say 3 aces and 2 kings (see Figure 1). There are four cards of each denomination and the order of selection can be ignored, so we use combinations:

*The order of selection **is not** important, so use **combinations**.* →

$$\text{number of ways to get 3 A's} = C(4, 3)$$
$$\text{number of ways to get 2 K's} = C(4, 2)$$

Now we must pick the two denominations. Since 3 A's and 2 K's is a different full house than 3 K's and 2 A's, the order of selecting the denominations is important, and so we use a permutation:

*The order of selection **is** important, so use a **permutation**.* →

$$\text{number of ways to select two denominations} = P(13, 2)$$

By the fundamental counting principle, the number of full houses is

$$C(4, 3) \cdot C(4, 2) \cdot P(13, 2) = 4 \cdot 6 \cdot 156 = 3744.$$

Note that if $r = n$, the formula for $C(n, r)$ becomes

$$C(n, n) = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = \frac{n!}{1 \cdot n!} = 1.$$

It is convenient to assign a meaning to $C(n, r)$ if $r = 0$. If the formula is to be true in this case, then we must have

$$C(n, 0) = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!\,0!} = \frac{n!}{n! \cdot 1} = 1.$$

Hence, we *define* $C(n, 0) = 1$, which is the same as $C(n, n)$. Finally, for consistency, we also *define* $C(0, 0) = 1$. Thus, $C(n, r)$ has meaning for all nonnegative integers n and r with $r \leq n$.

EXAMPLE 4 Finding the number of subsets of a set

Let S be a set of n elements. Find the number of distinct subsets of S.

SOLUTION Let r be any nonnegative integer such that $r \leq n$. From our previous work, the number of subsets of S that consist of r elements is $C(n, r)$, or $\binom{n}{r}$. Hence, to find the total number of subsets, it suffices to find the sum

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}. \qquad (*)$$

Recalling the formula for the binomial theorem,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

we can see that the indicated sum $(*)$ is precisely the binomial expansion of $(1 + 1)^n$. Thus, there are 2^n subsets of a set of n elements. In particular, a set of 3 elements has 2^3, or 8, different subsets. A set of 4 elements has 2^4, or 16, subsets. A set of 10 elements has 2^{10}, or 1024, subsets.

Pascal's triangle, introduced in Section 10.5, can easily be remembered by the following combination form:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Combining this information with that in Example 4, we conclude that the third coefficient in the expansion of $(a + b)^4$, $\binom{4}{2}$, is exactly the same as the number of two-element subsets of a set that contains four elements. We leave it as an exercise to find a generalization of the last statement (see Discussion Exercise 4 at the end of the chapter).

10.7 *Exercises*

Exer. 1–8: Find the number.

1 $C(7, 3)$

2 $C(8, 4)$

3 $C(9, 8)$

4 $C(6, 2)$

5 $C(n, n - 1)$

6 $C(n, 1)$

7 $C(7, 0)$

8 $C(5, 5)$

Exer. 9–10: Find the number of possible color arrangements for the 12 given disks, arranged in a row.

9 5 black, 3 red, 2 white, 2 green

10 3 black, 3 red, 3 white, 3 green

11 Find the number of distinguishable permutations of the letters in the word *bookkeeper*.

12 Find the number of distinguishable permutations of the letters in the word *moon*. List all the permutations.

13 **Choosing basketball teams** Ten people wish to play in a basketball game. In how many different ways can two teams of five players each be formed?

14 **Selecting test questions** A student may answer any six of ten questions on an examination.

(a) In how many ways can six questions be selected?

(b) How many selections are possible if the first two questions must be answered?

Exer. 15–16: Consider any eight points such that no three are collinear.

15 How many lines are determined?

16 How many triangles are determined?

17 **Book arrangement** A student has five mathematics books, four history books, and eight fiction books. In how many different ways can they be arranged on a shelf if books in the same category are kept next to one another?

18 **Selecting a basketball team** A basketball squad consists of twelve players.

(a) Disregarding positions, in how many ways can a team of five be selected?

(b) If the center of a team must be selected from two specific individuals on the squad and the other four members of the team from the remaining ten players, find the number of different teams possible.

19 **Selecting a football team** A football squad consists of three centers, ten linemen who can play either guard or tackle, three quarterbacks, six halfbacks, four ends, and four fullbacks. A team must have one center, two guards, two tackles, two ends, two halfbacks, a quarterback, and a fullback. In how many different ways can a team be selected from the squad?

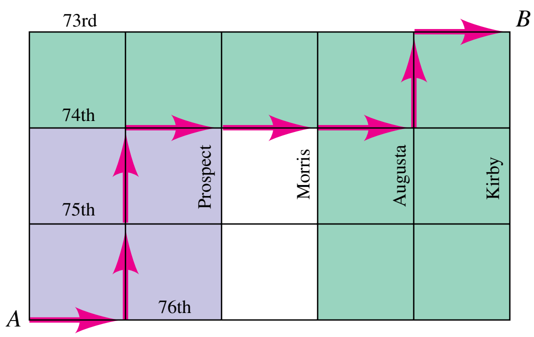
20 **Arranging keys on a ring** In how many different ways can seven keys be arranged on a key ring if the keys can slide completely around the ring?

21 **Committee selection** A committee of 3 men and 2 women is to be chosen from a group of 12 men and 8 women. Determine the number of different ways of selecting the committee.

22 **Birth order** Let the letters G and B denote a girl birth and a boy birth, respectively. For a family of three boys and three girls, one possible birth order is G G G B B B. How many birth orders are possible for these six children?

Exer. 23–24: Shown in each figure is a street map and a possible path from point *A* to point *B*. How many possible paths are there from *A* to *B* if moves are restricted to the right or up? (*Hint:* If R denotes a move one unit right and U denotes a move one unit up, then the path in Exercise 23 can be specified by R U U R R R U R.)

23



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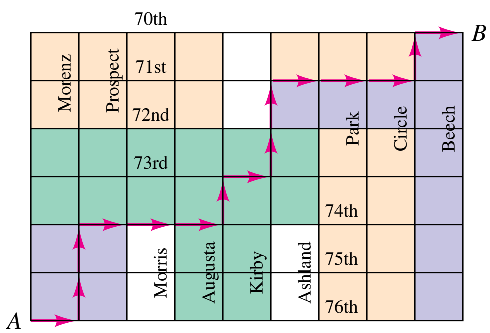


25 **Lotto selections** To win a state lottery game, a player must correctly select six numbers from the numbers 1 through 49.

(a) Find the total number of selections possible.

(b) Work part (a) if a player selects only even numbers.

26 **Office assignments** A mathematics department has ten faculty members but only nine offices, so one office must be shared by two individuals. In how many different ways can the offices be assigned?

27 **Tennis tournament** In a round-robin tennis tournament, every player meets every other player exactly once. How many players can participate in a tournament of 45 matches?

28 **True-or-false test** A true-or-false test has 20 questions.

(a) In how many different ways can the test be completed?

(b) In how many different ways can a student answer 10 questions correctly?

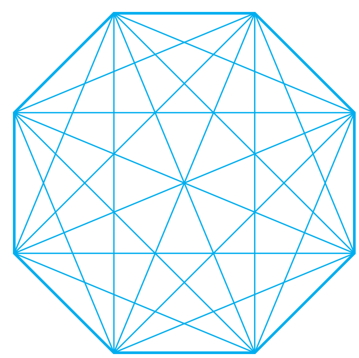
29 **Basketball championship series** The winner of the seven-game NBA championship series is the team that wins four games. In how many different ways can the series be extended to seven games?

30 A geometric design is determined by joining every pair of vertices of an octagon (see the figure).

(a) How many triangles in the design have their three vertices on the octagon?

(b) How many quadrilaterals in the design have their four vertices on the octagon?

Exercise 30



31 **Ice cream selections** An ice cream parlor stocks 31 different flavors and advertises that it serves almost 4500 different triple scoop cones, with each scoop being a different flavor. How was this number obtained?

32 **Choices of hamburger condiments** A fast food restaurant advertises that it offers any combination of 8 condiments on a hamburger, thus giving a customer 256 choices. How was this number obtained?

33 **Scholarship selection** A committee is going to select 30 students from a pool of 1000 to receive scholarships. How may ways could the students be selected if each scholarship is worth

(a) the same amount?

(b) a different amount?