Exer. 49–50: Simplify the expression using the binomial theorem.

49 $\dfrac{(x + h)^4 - x^4}{h}$

50 $\dfrac{(x + h)^5 - x^5}{h}$

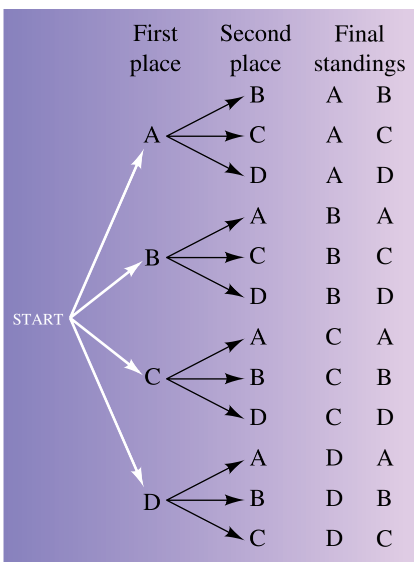
51 Show that $\dbinom{n}{1} = \dbinom{n}{n - 1}$ for $n \geq 1$.

52 Show that $\dbinom{n}{0} = \dbinom{n}{n}$ for $n \geq 0$.

10.6 Permutations

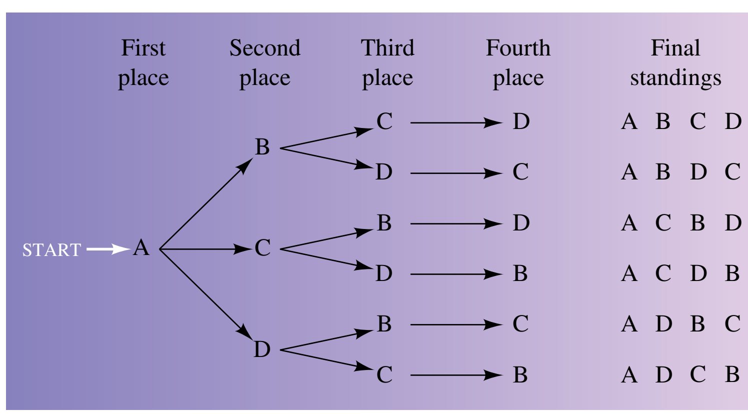
Suppose that four teams are involved in a tournament in which first, second, third, and fourth places will be determined. For identification purposes, we label the teams A, B, C, and D. Let us find the number of different ways that first and second place can be decided. It is convenient to use a **tree diagram,** as in Figure 1. After the word START, the four possibilities for first place are listed. From each of these an arrow points to a possible second-place finisher. The final standings list the possible outcomes, from left to right. They are found by following the different paths (*branches* of the tree) that lead from the word START to the second-place team. The total number of outcomes is 12, which is the product of the number of choices (4) for first place and the number of choices (3) for second place (after first has been determined).

Figure 1



Let us now find the total number of ways that first, second, third, and fourth positions can be filled. To sketch a tree diagram, we may begin by drawing arrows from the word START to each possible first-place finisher A, B, C, or D. Next we draw arrows from those to possible second-place finishers, as was done in Figure 1. From each second-place position we then draw arrows indicating the possible third-place positions. Finally, we draw arrows to the fourth-place team. If we consider only the case in which team A finishes in first place, we have the diagram shown in Figure 2.

Figure 2



Note that there are six possible final standings in which team A occupies first place. In a complete tree diagram there would also be three other branches of this type corresponding to first-place finishes for B, C, and D. A complete diagram would display the following 24 possibilities for the final standings:

A first	ABCD, ABDC, ACBD, ACDB, ADBC, ADCB,
B first	BACD, BADC, BCAD, BCDA, BDAC, BDCA,
C first	CABD, CADB, CBAD, CBDA, CDAB, CDBA,
D first	DABC, DACB, DBAC, DBCA, DCAB, DCBA.

Note that the number of possibilities (24) is the product of the number of ways (4) that first place may occur, the number of ways (3) that second place may occur (after first place has been determined), the number of possible outcomes (2) for third place (after the first two places have been decided), and the number of ways (1) that fourth place can occur (after the first three places have been taken).

The preceding discussion illustrates the following general rule, which we accept as a basic axiom of counting.

Fundamental Counting Principle

Let $E_1, E_2, \ldots, E_k$ be a sequence of k events. If, for each i, the event E_i can occur in m_i ways, then the total number of ways all the events may take place is the product $m_1 m_2 \cdots m_k$.

Returning to our first illustration, we let E_1 represent the determination of the first-place team, so that $m_1 = 4$. If E_2 denotes the determination of the second-place team, then $m_2 = 3$. Hence, the number of outcomes for the sequence E_1, E_2 is $4 \cdot 3 = 12$, which is the same as that found by means of the tree diagram. If we proceed to E_3, the determination of the third-place team, then $m_3 = 2$, and hence $m_1 m_2 m_3 = 24$. Finally, if E_1, E_2, and E_3 have occurred, there is only one possible outcome for E_4. Thus, $m_4 = 1$, and $m_1 m_2 m_3 m_4 = 24$.

Instead of teams, let us now regard *a, b, c,* and *d* merely as symbols and consider the various *orderings,* or *arrangements,* that may be assigned to these symbols, taking them either two at a time, three at a time, or four at a time. By abstracting in this way we may apply our methods to other similar situations. The arrangements we have discussed are **arrangements without repetitions,** since a symbol may not be used more than once in an arrangement. In Example 1 we shall consider arrangements in which repetitions *are* allowed.

Previously we defined ordered pairs and ordered triples. Similarly, an *ordered 4-tuple* is a set containing four elements x_1, x_2, x_3, x_4 in which an ordering has been specified, so that one of the elements may be referred to as the *first element,* another as the *second element,* and so on. The symbol (x_1, x_2, x_3, x_4) is used for the ordered 4-tuple having first element x_1, second element x_2, third element x_3, and fourth element x_4. In general, for any positive integer *r,* we speak of the **ordered *r*-tuple**

$$(x_1, x_2, \ldots, x_r)$$

as a set of r elements in which x_1 is designated as the first element, x_2 as the second element, and so on.

EXAMPLE 1 **Determining the number of *r*-tuples**

Using only the letters *a, b, c,* and *d,* determine how many of the following can be obtained:

(a) ordered triples **(b)** ordered 4-tuples **(c)** ordered r-tuples

SOLUTION

(a) We must determine the number of symbols of the form (x_1, x_2, x_3) that can be obtained using only the letters *a, b, c,* and *d.* This is not the same as listing first, second, and third place as in our previous illustration, since we have not ruled out the possibility of repetitions. For example, (a, b, a), (a, a, b), and (a, a, a) are different ordered triples. If, for $i = 1, 2, 3$, we let E_i represent the determination of x_i in the ordered triple (x_1, x_2, x_3), then, since repetitions are allowed, there are four possibilities—*a, b, c,* and *d*—for each of E_1, E_2, and E_3. Hence, by the fundamental counting principle, the total number of ordered triples is $4 \cdot 4 \cdot 4$, or 64.

(b) The number of possible ordered 4-tuples of the form (x_1, x_2, x_3, x_4) is $4 \cdot 4 \cdot 4 \cdot 4$, or 256.

(c) The number of ordered r-tuples is the product $4 \cdot 4 \cdot 4 \cdot \cdots \cdot 4$, with 4 appearing as a factor r times. That product equals 4^r.

EXAMPLE 2 **Choosing class officers**

A class consists of 60 girls and 40 boys. In how many ways can a president, vice-president, treasurer, and secretary be chosen if the treasurer must be a girl, the secretary must be a boy, and a student may not hold more than one office?

SOLUTION If an event is specialized in some way (for example, the treasurer *must* be a girl), then that event should be considered before any nonspecialized events. Thus, we let E_1 represent the choice of treasurer and E_2 the choice of secretary. Next we let E_3 and E_4 denote the choices for president and vice-president, respectively. As in the fundamental counting principle, we let m_i denote the number of different ways E_i can occur for $i = 1, 2, 3$, and 4. It follows that $m_1 = 60$, $m_2 = 40$, $m_3 = 60 + 40 - 2 = 98$, and $m_4 = 97$. By the fundamental counting principle, the total number of possibilities is

$$m_1 m_2 m_3 m_4 = 60 \cdot 40 \cdot 98 \cdot 97 = 22{,}814{,}400.$$

When working with sets, we are usually not concerned about the order or arrangement of the elements. In the remainder of this section, however, the arrangement of the elements will be our main concern.

Definition of Permutation	Let S be a set of n elements and let $1 \le r \le n$. A **permutation** of r elements of S is an arrangement, without repetitions, of r elements.

We also use the phrase **permutation of *n* elements taken *r* at a time.** The symbol $P(n, r)$ will denote the number of different permutations of r elements that can be obtained from a set of n elements. As a special case, $P(n, n)$ denotes the number of arrangements of n elements of S—that is, the number of ways of arranging *all* the elements of S.

In our first discussion involving the four teams A, B, C, and D, we had $P(4, 2) = 12$, since there are 12 different ways of arranging the four teams in groups of two. We also showed that the number of ways to arrange all the elements A, B, C, and D is 24. In permutation notation we would write this result as $P(4, 4) = 24$.

The next theorem gives us a general formula for $P(n, r)$.

Theorem on the Number of Different Permutations	Let S be a set of n elements and let $1 \le r \le n$. The number of different permutations of r elements of S is $$P(n, r) = n(n-1)(n-2)\cdots(n-r+1).$$

PROOF The problem of determining $P(n, r)$ is equivalent to determining the number of different r-tuples $(x_1, x_2, \ldots, x_r)$ such that each x_i is an element of S and no element of S appears twice in the same r-tuple. We may find this number by means of the fundamental counting principle. For each $i = 1, 2, \ldots, r$, let E_i represent the determination of the element x_i and let m_i be the number of different ways of choosing x_i. We wish to apply the sequence $E_1, E_2, \ldots, E_r$. We have n possible choices for x_1, and consequently $m_1 = n$. Since repetitions are not allowed, we have $n - 1$ choices for x_2, so $m_2 = n - 1$. Continuing in this manner, we successively obtain $m_3 = n - 2$, $m_4 = n - 3$, and ultimately $m_r = n - (r - 1)$ or, equivalently, $m_r = n - r + 1$. Hence, using the fundamental counting principle, we obtain the formula for $P(n, r)$.

Note that *the formula for $P(n, r)$ in the previous theorem contains exactly r factors on the right-hand side,* as shown in the following illustration.

ILLUSTRATION **Number of Different Permutations**

- $P(n, 1) = n$
- $P(n, 2) = n(n-1)$
- $P(n, 3) = n(n-1)(n-2)$
- $P(n, 4) = n(n-1)(n-2)(n-3)$

EXAMPLE 3 **Evaluating $P(n, r)$**

Find $P(5, 2)$, $P(6, 4)$, and $P(5, 5)$.

SOLUTION We will use the formula for $P(n, r)$ in the preceding theorem. In each case, we first calculate the value of $(n - r + 1)$.

$$5 - 2 + 1 = \underline{4}, \quad \text{so} \quad P(5, 2) = 5 \cdot \underline{4} = 20$$
$$6 - 4 + 1 = \underline{3}, \quad \text{so} \quad P(6, 4) = 6 \cdot 5 \cdot 4 \cdot \underline{3} = 360$$
$$5 - 5 + 1 = \underline{1}, \quad \text{so} \quad P(5, 5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot \underline{1} = 120$$

EXAMPLE 4 **Arranging the batting order for a baseball team**

A baseball team consists of nine players. Find the number of ways of arranging the first four positions in the batting order if the pitcher is excluded.

SOLUTION We wish to find the number of permutations of 8 objects taken 4 at a time. Using the formula for $P(n, r)$ with $n = 8$ and $r = 4$, we have $n - r + 1 = 5$, and it follows that

$$P(8, 4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

The next result gives us a form for $P(n, r)$ that involves the factorial symbol.

Factorial Form for $P(n, r)$

If n is a positive integer and $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n - r)!}.$$

PROOF If we let $r = n$ in the formula for $P(n, r)$ in the theorem on permutations, we obtain the number of different arrangements of *all* the elements of a set consisting of n elements. In this case,

$$n - r + 1 = n - n + 1 = 1$$

and

$$P(n, n) = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!.$$

Consequently, $P(n, n)$ is the product of the first n positive integers. This result is also given by the factorial form, for if $r = n$, then

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

If $1 \leq r < n$, then

$$\frac{n!}{(n - r)!} = \frac{n(n - 1)(n - 2) \cdots (n - r + 1) \cdot [(n - r)!]}{(n - r)!}$$
$$= n(n - 1)(n - 2) \cdots (n - r + 1).$$

This agrees with the formula for $P(n, r)$ in the theorem on permutations.

EXAMPLE 5 Evaluating $P(n, r)$ using factorials

Use the factorial form for $P(n, r)$ to find $P(5, 2)$, $P(6, 4)$, and $P(5, 5)$.

SOLUTION

$$P(5, 2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 5 \cdot 4 = 20$$

$$P(6, 4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

$$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

10.6 *Exercises*

Exer. 1–8: Find the number.

1 $P(7, 3)$ 2 $P(8, 5)$

3 $P(9, 6)$ 4 $P(5, 3)$

5 $P(5, 5)$ 6 $P(4, 4)$

7 $P(6, 1)$ 8 $P(5, 1)$

Exer. 9–12: Simplify the permutation.

9 $P(n, 0)$ 10 $P(n, 1)$

11 $P(n, n - 1)$ 12 $P(n, 2)$

13 How many three-digit numbers can be formed from the digits 1, 2, 3, 4, and 5 if repetitions

(a) are not allowed? (b) are allowed?

14 Work Exercise 13 for four-digit numbers.

15 How many numbers can be formed from the digits 1, 2, 3, and 4 if repetitions are not allowed? (*Note:* 42 and 231 are examples of such numbers.)

16 Determine the number of positive integers less than 10,000 that can be formed from the digits 1, 2, 3, and 4 if repetitions are allowed.

17 **Basketball standings** If eight basketball teams are in a tournament, find the number of different ways that first, second, and third place can be decided, assuming ties are not allowed.

18 **Basketball standings** Work Exercise 17 for 12 teams.

19 **Wardrobe mix 'n' match** A girl has four skirts and six blouses. How many different skirt-blouse combinations can she wear?

20 **Wardrobe mix 'n' match** Refer to Exercise 19. If the girl also has three sweaters, how many different skirt-blouse-sweater combinations can she wear?

21 **License plate numbers** In a certain state, automobile license plates start with one letter of the alphabet, followed by five digits $(0, 1, 2, \ldots, 9)$. Find how many different license plates are possible if

(a) the first digit following the letter cannot be 0

(b) the first letter cannot be O or I and the first digit cannot be 0

22 **Tossing dice** Two dice are tossed, one after the other. In how many different ways can they fall? List the number of different ways the sum of the dots can equal

(a) 3 (b) 5 (c) 7 (d) 9 (e) 11

23 **Seating arrangement** A row of six seats in a classroom is to be filled by selecting individuals from a group of ten students.

(a) In how many different ways can the seats be occupied?

(b) If there are six boys and four girls in the group and if boys and girls are to be alternated, find the number of different seating arrangements.

24 Scheduling courses A student in a certain college may take mathematics at 8, 10, 11, or 2 o'clock; English at 9, 10, 1, or 2; and history at 8, 11, 2, or 3. Find the number of different ways in which the student can schedule the three courses.

25 True-or-false test In how many different ways can a test consisting of ten true-or-false questions be completed?

26 Multiple-choice test A test consists of six multiple-choice questions, and there are five choices for each question. In how many different ways can the test be completed?

27 Seating arrangement In how many different ways can eight people be seated in a row?

28 Book arrangement In how many different ways can ten books be arranged on a shelf?

29 Semaphore With six different flags, how many different signals can be sent by placing three flags, one above the other, on a flag pole?

30 Selecting books In how many different ways can five books be selected from a twelve-volume set of books?

31 Radio call letters How many four-letter radio station call letters can be formed if the first letter must be K or W and repetitions

(a) are not allowed? (b) are allowed?

32 Fraternity designations There are 24 letters in the Greek alphabet. How many fraternities may be specified by choosing three Greek letters if repetitions

(a) are not allowed? (b) are allowed?

33 Phone numbers How many ten-digit phone numbers can be formed from the digits 0, 1, 2, 3, . . . , 9 if the first digit may not be 0?

34 Baseball batting order After selecting nine players for a baseball game, the manager of the team arranges the batting order so that the pitcher bats last and the best hitter bats third. In how many different ways can the remainder of the batting order be arranged?

35 ATM access code A customer remembers that 2, 4, 7, and 9 are the digits of a four-digit access code for an automatic bank-teller machine. Unfortunately, the customer has forgotten the order of the digits. Find the largest possible number of trials necessary to obtain the correct code.

36 ATM access code Work Exercise 35 if the digits are 2, 4, and 7 and one of these digits is repeated in the four-digit code.

37 Selecting theater seats Three married couples have purchased tickets for a play. Spouses are to be seated next to each other, and the six seats are in a row. In how many ways can the six people be seated?

38 Horserace results Ten horses are entered in a race. If the possibility of a tie for any place is ignored, in how many ways can the first-, second-, and third-place winners be determined?

39 Lunch possibilities Owners of a restaurant advertise that they offer 1,114,095 different lunches based on the fact that they have 16 "free fixins" to go along with any of their 17 menu items (sandwiches, hot dogs, and salads). How did they arrive at that number?

40 Shuffling cards

(a) In how many ways can a standard deck of 52 cards be shuffled?

(b) In how many ways can the cards be shuffled so that the four aces appear on the top of the deck?

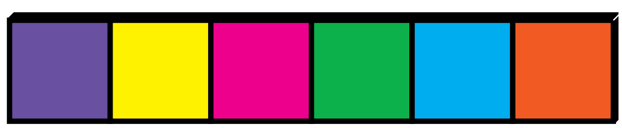
41 Numerical palindromes A palindrome is an integer, such as 45654, that reads the same backward and forward.

(a) How many five-digit palindromes are there?

(b) How many n-digit palindromes are there?

42 Color arrangements Each of the six squares shown in the figure is to be filled with any one of ten possible colors. How many ways are there of coloring the strip shown in the figure so that no two adjacent squares have the same color?

Exercise 42



43 The graph of

$$y = \frac{x!\, e^x}{x^x \sqrt{2\pi x}}$$

has a horizontal asymptote of $y = 1$. Use this fact to find an approximation for $n!$ if n is a large positive integer.

44 (a) What happens if a calculator is used to find $P(150, 50)$? Explain.

(b) Approximate r if $P(150, 50) = 10^r$ by using the following formula from advanced mathematics:

$$\log n! \approx \frac{n \ln n - n}{\ln 10}$$