B Binomial Theorem

In Chapter 4, when multiplying polynomials, we developed patterns for squaring and cubing binomials. Now we want to develop a general pattern that can be used to raise a binomial to any positive integral power. Let's begin by looking at some specific expansions that can be verified by direct multiplication. (Note that the patterns for squaring and cubing a binomial are a part of this list.)

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

First, note the pattern of the exponents for x and y on a term-by-term basis. The exponents of x begin with the exponent of the binomial and decrease by 1, term by term, until the last term has x^0, which is 1. The exponents of y begin with zero ($y^0 = 1$) and increase by 1, term by term, until the last term contains y to the power of the binomial. In other words, the variables in the expansion of $(x + y)^n$ have the following pattern.

$$x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad x^{n-3}y^3, \quad \ldots\ldots, \quad xy^{n-1}, \quad y^n$$

Note that for each term, the sum of the exponents of x and y is n.

Now let's look for a pattern for the coefficients by examining specifically the expansion of $(x + y)^5$.

$$(x + y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5$$

$C(5, 1)$ $C(5, 2)$ $C(5, 3)$ $C(5, 4)$ $C(5, 5)$

As indicated by the arrows, the coefficients are numbers that arise as different-sized combinations of five things. To see why this happens, consider the coefficient for the term containing x^3y^2. The two y's (for y^2) come from two of the factors of $(x + y)$, and therefore the three x's (for x^3) must come from the other three factors of $(x + y)$. In other words, the coefficient is $C(5, 2)$.

We can now state a general expansion formula for $(x + y)^n$; this formula is often called the **binomial theorem**. But before stating it, let's make a small switch in notation. Instead of $C(n, r)$, we shall write $\binom{n}{r}$, which will prove to be a little more convenient at this time. The symbol $\binom{n}{r}$, still refers to the number of combinations of n things taken r at a time, but in this context, it is called a **binomial coefficient**.

Binomial Theorem

For any binomial $(x + y)$ and any natural number n,

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

The binomial theorem can be proved by mathematical induction, but we will not do that in this text. Instead, we'll consider a few examples that put the binomial theorem to work.

EXAMPLE 1 Expand $(x + y)^7$.

Solution

$$(x + y)^7 = x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4$$
$$+ \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7$$
$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

EXAMPLE 2 Expand $(x - y)^5$.

Solution

We shall treat $(x - y)^5$ as $[x + (-y)]^5$:

$$[x + (-y)]^5 = x^5 + \binom{5}{1}x^4(-y) + \binom{5}{2}x^3(-y)^2 + \binom{5}{3}x^2(-y)^3$$
$$+ \binom{5}{4}x(-y)^4 + \binom{5}{5}(-y)^5$$
$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

EXAMPLE 3 Expand $(2a + 3b)^4$.

Solution

Let $x = 2a$ and $y = 3b$ in the binomial theorem:

$$(2a + 3b)^4 = (2a)^4 + \binom{4}{1}(2a)^3(3b) + \binom{4}{2}(2a)^2(3b)^2$$
$$+ \binom{4}{3}(2a)(3b)^3 + \binom{4}{4}(3b)^4$$
$$= 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$$

EXAMPLE 4 Expand $\left(a + \frac{1}{n}\right)^5$.

Solution

$$\left(a + \frac{1}{n}\right)^5 = a^5 + \binom{5}{1}a^4\left(\frac{1}{n}\right) + \binom{5}{2}a^3\left(\frac{1}{n}\right)^2 + \binom{5}{3}a^2\left(\frac{1}{n}\right)^3 + \binom{5}{4}a\left(\frac{1}{n}\right)^4 + \binom{5}{5}\left(\frac{1}{n}\right)^5$$
$$= a^5 + \frac{5a^4}{n} + \frac{10a^3}{n^2} + \frac{10a^2}{n^3} + \frac{5a}{n^4} + \frac{1}{n^5}$$

EXAMPLE 5 Expand $(x^2 - 2y^3)^6$.

Solution

$$[x^2 + (-2y^3)]^6 = (x^2)^6 + \binom{6}{1}(x^2)^5(-2y^3) + \binom{6}{2}(x^2)^4(-2y^3)^2$$
$$+ \binom{6}{3}(x^2)^3(-2y^3)^3 + \binom{6}{4}(x^2)^2(-2y^3)^4$$
$$+ \binom{6}{5}(x^2)(-2y^3)^5 + \binom{6}{6}(-2y^3)^6$$
$$= x^{12} - 12x^{10}y^3 + 60x^8y^6 - 160x^6y^9 + 240x^4y^{12} - 192x^2y^{15} + 64y^{18}$$

Finding Specific Terms

Sometimes it is convenient to be able to write down the specific term of a binomial expansion without writing out the entire expansion. For example, suppose that we want the sixth term of the expansion $(x + y)^{12}$. We can proceed as follows: The sixth term will contain y^5. (Note in the binomial theorem that the *exponent of y is always one less than the number of the term*.) Because the sum of the exponents for x and y must be 12 (the exponent of the binomial), the sixth term will also contain x^7. The coefficient is $\binom{12}{5}$, and the 5 agrees with the exponent of y^5. Therefore the sixth term of $(x + y)^{12}$ is

$$\binom{12}{5}x^7y^5 = 792x^7y^5$$

EXAMPLE 6 Find the fourth term of $(3a + 2b)^7$.

Solution

The fourth term will contain $(2b)^3$, and therefore it will also contain $(3a)^4$. The coefficient is $\binom{7}{3}$. Thus the fourth term is

$$\binom{7}{3}(3a)^4(2b)^3 = (35)(81a^4)(8b^3) = 22{,}680a^4b^3$$

EXAMPLE 7 Find the sixth term of $(4x - y)^9$.

Solution

The sixth term will contain $(-y)^5$, and therefore it will also contain $(4x)^4$. The coefficient is $\binom{9}{5}$. Thus the sixth term is

$$\binom{9}{5}(4x)^4(-y)^5 = (126)(256x^4)(-y^5) = -32{,}256x^4y^5$$

Practice Exercises

For Problems 1–26, expand and simplify each binomial.

1. $(x + y)^8$
2. $(x + y)^9$
3. $(x - y)^6$
4. $(x - y)^4$
5. $(a + 2b)^4$
6. $(3a + b)^4$
7. $(x - 3y)^5$
8. $(2x - y)^6$
9. $(2a - 3b)^4$
10. $(3a - 2b)^5$
11. $(x^2 + y)^5$
12. $(x + y^3)^6$
13. $(2x^2 - y^2)^4$
14. $(3x^2 - 2y^2)^5$
15. $(x + 3)^6$
16. $(x + 2)^7$
17. $(x - 1)^9$
18. $(x - 3)^4$
19. $\left(1 + \frac{1}{n}\right)^4$
20. $\left(2 + \frac{1}{n}\right)^5$
21. $\left(a - \frac{1}{n}\right)^6$
22. $\left(2a - \frac{1}{n}\right)^5$
23. $(1 + \sqrt{2})^4$
24. $(2 + \sqrt{3})^3$
25. $(3 - \sqrt{2})^5$
26. $(1 - \sqrt{3})^4$

For Problems 27–36, write the first four terms of each expansion.

27. $(x + y)^{12}$
28. $(x + y)^{15}$
29. $(x - y)^{20}$
30. $(a - 2b)^{13}$
31. $(x^2 - 2y^3)^{14}$
32. $(x^3 - 3y^2)^{11}$
33. $\left(a + \frac{1}{n}\right)^9$
34. $\left(2 - \frac{1}{n}\right)^6$
35. $(-x + 2y)^{10}$
36. $(-a - b)^{14}$

For Problems 37–46, find the specified term for each binomial expansion.

37. The fourth term of $(x + y)^8$
38. The seventh term of $(x + y)^{11}$
39. The fifth term of $(x - y)^9$
40. The fourth term of $(x - 2y)^6$
41. The sixth term of $(3a + b)^7$
42. The third term of $(2x - 5y)^5$
43. The eighth term of $(x^2 + y^3)^{10}$
44. The ninth term of $(a + b^3)^{12}$
45. The seventh term of $\left(1 - \frac{1}{n}\right)^{15}$
46. The eighth term of $\left(1 - \frac{1}{n}\right)^{13}$