12.1 Algebra of 2 × 2 Matrices

OBJECTIVES

1 Add and subtract matrices

2 Multiply a matrix by a scalar

3 Multiply 2 × 2 matrices

Throughout these next two sections, we will be working primarily with 2 × 2 matrices; therefore any reference to matrices means 2 × 2 matrices unless stated otherwise. The following 2 × 2 matrix notation will be used frequently.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Two matrices are **equal** if and only if all elements in corresponding positions are equal. Thus $A = B$ if and only if $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$, and $a_{22} = b_{22}$.

Addition of Matrices

To **add** two matrices, we add the elements that appear in corresponding positions. Therefore the sum of matrix A and matrix B is defined as follows:

Definition 12.1

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

For example,

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -4 & 11 \end{bmatrix}$$

It is not difficult to show that the **commutative** and **associative properties** are valid for the addition of matrices. Thus we can state that

$$A + B = B + A \qquad \text{and} \qquad (A + B) + C = A + (B + C)$$

Because

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we see that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is called the **zero matrix**, represented by O, is the **additive identity element**. Thus we can state that

$$A + O = O + A = A$$

Because every real number has an additive inverse, it follows that any matrix A has an **additive inverse**, $-A$, which is formed by taking the additive inverse of each element of A. For example, if

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 0 \end{bmatrix} \quad \text{then } -A = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$$

and

$$A + (-A) = \begin{bmatrix} 4 & -2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In general, we can state that every matrix A has an additive inverse $-A$ such that

$$A + (-A) = (-A) + A = O$$

Subtraction of Matrices

Like the algebra of real numbers, **subtraction** of matrices can be defined in terms of *adding the additive inverse*. Therefore we can define subtraction as follows:

Definition 12.2

$$A - B = A + (-B)$$

For example,

$$\begin{bmatrix} 2 & -7 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ -6 & 5 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -11 \\ -4 & 6 \end{bmatrix}$$

Scalar Multiplication

When we work with matrices, we commonly refer to a single real number as a **scalar** to distinguish it from a matrix. So, to get the **product** of a scalar and a matrix (often referred to as **scalar multiplication**) we would multiply each element of the matrix by the scalar. For example,

$$3\begin{bmatrix} -4 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3(-4) & 3(-6) \\ 3(1) & 3(-2) \end{bmatrix} = \begin{bmatrix} -12 & -18 \\ 3 & -6 \end{bmatrix}$$

In general, scalar multiplication can be defined as follows:

Definition 12.3

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

where k is any real number.

Classroom Example

If $A = \begin{bmatrix} -3 & 2 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -6 & 5 \end{bmatrix}$, find:

(a) $-3B$
(b) $2A + 5B$
(c) $3A - B$

EXAMPLE 1

If $A = \begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 7 & -6 \end{bmatrix}$, find (a) $-2A$, (b) $3A + 2B$, and (c) $A - 4B$.

Solutions

(a) $$-2A = -2\begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ -4 & 10 \end{bmatrix}$$

(b) $$\begin{aligned} 3A + 2B &= 3\begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix} + 2\begin{bmatrix} 2 & -3 \\ 7 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -12 & 9 \\ 6 & -15 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ 14 & -12 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 3 \\ 20 & -27 \end{bmatrix} \end{aligned}$$

(c) $$\begin{aligned} A - 4B &= \begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix} - 4\begin{bmatrix} 2 & -3 \\ 7 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix} - \begin{bmatrix} 8 & -12 \\ 28 & -24 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 3 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -8 & 12 \\ -28 & 24 \end{bmatrix} \\ &= \begin{bmatrix} -12 & 15 \\ -26 & 19 \end{bmatrix} \end{aligned}$$

The following properties, which are easy to check, pertain to scalar multiplication and matrix addition (where k and l represent any real numbers):

$$k(A + B) = kA + kB$$
$$(k + l)A = kA + lA$$
$$(kl)A = k(lA)$$

Multiplication of Matrices

At this time, it probably would seem quite natural to define matrix multiplication by multiplying corresponding elements of two matrices. However, such a definition does not have many worthwhile applications. Instead, we use a special type of **matrix multiplication**, sometimes referred to as a "row-by-column multiplication." We state the definition below, paraphrase what it says, and then give some examples.

Definition 12.4

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \end{aligned}$$

Note the row-by-column pattern of Definition 12.4. We multiply the rows of A times the columns of B in a pairwise entry fashion, adding the results. For example, the element in the first row and second column of the product is obtained by multiplying the elements of the first row of A times the elements of the second column of B and adding the results.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = [\qquad a_{11}b_{12} + a_{12}b_{22}]$$

Now let's look at some specific examples.

Classroom Example

If $A = \begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ -5 & 1 \end{bmatrix}$, find (a) AB and (b) BA.

EXAMPLE 2 If $A = \begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ -1 & 7 \end{bmatrix}$, find (a) AB and (b) BA.

Solutions

(a) $AB = \begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 7 \end{bmatrix}$

$$= \begin{bmatrix} (-2)(3) + (1)(-1) & (-2)(-2) + (1)(7) \\ (4)(3) + (5)(-1) & (4)(-2) + (5)(7) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 11 \\ 7 & 27 \end{bmatrix}$$

(b) $BA = \begin{bmatrix} 3 & -2 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} (3)(-2) + (-2)(4) & (3)(1) + (-2)(5) \\ (-1)(-2) + (7)(4) & (-1)(1) + (7)(5) \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -7 \\ 30 & 34 \end{bmatrix}$$

Example 2 makes it immediately apparent that matrix multiplication is *not* a **commutative** operation.

Classroom Example

If $A = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$, find AB.

EXAMPLE 3 If $A = \begin{bmatrix} 2 & -6 \\ -3 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}$, find AB.

Solution

Once you feel comfortable with Definition 12.4, you can do the addition mentally.

$$AB = \begin{bmatrix} 2 & -6 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 3 illustrates that the product of two matrices can be the zero matrix, even though neither of the two matrices is the zero matrix. This is different from the property of real numbers that states $ab = 0$ if and only if $a = 0$ or $b = 0$.

As we illustrated and stated earlier, matrix multiplication is *not* a commutative operation. However, it is an **associative** operation and it does exhibit two **distributive properties**. These properties can be stated as follows:

$$(AB)C = A(BC)$$
$$A(B + C) = AB + AC$$
$$(B + C)A = BA + CA$$

We will ask you to verify these properties in the next set of problems.

Concept Quiz 12.1

For the following problems, given that A, B, and C are 2×2 matrices and k and l are real numbers, answer true or false.

1. The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity element.
2. If $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then A and B are additive inverses.
3. $A + B = B + A$.
4. $AB = BA$.
5. If $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then either $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
6. The product of A times B can never equal $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
7. To perform the scalar multiplication kA, only the elements in the first row of A are multiplied by k.
8. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^2 = AA = \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$.

Problem Set 12.1

For Problems 1–12, compute the indicated matrix by using the following matrices: **(Objectives 1 and 2)**

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 6 \\ -4 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}$$

1. $A + B$
2. $B - C$
3. $3C + D$
4. $2D - E$
5. $4A - 3B$
6. $2B + 3D$
7. $(A - B) - C$
8. $B - (D - E)$
9. $2D - 4E$
10. $3A - 4E$
11. $B - (D + E)$
12. $A - (B + C)$

For Problems 13–26, compute AB and BA. **(Objective 3)**

13. $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$
14. $A = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix}$
15. $A = \begin{bmatrix} 1 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -3 \\ 4 & 5 \end{bmatrix}$
16. $A = \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$
17. $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$
18. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$
19. $A = \begin{bmatrix} -3 & -2 \\ -4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$
20. $A = \begin{bmatrix} -2 & 3 \\ -1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ -5 & -7 \end{bmatrix}$
21. $A = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$
22. $A = \begin{bmatrix} -8 & -5 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -5 \\ 3 & 8 \end{bmatrix}$
23. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ 6 & -4 \end{bmatrix}$
24. $A = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{2}{3} \end{bmatrix}$, $B = \begin{bmatrix} -6 & -18 \\ 12 & -12 \end{bmatrix}$

25. $A = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$

26. $A = \begin{bmatrix} -3 & -5 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -\frac{5}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$

For Problems 27–30, use the following matrices. (Objective 3)

$$A = \begin{bmatrix} -2 & 3 \\ 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

27. Compute AB and BA.

28. Compute AC and CA.

29. Compute AD and DA.

30. Compute AI and IA.

For Problems 31–34, use the following matrices. (Objective 3)

$$A = \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$$

31. Show that $(AB)C = A(BC)$.

32. Show that $A(B + C) = AB + AC$.

33. Show that $(A + B)C = AC + BC$.

34. Show that $(3 + 2)A = 3A + 2A$.

For Problems 35–43, use the following matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

35. Show that $A + B = B + A$.

36. Show that $(A + B) + C = A + (B + C)$.

37. Show that $A + (-A) = O$.

38. Show that $k(A + B) = kA + kB$ for any real number k.

39. Show that $(k + l)A = kA + lA$ for any real numbers k and l.

40. Show that $(kl)A = k(lA)$ for any real numbers k and l.

41. Show that $(AB)C = A(BC)$.

42. Show that $A(B + C) = AB + AC$.

43. Show that $(A + B)C = AC + BC$.

Thoughts Into Words

44. How would you show that addition of 2 × 2 matrices is a commutative operation?

45. How would you show that subtraction of 2 × 2 matrices is not a commutative operation?

46. How would you explain matrix multiplication to someone who missed class the day it was discussed?

47. Your friend says that because multiplication of real numbers is a commutative operation, it seems reasonable that multiplication of matrices should also be a commutative operation. How would you react to that statement?

Further Investigations

48. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, calculate A^2 and A^3, where A^2 means AA, and A^3 means AAA.

49. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, calculate A^2 and A^3.

50. Does $(A + B)(A - B) = A^2 - B^2$ for all 2 × 2 matrices? Defend your answer.

Graphing Calculator Activities

51. Use a calculator to check the answers to all three parts of Example 1.

52. Use a calculator to check your answers for Problems 21–26.

53. Use the following matrices:

$$A = \begin{bmatrix} 7 & -4 \\ 6 & 9 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 8 \\ -5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 8 & -2 \\ 4 & -7 \end{bmatrix}$$

(a) Show that $(AB)C = A(BC)$.
(b) Show that $A(B + C) = AB + AC$.
(c) Show that $(B + C)A = BA + CA$.

Answers to the Concept Quiz

1. True **2.** True **3.** True **4.** False **5.** False **6.** False **7.** False **8.** False

12.2 Multiplicative Inverses

OBJECTIVES

1. Find the multiplicative inverse of a 2×2 matrix
2. Find the product of a 2×2 and a 2×1 matrix
3. Solve a system of two linear equations by using matrices

We know that 1 is a multiplicative identity element for the set of real numbers. That is, $a(1) = 1(a) = a$ for any real number a. Is there a multiplicative identity element for 2×2 matrices? Yes. The matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the **multiplicative identity element** because

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Therefore we can state that

$$AI = IA = A$$

for all 2×2 matrices.

Again, refer to the set of real numbers, in which every nonzero real number a has a multiplicative inverse $1/a$ such that $a(1/a) = (1/a)\,a = 1$. Does every 2×2 matrix have a multiplicative inverse? To help answer this question, let's think about finding the multiplicative inverse (if one exists) for a specific matrix. This should give us some clues about a general approach.

Classroom Example
Find the multiplicative inverse of

$$A = \begin{bmatrix} 9 & 3 \\ 4 & 2 \end{bmatrix}.$$

EXAMPLE 1 Find the multiplicative inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.

Solution

We are looking for a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. In other words, we want to solve the following matrix equation:

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We need to multiply the two matrices on the left side of this equation and then set the elements of the product matrix equal to the corresponding elements of the identity matrix. We obtain the following system of equations:

$$\begin{pmatrix} 3x + 5z = 1 \\ 3y + 5w = 0 \\ 2x + 4z = 0 \\ 2y + 4w = 1 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

Solving equations (1) and (3) simultaneously produces values for x and z.

$$x = \frac{\begin{vmatrix} 1 & 5 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{1(4) - 5(0)}{3(4) - 5(2)} = \frac{4}{2} = 2$$

$$z = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{3(0) - 1(2)}{3(4) - 5(2)} = \frac{-2}{2} = -1$$

Likewise, solving equations (2) and (4) simultaneously produces values for y and w.

$$y = \frac{\begin{vmatrix} 0 & 5 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{0(4) - 5(1)}{3(4) - 5(2)} = \frac{-5}{2} = -\frac{5}{2}$$

$$w = \frac{\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{3(1) - 0(2)}{3(4) - 5(2)} = \frac{3}{2}$$

Therefore

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

To check this, we perform the following multiplication:

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now let's use the approach in Example 1 on the general matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We want to find

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

such that $AA^{-1} = I$. Therefore we need to solve the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for x, y, z, and w. Once again, we multiply the two matrices on the left side of the equation and set the elements of this product matrix equal to the corresponding elements of the identity matrix. We then obtain the following system of equations:

$$\begin{pmatrix} a_{11}x + a_{12}z = 1 \\ a_{11}y + a_{12}w = 0 \\ a_{21}x + a_{22}z = 0 \\ a_{21}y + a_{22}w = 1 \end{pmatrix}$$

Solving this system produces

$$x = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \qquad y = \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$z = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \qquad w = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}$$

Note that the number in each denominator, $a_{11}a_{22} - a_{12}a_{21}$, is the determinant of the matrix A. Thus, if $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Matrix multiplication will show that $AA^{-1} = A^{-1}A = I$. If $|A| = 0$, then the matrix A has *no* multiplicative inverse.

Classroom Example

Find A^{-1} if $A = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$.

EXAMPLE 2

Find A^{-1} if $A = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$.

Solution

First let's find $|A|$.

$$|A| = (3)(-4) - (5)(-2) = -2$$

Therefore

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{2} \\ -1 & -\frac{3}{2} \end{bmatrix}$$

It is easy to check that $AA^{-1} = A^{-1}A = I$.

Classroom Example

Find A^{-1} if $A = \begin{bmatrix} -2 & -3 \\ 6 & 9 \end{bmatrix}$.

EXAMPLE 3

Find A^{-1} if $A = \begin{bmatrix} 8 & -2 \\ -12 & 3 \end{bmatrix}$.

Solution

$$|A| = (8)(3) - (-2)(-12) = 0$$

Therefore A has no multiplicative inverse.

More about the Multiplication of Matrices

Thus far we have found the products of only 2×2 matrices. The row-by-column multiplication pattern can be applied to many different kinds of matrices, which we shall see in the next section. For now, let's find the product of a 2×2 matrix and a 2×1 matrix, with the 2×2 matrix on the left, as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{bmatrix}$$

Note that the product matrix is a 2×1 matrix. The following example illustrates this pattern:

$$\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} (-2)(5) + (3)(7) \\ (1)(5) + (-4)(7) \end{bmatrix} = \begin{bmatrix} 11 \\ -23 \end{bmatrix}$$

Back to Solving Systems of Equations

The linear system of equations

$$\begin{pmatrix} a_{11}x + a_{12}y = d_1 \\ a_{21}x + a_{22}y = d_2 \end{pmatrix}$$

can be represented by the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

If we let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

then the previous matrix equation can be written $AX = B$.

If A^{-1} exists, then we can multiply both sides of $AX = B$ by A^{-1} (on the left) and simplify as follows:

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}(B) \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore the product $A^{-1}B$ is the solution of the system.

Classroom Example

Solve the system $\begin{pmatrix} 8x + 3y = 12 \\ 5x + 7y = -13 \end{pmatrix}$.

EXAMPLE 4 Solve the system $\begin{pmatrix} 5x + 4y = 10 \\ 6x + 5y = 13 \end{pmatrix}$.

Solution

If we let

$$A = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

then the given system can be represented by the matrix equation $AX = B$. From our previous discussion, we know that the solution of this equation is $X = A^{-1}B$, so we need to find A^{-1} and the product $A^{-1}B$.

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$$

Therefore

$$A^{-1}B = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

The solution set of the given system is $\{(-2, 5)\}$.

Classroom Example

Solve the system $\begin{pmatrix} -2x - 3y = 2 \\ 2x - 5y = -18 \end{pmatrix}$.

EXAMPLE 5

Solve the system $\begin{pmatrix} 3x - 2y = 9 \\ 4x + 7y = -17 \end{pmatrix}$.

Solution

If we let

$$A = \begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 9 \\ -17 \end{bmatrix}$$

then the system is represented by $AX = B$, where $X = A^{-1}B$ and

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 7 & 2 \\ -4 & 3 \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 7 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{29} & \frac{2}{29} \\ -\frac{4}{29} & \frac{3}{29} \end{bmatrix}$$

Therefore

$$A^{-1}B = \begin{bmatrix} \frac{7}{29} & \frac{2}{29} \\ -\frac{4}{29} & \frac{3}{29} \end{bmatrix} \begin{bmatrix} 9 \\ -17 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The solution set of the given system is $\{(1, -3)\}$.

This technique of using matrix inverses to solve systems of linear equations is especially useful when there are many systems to be solved that have the same coefficients but different constant terms.

Concept Quiz 12.2

For the following problems, answer true or false.

1. Every 2×2 matrix has a multiplicative inverse.

2. If $A = \begin{bmatrix} 4 & 7 \\ 2 & 5 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{7} \\ \frac{1}{2} & \frac{1}{5} \end{bmatrix}$.

3. If $|A| = 0$, then A does not have a multiplicative inverse.

4. If $|A| = 1$, then A does have a multiplicative inverse.

5. The multiplicative identity element for 2×2 matrices is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6. If A has an inverse A^{-1}, then $AA^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

7. The commutative property holds for the multiplication of a matrix and its inverse.

8. The commutative property holds for the multiplication of a matrix and the multiplicative identity element.

Problem Set 12.2

For Problems 1–18, find the multiplicative inverse (if one exists) of each matrix. **(Objective 1)**

1. $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & 9 \\ 3 & 13 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$

7. $\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$ **8.** $\begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$

9. $\begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & -4 \\ 6 & -8 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ **12.** $\begin{bmatrix} -2 & 0 \\ -3 & 5 \end{bmatrix}$

13. $\begin{bmatrix} -2 & -3 \\ -1 & -4 \end{bmatrix}$ **14.** $\begin{bmatrix} -2 & -5 \\ -3 & -6 \end{bmatrix}$

15. $\begin{bmatrix} -2 & 5 \\ -3 & 6 \end{bmatrix}$ **16.** $\begin{bmatrix} -3 & 4 \\ 1 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

For Problems 19–26, compute AB. **(Objective 2)**

19. $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

20. $A = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

21. $A = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

22. $A = \begin{bmatrix} 5 & 2 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

23. $A = \begin{bmatrix} -4 & 2 \\ 7 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & -3 \\ 2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$

25. $A = \begin{bmatrix} -2 & -3 \\ -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

26. $A = \begin{bmatrix} -3 & -5 \\ 4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ -10 \end{bmatrix}$

For Problems 27–40, use the method of matrix inverses to solve each system. **(Objective 3)**

27. $\begin{pmatrix} 2x + 3y = 13 \\ x + 2y = 8 \end{pmatrix}$ **28.** $\begin{pmatrix} 3x + 2y = 10 \\ 7x + 5y = 23 \end{pmatrix}$

29. $\begin{pmatrix} 4x - 3y = -23 \\ -3x + 2y = 16 \end{pmatrix}$ **30.** $\begin{pmatrix} 6x - y = -14 \\ 3x + 2y = -17 \end{pmatrix}$

31. $\begin{pmatrix} x - 7y = 7 \\ 6x + 5y = -5 \end{pmatrix}$ **32.** $\begin{pmatrix} x + 9y = -5 \\ 4x - 7y = -20 \end{pmatrix}$

33. $\begin{pmatrix} 3x - 5y = 2 \\ 4x - 3y = -1 \end{pmatrix}$ **34.** $\begin{pmatrix} 5x - 2y = 6 \\ 7x - 3y = 8 \end{pmatrix}$

35. $\begin{pmatrix} y = 19 - 3x \\ 9x - 5y = 1 \end{pmatrix}$ **36.** $\begin{pmatrix} 4x + 3y = 31 \\ x = 5y + 2 \end{pmatrix}$

37. $\begin{pmatrix} 3x + 2y = 0 \\ 30x - 18y = -19 \end{pmatrix}$ **38.** $\begin{pmatrix} 12x + 30y = 23 \\ 12x - 24y = -13 \end{pmatrix}$

39. $\begin{pmatrix} \frac{1}{3}x + \frac{3}{4}y = 12 \\ \frac{2}{3}x + \frac{1}{5}y = -2 \end{pmatrix}$ **40.** $\begin{pmatrix} \frac{3}{2}x + \frac{1}{6}y = 11 \\ \frac{2}{3}x - \frac{1}{4}y = 1 \end{pmatrix}$

Thoughts Into Words

41. Describe how to solve the system $\begin{pmatrix} x - 2y = -10 \\ 3x + 5y = 14 \end{pmatrix}$

using each of the following techniques.

(a) substitution method
(b) elimination-by-addition method
(c) reduced echelon form of the augmented matrix
(d) determinants
(e) the method of matrix inverses

Graphing Calculator Activities

42. Use your calculator to find the multiplicative inverse (if one exists) of each of the following matrices. Be sure to check your answers by showing that $A^{-1}A = I$.

(a) $\begin{bmatrix} 7 & 6 \\ 8 & 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} -12 & 5 \\ -19 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} -7 & 9 \\ 6 & -8 \end{bmatrix}$ **(d)** $\begin{bmatrix} -6 & -11 \\ -4 & -8 \end{bmatrix}$

(e) $\begin{bmatrix} 13 & 12 \\ 4 & 4 \end{bmatrix}$ **(f)** $\begin{bmatrix} 15 & -8 \\ -9 & 5 \end{bmatrix}$

(g) $\begin{bmatrix} 9 & 36 \\ 3 & 12 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1.2 & 1.5 \\ 7.6 & 4.5 \end{bmatrix}$

43. Use your calculator to find the multiplicative inverse of $\begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ What difficulty did you encounter?

44. Use your calculator and the method of matrix inverses to solve each of the following systems. Be sure to check your solutions.

(a) $\begin{pmatrix} 5x + 7y = 82 \\ 7x + 10y = 116 \end{pmatrix}$ **(b)** $\begin{pmatrix} 9x - 8y = -150 \\ -10x + 9y = 168 \end{pmatrix}$

(c) $\begin{pmatrix} 15x - 8y = -15 \\ -9x + 5y = 12 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1.2x + 1.5y = 5.85 \\ 7.6x + 4.5y = 19.55 \end{pmatrix}$

(e) $\begin{pmatrix} 12x - 7y = -34.5 \\ 8x + 9y = 79.5 \end{pmatrix}$ **(f)** $\begin{pmatrix} \frac{3x}{2} + \frac{y}{6} = 11 \\ \frac{2x}{3} - \frac{y}{4} = 1 \end{pmatrix}$

(g) $\begin{pmatrix} 114x + 129y = 2832 \\ 127x + 214y = 4139 \end{pmatrix}$

(h) $\begin{pmatrix} \frac{x}{2} + \frac{2y}{5} = 14 \\ \frac{3x}{4} + \frac{y}{4} = 14 \end{pmatrix}$

Answers to the Concept Quiz

1. False **2.** False **3.** True **4.** True **5.** False **6.** False **7.** True **8.** True

12.3 $m \times n$ Matrices

OBJECTIVES

1. Add and subtract general $m \times n$ matrices
2. Multiply an $m \times n$ matrix by a scalar
3. Multiply an $m \times n$ matrix by an $n \times p$ matrix
4. Find the inverse of a square $m \times m$ matrix
5. Solve systems of linear equations using matrices

Now let's see how much of the algebra of 2×2 matrices extends to $m \times n$ matrices—that is, to matrices of any dimension. In Section 11.4 we represented a general $m \times n$ matrix by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

We denote the element at the intersection of row i and column j by a_{ij}. It is also customary to denote a matrix A with the abbreviated notation (a_{ij}).

Addition of matrices can be extended to matrices of any dimension by the following definition:

Definition 12.5

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of the *same dimension*. Then

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

Definition 12.5 states that to add two matrices, we add the elements that appear in corresponding positions in the matrices. For this to work, the matrices must be of the same dimension. An example of the sum of two 3×2 matrices is

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -3 & -7 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -8 \\ 2 & 17 \end{bmatrix}$$

The **commutative** and **associative properties** hold for any matrices that can be added. The $m \times n$ **zero matrix**, denoted by O, is the matrix that contains all zeros. It is the **identity element for addition**. For example,

$$\begin{bmatrix} 2 & 3 & -1 & -5 \\ -7 & 6 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 & -5 \\ -7 & 6 & 2 & 8 \end{bmatrix}$$

Every matrix A has an **additive inverse**, $-A$, that can be found by changing the sign of each element of A. For example, if

$$A = [2 \quad -3 \quad 0 \quad 4 \quad -7]$$

then

$$-A = [-2 \quad 3 \quad 0 \quad -4 \quad 7]$$

Furthermore, $A + (-A) = O$ for all matrices.

The definition we gave earlier for subtraction, $A - B = A + (-B)$, can be extended to any two matrices of the same dimension. For example,

$$\begin{aligned}[-4 \quad 3 \quad -5] - [7 \quad -4 \quad -1] &= [-4 \quad 3 \quad -5] + [-7 \quad 4 \quad 1] \\ &= [-11 \quad 7 \quad -4]\end{aligned}$$

The **scalar product** of any real number k and any $m \times n$ matrix $A = (a_{ij})$ is defined by

$$kA = (ka_{ij})$$

In other words, to find kA, we simply multiply each element of A by k. For example,

$$(-4)\begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 4 & 5 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 8 & -12 \\ -16 & -20 \\ 0 & 32 \end{bmatrix}$$

The properties $k(A + B) = kA + kB$, $(k + l)A = kA + lA$, and $(kl)A = k(lA)$ hold for all matrices. The matrices A and B must be of the same dimension to be added.

The row-by-column definition for multiplying two matrices can be extended, but we must take care. In order for us to define the product AB of two matrices A and B, **the number of columns of A must equal the number of rows of B**. Suppose $A = (a_{ij})$ is $m \times n$, and $B = (b_{ij})$ is $n \times p$. Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = C$$

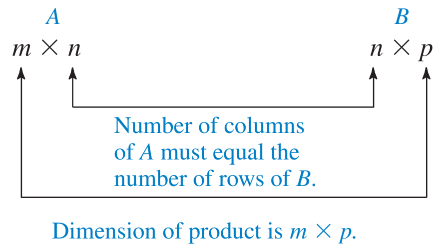
The product matrix C is of the dimension $m \times p$, and the general element, c_{ij}, is determined as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

A specific element of the product matrix, such as c_{23}, is the result of multiplying the elements in row 2 of matrix A by the elements in column 3 of matrix B and adding the results. Therefore

$$c_{23} = a_{21}b_{13} + a_{22}b_{23} + \cdots + a_{2n}b_{n3}$$

The following example illustrates the product of a 2×3 matrix and a 3×2 matrix:



$$\overset{A}{\begin{bmatrix} 2 & -3 & 1 \\ -4 & 0 & 5 \end{bmatrix}} \overset{B}{\begin{bmatrix} -1 & -5 \\ 4 & -2 \\ 6 & 1 \end{bmatrix}} = \overset{C}{\begin{bmatrix} -8 & -3 \\ 34 & 25 \end{bmatrix}}$$

$$c_{11} = (2)(-1) + (-3)(4) + (1)(6) = -8$$

$$c_{12} = (2)(-5) + (-3)(-2) + (1)(1) = -3$$

$$c_{21} = (-4)(-1) + (0)(4) + (5)(6) = 34$$

$$c_{22} = (-4)(-5) + (0)(-2) + (5)(1) = 25$$

Recall that matrix multiplication is *not* commutative. In fact, it may be that AB is defined and BA is not defined. For example, if A is a 2×3 matrix and B is a 3×4 matrix, then the product AB is a 2×4 matrix, but the product BA is not defined because the number of columns of B does not equal the number of rows of A.

The **associative property for multiplication** and the two **distributive properties** hold if the matrices have the proper number of rows and columns for the operations to be defined. In that case, we have $(AB)C = A(BC)$, $A(B + C) = AB + AC$, and $(A + B)C = AC + BC$.

Square Matrices

Now let's extend some of the algebra of 2×2 matrices to all square matrices (where the number of rows equals the number of columns). For example, the general **multiplicative identity element** for square matrices contains 1s in the main diagonal from the upper left-hand corner to the lower right-hand corner and 0s elsewhere. Therefore, for 3×3 and 4×4 matrices, the multiplicative identity elements are as follows:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We saw in Section 12.2 that some, but not all, 2×2 matrices have multiplicative inverses. In general, some, but not all, square matrices of a particular dimension have multiplicative inverses. If an $n \times n$ square matrix A does have a multiplicative inverse A^{-1}, then

$$AA^{-1} = A^{-1}A = I_n$$

The technique used in Section 12.2 for finding multiplicative inverses of 2×2 matrices does generalize, but it becomes quite complicated. Therefore, we shall now describe another technique that works for all square matrices. Given an $n \times n$ matrix A, we begin by forming the $n \times 2n$ matrix

$$\left[\begin{array}{cccc|ccccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & 0 & \ldots & 1 \end{array}\right]$$

where the identity matrix I_n appears to the right of A. Now we apply a succession of elementary row transformations to this double matrix until we obtain a matrix of the form

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & 1 & 0 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{array}\right]$$

The B matrix in this matrix is the desired inverse A^{-1}. If A does not have an inverse, then it is impossible to change the original matrix to this final form.

Classroom Example

Find A^{-1} if $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

EXAMPLE 1 Find A^{-1} if $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

Solution

First form the matrix

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

Now multiply row 1 by $\frac{1}{2}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

Next, add -3 times row 1 to row 2 to form a new row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -1 & -\frac{3}{2} & 1 \end{array}\right]$$

Then multiply row 2 by -1.

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & -1 \end{array}\right]$$

Finally, add -2 times row 2 to row 1 to form a new row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & 2 \\ 0 & 1 & \frac{3}{2} & -1 \end{array}\right]$$

The matrix inside the box is A^{-1}; that is,

$$A^{-1} = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{bmatrix}$$

This can be checked, as always, by showing that $AA^{-1} = A^{-1}A = I_2$.

Classroom Example

Find A^{-1} if $A = \begin{bmatrix} -2 & 5 & 3 \\ -3 & 1 & -1 \\ 1 & 4 & 5 \end{bmatrix}$.

EXAMPLE 2

Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ -3 & 1 & -2 \end{bmatrix}$.

Solution

Form the matrix $\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ -3 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$.

Add -2 times row 1 to row 2, and add 3 times row 1 to row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 4 & 4 & 3 & 0 & 1 \end{array}\right]$$

Add -1 times row 2 to row 1, and add -4 times row 2 to row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 7 & 3 & -1 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 24 & 11 & -4 & 1 \end{array}\right]$$

Multiply row 3 by $\frac{1}{24}$.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 7 & 3 & -1 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{24} & -\frac{1}{6} & \frac{1}{24} \end{array}\right]$$

Add -7 times row 3 to row 1, and add 5 times row 3 to row 2.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{5}{24} & \frac{1}{6} & -\frac{7}{24} \\ 0 & 1 & 0 & \frac{7}{24} & \frac{1}{6} & \frac{5}{24} \\ 0 & 0 & 1 & \frac{11}{24} & -\frac{1}{6} & \frac{1}{24} \end{array}\right]$$

Therefore

$$A^{-1} = \begin{bmatrix} -\frac{5}{24} & \frac{1}{6} & -\frac{7}{24} \\ \frac{7}{24} & \frac{1}{6} & \frac{5}{24} \\ \frac{11}{24} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix}$$

Be sure to check this!

Systems of Equations

In Section 12.2 we used the concept of the multiplicative inverse to solve systems of two linear equations in two variables. This same technique can be applied to general systems of n linear equations in n variables. Let's consider one such example involving three equations in three variables.

Classroom Example
Solve the system

$$\begin{pmatrix} x - y + 4z = 11 \\ 4x - 2y + 3z = 3 \\ 2x + y + 4z = 7 \end{pmatrix}, \text{ given that}$$

the inverse of the coefficient matrix is

$$\begin{bmatrix} -\frac{11}{31} & \frac{8}{31} & \frac{5}{31} \\ -\frac{10}{31} & -\frac{4}{31} & \frac{13}{31} \\ \frac{8}{31} & -\frac{3}{31} & \frac{2}{31} \end{bmatrix}.$$

EXAMPLE 3 Solve the system $\begin{pmatrix} x + y + 2z = -8 \\ 2x + 3y - z = 3 \\ -3x + y - 2z = 4 \end{pmatrix}$.

Solution

If we let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ -3 & 1 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -8 \\ 3 \\ 4 \end{bmatrix}$$

then the given system can be represented by the matrix equation $AX = B$. Therefore, we know that $X = A^{-1}B$, so we need to find A^{-1} and the product $A^{-1}B$. The matrix A^{-1} was found in Example 2, so let's use that result and find $A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} -\frac{5}{24} & \frac{1}{6} & -\frac{7}{24} \\ \frac{7}{24} & \frac{1}{6} & \frac{5}{24} \\ \frac{11}{24} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

The solution set of the given system is $\{(1, -1, -4)\}$.

Concept Quiz 12.3

For the following problems, answer true or false.

1. If A is a 5×2 matrix, then it has 5 rows and 2 columns of elements.
2. If $B = \begin{bmatrix} 3 & 2 & 5 & 7 \\ 4 & 0 & 8 & -1 \end{bmatrix}$, then B is a 4×2 matrix.
3. Only square matrices have an additive inverse.
4. For matrices that can be added, the commutative property holds.
5. If A is a 3×3 matrix, then $AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
6. Every square matrix has a multiplicative inverse matrix.
7. Given that a_{14} is an element of matrix A, then the element is in the first row and fourth column.
8. If $A = \begin{bmatrix} 2 & -4 & 5 \\ -1 & 0 & 3 \end{bmatrix}$, then $3A = \begin{bmatrix} 6 & -4 & 5 \\ -3 & 0 & 3 \end{bmatrix}$.

Problem Set 12.3

For Problems 1–8, find $A + B$, $A - B$, $2A + 3B$, and $4A - 2B$. **(Objectives 1 and 2)**

1. $A = \begin{bmatrix} 2 & -1 & 4 \\ -2 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 & -7 \\ 5 & -6 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & -6 \\ 2 & -1 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & -7 \\ -6 & 9 \end{bmatrix}$

3. $A = [2 \quad -1 \quad 4 \quad 12], \qquad B = [-3 \quad -6 \quad 9 \quad -5]$

4. $A = \begin{bmatrix} 3 \\ -9 \\ 7 \end{bmatrix}, \qquad B = \begin{bmatrix} -6 \\ 12 \\ 9 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 4 & -7 \\ 0 & 5 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & -1 & -3 \\ 10 & -2 & 4 \\ 7 & 0 & 12 \end{bmatrix}$

6. $A = \begin{bmatrix} 7 & -4 \\ -5 & 9 \\ -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 12 & 3 \\ -2 & -4 \\ -6 & 7 \end{bmatrix}$

7. $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ -5 & -4 \\ -7 & 11 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ -3 & 7 \\ 6 & -5 \\ 9 & -2 \end{bmatrix}$

8. $A = \begin{bmatrix} 0 & -1 & -2 \\ 3 & -4 & 6 \\ 5 & 4 & -9 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & -7 \\ -6 & 4 & 5 \\ 3 & -2 & -1 \end{bmatrix}$

For Problems 9–20, find AB and BA, whenever they exist. **(Objective 3)**

9. $A = \begin{bmatrix} 2 & -1 \\ 0 & -4 \\ -5 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & -2 & 6 \\ -1 & 4 & -2 \end{bmatrix}$

10. $A = \begin{bmatrix} -2 & 3 & -1 \\ 7 & -4 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ -5 & -6 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & -1 & -3 \\ 0 & -4 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & -1 & 4 \\ 0 & -2 & 3 & 5 \\ -6 & 4 & -2 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 & -1 & -4 \\ -5 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 \\ -4 & -1 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 3 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ -5 & 1 & -1 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

15. $A = [2 \quad -1 \quad 3 \quad 4], \qquad B = \begin{bmatrix} -1 \\ -3 \\ 2 \\ -4 \end{bmatrix}$

16. $A = \begin{bmatrix} -2 \\ 3 \\ -5 \end{bmatrix}, \qquad B = [3 \quad -4 \quad -5]$

17. $A = \begin{bmatrix} 2 \\ -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ -1 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} 3 & -2 & 2 & -4 \\ 1 & 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 1 \\ -3 & 1 & 4 \\ 5 & 2 & 0 \\ -4 & -1 & -2 \end{bmatrix}$

19. $A = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \qquad B = [3 \quad -4]$

20. $A = [3 \quad -7], \qquad B = \begin{bmatrix} 8 \\ -9 \end{bmatrix}$

For Problems 21–36, use the technique discussed in this section to find the multiplicative inverse (if one exists) of each matrix. **(Objective 4)**

21. $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

24. $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

25. $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$

26. $\begin{bmatrix} -3 & 1 \\ 3 & -2 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 3 & -2 \\ 1 & 4 & -1 \\ -2 & -7 & 5 \end{bmatrix}$

29. $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & 3 \\ 3 & -5 & 7 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 4 & -2 \\ -3 & -11 & 1 \\ 2 & 7 & 3 \end{bmatrix}$

31. $\begin{bmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 1 & -4 & 2 \end{bmatrix}$

32. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 2 & 3 \\ -3 & -4 & 3 \\ 2 & 4 & -1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -2 \\ -2 & 6 & 1 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

36. $\begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

For Problems 37–46, use the method of matrix inverses to solve each system. The required multiplicative inverses were found in Problems 21–36. **(Objective 5)**

37. $\begin{pmatrix} 2x + y = -4 \\ 7x + 4y = -13 \end{pmatrix}$

38. $\begin{pmatrix} 3x + 7y = -38 \\ 2x + 5y = -27 \end{pmatrix}$

39. $\begin{pmatrix} -2x + y = 1 \\ 3x - 4y = -14 \end{pmatrix}$

40. $\begin{pmatrix} -3x + y = -18 \\ 3x - 2y = 15 \end{pmatrix}$

41. $\begin{pmatrix} x + 2y + 3z = -2 \\ x + 3y + 4z = -3 \\ x + 4y + 3z = -6 \end{pmatrix}$

42. $\begin{pmatrix} x + 3y - 2z = 5 \\ x + 4y - z = 3 \\ -2x - 7y + 5z = -12 \end{pmatrix}$

43. $\begin{pmatrix} x - 2y + z = -3 \\ -2x + 5y + 3z = 34 \\ 3x - 5y + 7z = 14 \end{pmatrix}$

44. $\begin{pmatrix} x + 4y - 2z = 2 \\ -3x - 11y + z = -2 \\ 2x + 7y + 3z = -2 \end{pmatrix}$

45. $\begin{pmatrix} x + 2y + 3z = 2 \\ -3x - 4y + 3z = 0 \\ 2x + 4y - z = 4 \end{pmatrix}$

46. $\begin{pmatrix} x - 2y + 3z = -39 \\ -x + 3y - 2z = 40 \\ -2x + 6y + z = 45 \end{pmatrix}$

47. We can generate five systems of linear equations from the system

$$\begin{pmatrix} x + y + 2z = a \\ 2x + 3y - z = b \\ -3x + y - 2z = c \end{pmatrix}$$

by letting a, b, and c assume five different sets of values. Solve the system for each set of values. The inverse of the coefficient matrix of these systems is given in Example 2 of this section.

(a) $a = 7$, $b = 1$, and $c = -1$

(b) $a = -7$, $b = 5$, and $c = 1$

(c) $a = -9$, $b = -8$, and $c = 19$

(d) $a = -1$, $b = -13$, and $c = -17$

(e) $a = -2$, $b = 0$, and $c = -2$

Thoughts Into Words

48. How would you describe row-by-column multiplication of matrices?

49. Give a step-by-step explanation of how to find the multiplicative inverse of the matrix $\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ by using the technique of Section 12.3.

50. Explain how to find the multiplicative inverse of the matrix in Problem 49 by using the technique discussed in Section 12.2.

Further Investigations

51. Matrices can be used to code and decode messages. For example, suppose that we set up a one-to-one correspondence between the letters of the alphabet and the first 26 counting numbers, as follows:

A	B	C		Z
↕	↕	↕	. . .	↕
1	2	3		26

Now suppose that we want to code the message PLAY IT BY EAR. We can partition the letters of the message into groups of two. Because the last group will contain only one letter, let's arbitrarily stick in a Z to form a group of two. Let's also assign a number to each letter on the basis of the letter/number association we exhibited.

P	L	A	Y	I	T	B	Y	E	A	R	Z
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
16	12	1	25	9	20	2	25	5	1	18	26

Each pair of numbers can be recorded as columns in a 2×6 matrix B.

$$B = \begin{bmatrix} 16 & 1 & 9 & 2 & 5 & 18 \\ 12 & 25 & 20 & 25 & 1 & 26 \end{bmatrix}$$

Now let's choose a 2×2 matrix such that the matrix contains only integers, and its inverse also contains only integers. For example, we can use $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$; then $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.