27. A tank contains 20 gallons of water. One-half of the water is removed and replaced with antifreeze. Then one-half of this mixture is removed and replaced with antifreeze. This process is continued eight times. How much water remains in the tank after the eighth replacement process?

28. The radiator of a truck contains 10 gallons of water. Suppose we remove 1 gallon of water and replace it with antifreeze. Then we remove 1 gallon of this mixture and replace it with antifreeze. This process is carried out seven times. To the nearest tenth of a gallon, how much antifreeze is in the final mixture?

Thoughts Into Words

29. Your friend solves Problem 6 as follows: If the car depreciates 20% per year, then at the end of 5 years it will have depreciated 100% and be worth zero dollars. How would you convince him that his reasoning is incorrect?

30. A contractor wants you to clear some land for a housing project. He anticipates that it will take 20 working days to do the job. He offers to pay you one of two ways: (1) a fixed amount of $3000 or (2) a penny the first day, 2 cents the second day, 4 cents the third day, and so on, doubling your daily wages each day for the 20 days. Which offer should you take and why?

14.4 Mathematical Induction

OBJECTIVE 1 Use mathematical induction to prove mathematical statements

Is $2^n > n$ for all positive integer values of n? In an attempt to answer this question, we might proceed as follows:

If $n = 1$, then $2^n > n$ becomes $2^1 > 1$, a true statement.

If $n = 2$, then $2^n > n$ becomes $2^2 > 2$, a true statement.

If $n = 3$, then $2^n > n$ becomes $2^3 > 3$, a true statement.

We can continue in this way as long as we want, but obviously we can never show in this manner that $2^n > n$ for *every* positive integer n. However, we do have a form of proof, called **proof by mathematical induction**, that can be used to verify the truth of many mathematical statements involving positive integers. This form of proof is based on the following principle.

Principle of Mathematical Induction

Let P_n be a statement in terms of n, where n is a positive integer. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1} for every positive integer k,

then P_n is true for every positive integer n.

The principle of mathematical induction, a proof that some statement is true for all positive integers, consists of two parts. First, we must show that the statement is true for the positive integer 1. Second, we must show that if the statement is true for some positive integer, then it follows that it is also true for the next positive integer. Let's illustrate what this means.

Classroom Example
Prove that $3^n > n$ for all positive values of n.

EXAMPLE 1 Prove that $2^n > n$ for all positive integer values of n.

Proof

Part 1 If $n = 1$, then $2^n > n$ becomes $2^1 > 1$, which is a true statement.

Part 2 We must prove that if $2^k > k$, then $2^{k+1} > k + 1$ for all positive integer values of k. In other words, we should be able to start with $2^k > k$ and from that deduce $2^{k+1} > k + 1$. This can be done as follows:

$$2^k > k$$

$$2(2^k) > 2(k) \qquad \text{Multiply both sides by 2}$$

$$2^{k+1} > 2k$$

We know that $k \geq 1$ because we are working with positive integers. Therefore

$$k + k \geq k + 1 \qquad \text{Add } k \text{ to both sides}$$

$$2k \geq k + 1$$

Because $2^{k+1} > 2k$ and $2k \geq k + 1$, by the transitive property we conclude that

$$2^{k+1} > k + 1$$

Therefore, using parts 1 and 2, we proved that $2^n > n$ for *all* positive integers.

It will be helpful for you to look back over the proof in Example 1. Note that in part 1, we established that $2^n > n$ is true for $n = 1$. Then, in part 2, we established that if $2^n > n$ is true for any positive integer, then it must be true for the next consecutive positive integer. Therefore, because $2^n > n$ is true for $n = 1$, it must be true for $n = 2$. Likewise, if $2^n > n$ is true for $n = 2$, then it must be true for $n = 3$, and so on, for *all* positive integers.

We can depict proof by mathematical induction with dominoes. Suppose that in Figure 14.1, we have infinitely many dominoes lined up. If we can push the first domino over (part 1 of a mathematical induction proof), and if the dominoes are spaced so that each time one falls over, it causes the next one to fall over (part 2 of a mathematical induction proof), then by pushing the first one over we will cause a chain reaction that will topple all of the dominoes (Figure 14.2).

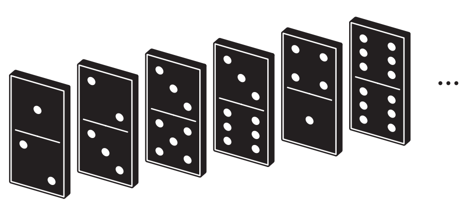

Figure 14.1

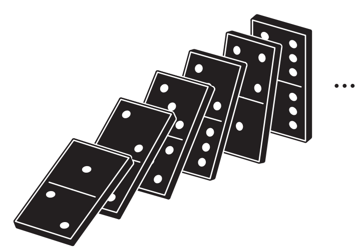

Figure 14.2

Recall that in the first three sections of this chapter, we used a_n to represent the nth term of a sequence and S_n to represent the sum of the first n terms of a sequence. For example, if $a_n = 2n$, then the first three terms of the sequence are $a_1 = 2(1) = 2$, $a_2 = 2(2) = 4$, and $a_3 = 2(3) = 6$. Furthermore, the kth term is $a_k = 2(k) = 2k$, and the $(k + 1)$ term is $a_{k+1} = 2(k + 1) = 2k + 2$. Relative to this same sequence, we can state that $S_1 = 2$, $S_2 = 2 + 4 = 6$, and $S_3 = 2 + 4 + 6 = 12$.

There are numerous sum formulas for sequences that can be verified by mathematical induction. For such proofs, the following property of sequences is used:

$$S_{k+1} = S_k + a_{k+1}$$

This property states that *the sum of the first* $k + 1$ *terms is equal to the sum of the first k terms plus the* $(k + 1)$ *term*. Let's see how this can be used in a specific example.

Classroom Example
Prove $S_n = 2n(n + 1)$ for the sequence $a_n = 4n$, if n is any positive integer.

EXAMPLE 2

Prove that $S_n = n(n + 1)$ for the sequence $a_n = 2n$, when n is any positive integer.

Proof

Part 1 If $n = 1$, then $S_1 = 1(1 + 1) = 2$, and 2 is the first term of the sequence $a_n = 2n$, so $S_1 = a_1 = 2$.

Part 2 Now we need to prove that if $S_k = k(k + 1)$, then $S_{k+1} = (k + 1)(k + 2)$. Using the property $S_{k+1} = S_k + a_{k+1}$, we can proceed as follows:

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= k(k + 1) + 2(k + 1) \\ &= (k + 1)(k + 2) \end{aligned}$$

Therefore, using parts 1 and 2, we proved that $S_n = n(n + 1)$ will yield the correct sum for any number of terms of the sequence $a_n = 2n$.

Classroom Example
Prove that $S_n = \dfrac{n(3n + 5)}{2}$ for the sequence $a_n = 3n + 1$, if n is any positive integer.

EXAMPLE 3

Prove that $S_n = 5n(n + 1)/2$ for the sequence $a_n = 5n$, when n is any positive integer.

Proof

Part 1 Because $S_1 = 5(1)\ (1 + 1)/2 = 5$, and 5 is the first term of the sequence $a_n = 5n$, we have $S_1 = a_1 = 5$.

Part 2 We need to prove that if $S_k = 5k(k + 1)/2$, then $S_{k+1} = \dfrac{5(k + 1)(k + 2)}{2}$.

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= \frac{5k(k + 1)}{2} + 5(k + 1) \\ &= \frac{5k(k + 1)}{2} + 5k + 5 \\ &= \frac{5k(k + 1) + 2(5k + 5)}{2} \\ &= \frac{5k^2 + 5k + 10k + 10}{2} \\ &= \frac{5k^2 + 15k + 10}{2} \\ &= \frac{5(k^2 + 3k + 2)}{2} \\ &= \frac{5(k + 1)(k + 2)}{2} \end{aligned}$$

Therefore, using parts 1 and 2, we proved that $S_n = 5n(n + 1)/2$ yields the correct sum for any number of terms of the sequence $a_n = 5n$.

Classroom Example
Prove that $S_n = \dfrac{3^n - 1}{2}$ for the sequence $a_n = 3^{n-1}$, if n is any positive integer.

EXAMPLE 4

Prove that $S_n = (4^n - 1)/3$ for the sequence $a_n = 4^{n-1}$, where n is any positive integer.

Proof

Part 1 Because $S_1 = (4^1 - 1)/3 = 1$, and 1 is the first term of the sequence $a_n = 4^{n-1}$, we have $S_1 = a_1 = 1$.

Part 2 We need to prove that if $S_k = (4^k - 1)/3$, then $S_{k+1} = (4^{k+1} - 1)/3$:

$$\begin{aligned}
S_{k+1} &= S_k + a_{k+1} \\
&= \frac{4^k - 1}{3} + 4^k \\
&= \frac{4^k - 1 + 3(4^k)}{3} \\
&= \frac{4^k + 3(4^k) - 1}{3} \\
&= \frac{4^k(1 + 3) - 1}{3} \\
&= \frac{4^k(4) - 1}{3} \\
&= \frac{4^{k+1} - 1}{3}
\end{aligned}$$

Therefore, using parts 1 and 2, we proved that $S_n = (4^n - 1)/3$ yields the correct sum for any number of terms of the sequence $a_n = 4^{n-1}$.

As our final example of this section, let's consider a proof by mathematical induction involving the concept of divisibility.

Classroom Example
Prove that for all positive integers n, the number $3^n - 1$ is divisible by 2.

EXAMPLE 5

Prove that for all positive integers n, the number $3^{2n} - 1$ is divisible by 8.

Proof

Part 1 If $n = 1$, then $3^{2n} - 1$ becomes $3^{2(1)} - 1 = 3^2 - 1 = 8$, and of course 8 is divisible by 8.

Part 2 We need to prove that if $3^{2k} - 1$ is divisible by 8, then $3^{2k+2} - 1$ is divisible by 8 for all integer values of k. This can be verified as follows. If $3^{2k} - 1$ is divisible by 8, then for some integer x, we have $3^{2k} - 1 = 8x$. Therefore

$$\begin{aligned}
3^{2k} - 1 &= 8x \\
3^{2k} &= 1 + 8x \\
3^2(3^{2k}) &= 3^2(1 + 8x) && \text{Multiply both sides by } 3^2 \\
3^{2k+2} &= 9(1 + 8x) \\
3^{2k+2} &= 9 + 9(8x) \\
3^{2k+2} &= 1 + 8 + 9(8x) && 9 = 1 + 8 \\
3^{2k+2} &= 1 + 8(1 + 9x) && \text{Apply distributive property to } 8 + 9(8x) \\
3^{2k+2} - 1 &= 8(1 + 9x)
\end{aligned}$$

Therefore $3^{2k+2} - 1$ is divisible by 8.

Thus using parts 1 and 2, we proved that $3^{2n} - 1$ is divisible by 8 for all positive integers n.

We conclude this section with a few final comments about proof by mathematical induction. Every mathematical induction proof is a two-part proof, and both parts are absolutely necessary. There can be mathematical statements that hold for one or the other of the two parts but not for both. For example, $(a + b)^n = a^n + b^n$ is true for $n = 1$, but it is false for every positive integer greater than 1. Therefore, if we were to attempt a mathematical induction proof for $(a + b)^n = a^n + b^n$, we could establish part 1 but not part 2. Another example of this type is the statement that $n^2 - n + 41$ produces a prime number for all positive integer values of n. This statement is true for $n = 1, 2, 3, 4, \ldots, 40$, but it is false when $n = 41$ (because $41^2 - 41 + 41 = 41^2$, which is not a prime number).

It is also possible that part 2 of a mathematical induction proof can be established but not part 1. For example, consider the sequence $a_n = n$ and the sum formula $S_n = (n + 3)(n - 2)/2$. If $n = 1$, then $a_1 = 1$ but $S_1 = (4)(-1)/2 = -2$, so part 1 does not hold. However, it is possible to show that $S_k = (k + 3)(k - 2)/2$ implies $S_{k+1} = (k + 4)(k - 1)/2$. We will leave the details of this for you to do.

Finally, it is important to realize that some mathematical statements are true for all positive integers greater than some fixed positive integer other than 1. (In Figure 14.1, this implies that we cannot knock down the first four dominoes; however, we can knock down the fifth domino and every one thereafter.) For example, we can prove by mathematical induction that $2^n > n^2$ for all positive integers $n > 4$. It requires a slight variation in the statement of the principle of mathematical induction. We will not concern ourselves with such problems in this text, but we want you to be aware of their existence.

Concept Quiz 14.4

For Problems 1–4, answer true or false.

1. Mathematical induction is used to prove mathematical statements involving positive integers.
2. A proof by mathematical induction consists of two parts.
3. Because $(a + b)^n = a^n + b^n$ is true for $n = 1$, it is true for all positive integer values of n.
4. To prove a mathematical statement involving positive integers by mathematical induction, the statement must be true for $n = 1$.

Problem Set 14.4

For Problems 1–10, use mathematical induction to prove each of the sum formulas for the indicated sequences. They are to hold for all positive integers n. **(Objective 1)**

1. $S_n = \dfrac{n(n + 1)}{2}$ for $a_n = n$
2. $S_n = n^2$ for $a_n = 2n - 1$
3. $S_n = \dfrac{n(3n + 1)}{2}$ for $a_n = 3n - 1$
4. $S_n = \dfrac{n(5n + 9)}{2}$ for $a_n = 5n + 2$
5. $S_n = 2(2^n - 1)$ for $a_n = 2^n$
6. $S_n = \dfrac{3(3^n - 1)}{2}$ for $a_n = 3^n$
7. $S_n = \dfrac{n(n + 1)(2n + 1)}{6}$ for $a_n = n^2$
8. $S_n = \dfrac{n^2(n + 1)^2}{4}$ for $a_n = n^3$
9. $S_n = \dfrac{n}{n + 1}$ for $a_n = \dfrac{1}{n(n + 1)}$
10. $S_n = \dfrac{n(n + 1)(n + 2)}{3}$ for $a_n = n(n + 1)$

In Problems 11–20, use mathematical induction to prove that each statement is true for all positive integers n.

11. $3^n \geq 2n + 1$

12. $4^n \geq 4n$

13. $n^2 \geq n$

14. $2^n \geq n + 1$

15. $4^n - 1$ is divisible by 3

16. $5^n - 1$ is divisible by 4

17. $6^n - 1$ is divisible by 5

18. $9^n - 1$ is divisible by 4

19. $n^2 + n$ is divisible by 2

20. $n^2 - n$ is divisible by 2

Thoughts Into Words

21. How would you describe proof by mathematical induction?

22. Compare inductive reasoning to prove by mathematical induction.

Answers to the Concept Quiz

1. True **2.** True **3.** False **4.** False