Graphing Calculator Activities

40. Use determinants and your calculator to solve each of the following systems:

(a) $\begin{pmatrix} 4x - 3y + z = 10 \\ 8x + 5y - 2z = -6 \\ -12x - 2y + 3z = -2 \end{pmatrix}$

(b) $\begin{pmatrix} 2x + y - z + w = -4 \\ x + 2y + 2z - 3w = 6 \\ 3x - y - z + 2w = 0 \\ 2x + 3y + z + 4w = -5 \end{pmatrix}$

(c) $\begin{pmatrix} x - 2y + z - 3w = 4 \\ 2x + 3y - z - 2w = -4 \\ 3x - 4y + 2z - 4w = 12 \\ 2x - y - 3z + 2w = -2 \end{pmatrix}$

(d) $\begin{pmatrix} 1.98x + 2.49y + 3.45z = 80.10 \\ 2.15x + 3.20y + 4.19z = 97.16 \\ 1.49x + 4.49y + 2.79z = 83.92 \end{pmatrix}$

Answers to the Concept Quiz

1. True **2.** False **3.** True **4.** True **5.** True **6.** True **7.** False **8.** True

11.6 Partial Fractions (Optional)

OBJECTIVE 1 Find partial fraction decompositions for rational expressions

In Chapter 4, we reviewed the process of adding rational expressions. For example,

$$\frac{3}{x-2} + \frac{2}{x+3} = \frac{3(x+3) + 2(x-2)}{(x-2)(x+3)} = \frac{3x + 9 + 2x - 4}{(x-2)(x+3)} = \frac{5x+5}{(x-2)(x+3)}$$

Now suppose that we want to reverse the process. That is, suppose we are given the rational expression

$$\frac{5x+5}{(x-2)(x+3)}$$

and we want to express it as the sum of two simpler rational expressions called **partial fractions**. This process, called **partial fraction decomposition**, has several applications in calculus and differential equations. The following property provides the basis for partial fraction decomposition.

Property 11.6

Let $f(x)$ and $g(x)$ be polynomials with real coefficients, such that the degree of $f(x)$ is less than the degree of $g(x)$. The indicated quotient $f(x)/g(x)$ can be decomposed into partial fractions as follows.

1. If $g(x)$ has a linear factor of the form $ax + b$, then the partial fraction decomposition will contain a term of the form

   $$\frac{A}{ax+b} \quad \text{where } A \text{ is a constant}$$

2. If $g(x)$ has a linear factor of the form $ax + b$ raised to the kth power, then the partial fraction decomposition will contain terms of the form

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

   where $A_1, A_2, \ldots, A_k$ are constants.

(Continued)

3. If $g(x)$ has a quadratic factor of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the partial fraction decomposition will contain a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \quad \text{where } A \text{ and } B \text{ are constants.}$$

4. If $g(x)$ has a quadratic factor of the form $ax^2 + bx + c$ raised to the kth power, where $b^2 - 4ac < 0$, then the partial fraction decomposition will contain terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_kx}{(ax^2 + bx + c)^k}$$

where $A_1, A_2, \ldots, A_k$, and $B_1, B_2, \ldots, B_k$ are constants.

Note that Property 11.6 applies only to **proper fractions**—that is, fractions in which the degree of the numerator is less than the degree of the denominator. If the numerator is not of lower degree, we can divide and then apply Property 11.6 to the remainder, which will be a proper fraction. For example,

$$\frac{x^3 - 3x^2 - 3x - 5}{x^2 - 4} = x - 3 + \frac{x - 17}{x^2 - 4}$$

and the proper fraction $\frac{x - 17}{x^2 - 4}$ can be decomposed into partial fractions by applying Property 11.6. Now let's consider some examples to illustrate the four cases in Property 11.6.

Classroom Example
Find the partial fraction decomposition of $\frac{13x - 5}{5x^2 - 13x - 6}$.

EXAMPLE 1

Find the partial fraction decomposition of $\frac{11x + 2}{2x^2 + x - 1}$.

Solution

The denominator can be expressed as $(x + 1)(2x - 1)$. Therefore, according to part 1 of Property 11.6, each of the linear factors produces a partial fraction of the form *constant over linear factor*. In other words, we can write

$$\frac{11x + 2}{(x + 1)(2x - 1)} = \frac{A}{x + 1} + \frac{B}{2x - 1} \qquad (1)$$

for some constants A and B. To find A and B, we multiply both sides of equation (1) by the least common denominator $(x + 1)(2x - 1)$:

$$11x + 2 = A(2x - 1) + B(x + 1) \qquad (2)$$

Equation (2) is an **identity**: *It is true for all values of* x. Therefore, let's choose some convenient values for x that will determine the values for A and B. If we let $x = -1$, then equation (2) becomes an equation in only A.

$$11(-1) + 2 = A[2(-1) - 1] + B(-1 + 1)$$
$$-9 = -3A$$
$$3 = A$$

If we let $x = \frac{1}{2}$, then equation (2) becomes an equation only in B.

$$11\left(\frac{1}{2}\right) + 2 = A\left[2\left(\frac{1}{2}\right) - 1\right] + B\left(\frac{1}{2} + 1\right)$$
$$\frac{15}{2} = \frac{3}{2}B$$
$$5 = B$$

Therefore, the given rational expression can now be written

$$\frac{11x + 2}{2x^2 + x - 1} = \frac{3}{x + 1} + \frac{5}{2x - 1}$$

The key idea in Example 1 is the statement that equation (2) is true for all values of x. If we had chosen *any* two values for x, we still would have been able to determine the values for A and B. For example, letting $x = 1$ and then $x = 2$ produces the equations $13 = A + 2B$ and $24 = 3A + 3B$. Solving this system of two equations in two unknowns produces $A = 3$ and $B = 5$. In Example 1, our choices of letting $x = -1$ and then $x = \frac{1}{2}$ simply eliminated the need for solving a system of equations to find A and B.

Classroom Example
Find the partial fraction decomposition of $\frac{6x^2 + 25x + 36}{x(x + 3)^2}$.

EXAMPLE 2 Find the partial fraction decomposition of $\frac{-2x^2 + 7x + 2}{x(x - 1)^2}$.

Solution

Apply part 1 of Property 11.6 to determine that there is a partial fraction of the form A/x corresponding to the factor of x. Next, applying part 2 of Property 11.6 and the squared factor $(x - 1)^2$ gives rise to a sum of partial fractions of the form

$$\frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

Therefore, the complete partial fraction decomposition is of the form

$$\frac{-2x^2 + 7x + 2}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} \tag{1}$$

Multiply both sides of equation (1) by $x(x - 1)^2$ to produce

$$-2x^2 + 7x + 2 = A(x - 1)^2 + Bx(x - 1) + Cx \tag{2}$$

which is true for all values of x. If we let $x = 1$, then equation (2) becomes an equation in only C.

$$\begin{aligned} -2(1)^2 + 7(1) + 2 &= A(1 - 1)^2 + B(1)(1 - 1) + C(1) \\ 7 &= C \end{aligned}$$

If we let $x = 0$, then equation (2) becomes an equation in just A.

$$\begin{aligned} -2(0)^2 + 7(0) + 2 &= A(0 - 1)^2 + B(0)(0 - 1) + C(0) \\ 2 &= A \end{aligned}$$

If we let $x = 2$, then equation (2) becomes an equation in A, B, and C.

$$\begin{aligned} -2(2)^2 + 7(2) + 2 &= A(2 - 1)^2 + B(2)(2 - 1) + C(2) \\ 8 &= A + 2B + 2C \end{aligned}$$

But we already know that $A = 2$ and $C = 7$, so we can easily determine B.

$$\begin{aligned} 8 &= 2 + 2B + 14 \\ -8 &= 2B \\ -4 &= B \end{aligned}$$

Therefore, the original rational expression can be written

$$\frac{-2x^2 + 7x + 2}{x(x - 1)^2} = \frac{2}{x} - \frac{4}{x - 1} + \frac{7}{(x - 1)^2}$$

Classroom Example
Find the partial fraction decomposition of $\dfrac{5x^2 + 21x + 4}{(x + 2)(x^2 + 6x + 2)}$.

EXAMPLE 3 Find the partial fraction decomposition of $\dfrac{4x^2 + 6x - 10}{(x + 3)(x^2 + x + 2)}$.

Solution

Apply part 1 of Property 11.6 to determine that there is a partial fraction of the form $A/(x + 3)$ that corresponds to the factor $x + 3$. Apply part 3 of Property 11.6 to determine that there is also a partial fraction of the form

$$\frac{Bx + C}{x^2 + x + 2}$$

Thus the complete partial fraction decomposition is of the form

$$\frac{4x^2 + 6x - 10}{(x + 3)(x^2 + x + 2)} = \frac{A}{x + 3} + \frac{Bx + C}{x^2 + x + 2} \tag{1}$$

Multiply both sides of equation (1) by $(x + 3)(x^2 + x + 2)$ to produce

$$4x^2 + 6x - 10 = A(x^2 + x + 2) + (Bx + C)(x + 3) \tag{2}$$

which is true for all values of x. If we let $x = -3$, then equation (2) becomes an equation in A alone.

$$\begin{aligned} 4(-3)^2 + 6(-3) - 10 &= A[(-3)^2 + (-3) + 2] + [B(-3) + C][(-3) + 3] \\ 8 &= 8A \\ 1 &= A \end{aligned}$$

If we let $x = 0$, then equation (2) becomes an equation in A and C.

$$\begin{aligned} 4(0)^2 + 6(0) - 10 &= A(0^2 + 0 + 2) + [B(0) + C](0 + 3) \\ -10 &= 2A + 3C \end{aligned}$$

Because $A = 1$, we obtain the value of C.

$$\begin{aligned} -10 &= 2 + 3C \\ -12 &= 3C \\ -4 &= C \end{aligned}$$

If we let $x = 1$, then equation (2) becomes an equation in A, B, and C.

$$\begin{aligned} 4(1)^2 + 6(1) - 10 &= A(1^2 + 1 + 2) + [B(1) + C](1 + 3) \\ 0 &= 4A + 4B + 4C \\ 0 &= A + B + C \end{aligned}$$

But because $A = 1$ and $C = -4$, we obtain the value of B.

$$\begin{aligned} 0 &= A + B + C \\ 0 &= 1 + B + (-4) \\ 3 &= B \end{aligned}$$

Therefore, the original rational expression can now be written

$$\frac{4x^2 + 6x - 10}{(x + 3)(x^2 + x + 2)} = \frac{1}{x + 3} + \frac{3x - 4}{x^2 + x + 2}$$

Classroom Example
Find the partial fraction decomposition of $\frac{x^3 + 2x^2 + 3x + 9}{(x^2 + 3)^2}$.

EXAMPLE 4

Find the partial fraction decomposition of $\frac{x^3 + x^2 + x + 3}{(x^2 + 1)^2}$.

Solution

Apply part 4 of Property 11.6 to determine that the partial fraction decomposition of this fraction is of the form

$$\frac{x^3 + x^2 + x + 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \tag{1}$$

Multiply both sides of equation (1) by $(x^2 + 1)^2$ to produce

$$x^3 + x^2 + x + 3 = (Ax + B)(x^2 + 1) + Cx + D \tag{2}$$

which is true for all values of x. Equation (2) is an identity, so we know that the coefficients of similar terms on both sides of the equation must be equal. Therefore, let's collect similar terms on the right side of equation (2).

$$\begin{aligned} x^3 + x^2 + x + 3 &= Ax^3 + Ax + Bx^2 + B + Cx + D \\ &= Ax^3 + Bx^2 + (A + C)x + B + D \end{aligned}$$

Now we can equate coefficients from both sides:

$$1 = A \qquad 1 = B \qquad 1 = A + C \qquad \text{and} \qquad 3 = B + D$$

From these equations, we can determine that $A = 1$, $B = 1$, $C = 0$, and $D = 2$. Therefore, the original rational expression can be written

$$\frac{x^3 + x^2 + x + 3}{(x^2 + 1)^2} = \frac{x + 1}{x^2 + 1} + \frac{2}{(x^2 + 1)^2}$$

Concept Quiz 11.6

For Problems 1–8, answer true or false.

1. The process of partial fraction decomposition expresses a rational expression as the sum of two or more simpler rational expressions.
2. A rational expression is considered a proper fraction if the degree of the numerator is equal to or less than the degree of the denominator.
3. The process of partial fraction decomposition applies only to proper fractions.
4. To apply partial fraction decomposition to a rational expression that is not a proper fraction, use long division to obtain a remainder that is a proper fraction.
5. If an equation is an identity, any value of x substituted into the equation produces an equivalent equation.
6. A quadratic expression such as $ax^2 + bx + c$ is not factorable over the real numbers if $b^2 - 4ac < 0$.
7. Given that $5x + 3 = A(x - 4) + B(x + 7)$ is an identity, the value of A can be determined by substituting any value of x into the identity.
8. Given that $3x - 1 = A(x - 2) + B(x + 5)$ is an identity, the value of B can be determined by substituting -5 for x into the identity.

Problem Set 11.6

For Problems 1–22, find the partial fraction decomposition for each rational expression. **(Objective 1)**

1. $\dfrac{11x - 10}{(x - 2)(x + 1)}$

2. $\dfrac{11x - 2}{(x + 3)(x - 4)}$

3. $\dfrac{-2x - 8}{x^2 - 1}$

4. $\dfrac{-2x + 32}{x^2 - 4}$

5. $\dfrac{20x - 3}{6x^2 + 7x - 3}$

6. $\dfrac{-2x - 8}{10x^2 - x - 2}$

7. $\dfrac{x^2 - 18x + 5}{(x - 1)(x + 2)(x - 3)}$

8. $\dfrac{-9x^2 + 7x - 4}{x^3 - 3x^2 - 4x}$

9. $\dfrac{-6x^2 + 7x + 1}{x(2x - 1)(4x + 1)}$

10. $\dfrac{15x^2 + 20x + 30}{(x + 3)(3x + 2)(2x + 3)}$

11. $\dfrac{2x + 1}{(x - 2)^2}$

12. $\dfrac{-3x + 1}{(x + 1)^2}$

13. $\dfrac{-6x^2 + 19x + 21}{x^2(x + 3)}$

14. $\dfrac{10x^2 - 73x + 144}{x(x - 4)^2}$

15. $\dfrac{-2x^2 - 3x + 10}{(x^2 + 1)(x - 4)}$

16. $\dfrac{8x^2 + 15x + 12}{(x^2 + 4)(3x - 4)}$

17. $\dfrac{3x^2 + 10x + 9}{(x + 2)^3}$

18. $\dfrac{2x^3 + 8x^2 + 2x + 4}{(x + 1)^2(x^2 + 3)}$

19. $\dfrac{5x^2 + 3x + 6}{x(x^2 - x + 3)}$

20. $\dfrac{x^3 + x^2 + 2}{(x^2 + 2)^2}$

21. $\dfrac{2x^3 + x + 3}{(x^2 + 1)^2}$

22. $\dfrac{4x^2 + 3x + 14}{x^3 - 8}$

Thoughts Into Words

23. Give a general description of partial fraction decomposition for someone who missed class the day it was discussed.

24. Give a step-by-step explanation of how to find the partial fraction decomposition of $\dfrac{11x + 5}{2x^2 + 5x - 3}$.

Answers to the Concept Quiz

1. True **2.** False **3.** True **4.** True **5.** True **6.** True **7.** False **8.** False