Functions of a Complex Variable I

Analytic Properties, Mapping

The imaginary numbers are a wonderful flight of God's spirit;
they are almost an amphibian between being and not being.

Gottfried Wilhelm von Leibniz, 1702

We turn now to a study of functions of a complex variable. In this area we develop some of the most powerful and widely useful tools in all of analysis. To indicate, at least partly, why complex variables are important, we mention briefly several areas of application.

1. For many pairs of functions u and v, both u and v satisfy Laplace's equation,

$$\nabla^2\psi = \frac{\partial^2\psi(x, y)}{\partial x^2} + \frac{\partial^2\psi(x, y)}{\partial y^2} = 0.$$

Hence either u or v may be used to describe a two-dimensional electrostatic potential. The other function, which gives a family of curves orthogonal to those of the first function, may then be used to describe the electric field $\mathbf{E}$. A similar situation holds for the hydrodynamics of an ideal fluid in irrotational motion. The function u might describe the velocity potential, whereas the function v would then be the stream function.

In many cases in which the functions u and v are unknown, mapping or transforming in the complex plane permits us to create a coordinate system tailored to the particular problem.

2. In Chapter 9 we shall see that the second-order differential equations of interest in physics may be solved by power series. The same power series may be used in the complex plane to replace x by the complex variable z. The dependence of the solution $f(z)$ at a given z_0 on the behavior of $f(z)$ elsewhere gives us greater insight into the behavior of our

solution and a powerful tool (analytic continuation) for extending the region in which the solution is valid.

3. The change of a parameter k from real to imaginary, $k \to ik$, transforms the Helmholtz equation into the diffusion equation. The same change transforms the Helmholtz equation solutions (Bessel and spherical Bessel functions) into the diffusion equation solutions (modified Bessel and modified spherical Bessel functions).

4. Integrals in the complex plane have a wide variety of useful applications:

- Evaluating definite integrals;
- Inverting power series;
- Forming infinite products;
- Obtaining solutions of differential equations for large values of the variable (asymptotic solutions);
- Investigating the stability of potentially oscillatory systems;
- Inverting integral transforms.

5. Many physical quantities that were originally real become complex as a simple physical theory is made more general. The real index of refraction of light becomes a complex quantity when absorption is included. The real energy associated with an energy level becomes complex when the finite lifetime of the level is considered.

6.1 COMPLEX ALGEBRA

A complex number is nothing more than an ordered pair of two real numbers, (a, b). Similarly, a complex variable is an ordered pair of two real variables,[1]

$$z \equiv (x, y). \tag{6.1}$$

The ordering is significant. In general (a, b) is not equal to (b, a) and (x, y) is not equal to (y, x). As usual, we continue writing a real number $(x, 0)$ simply as x, and we call $i \equiv (0, 1)$ the imaginary unit.

All our complex variable analysis can be developed in terms of ordered pairs of numbers (a, b), variables (x, y), and functions $(u(x, y), v(x, y))$.

We now define **addition** of complex numbers in terms of their Cartesian components as

$$\boxed{z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),} \tag{6.2a}$$

that is, two-dimensional vector addition. In Chapter 1 the points in the xy-plane are identified with the two-dimensional displacement vector $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. As a result, two-dimensional vector analogs can be developed for much of our complex analysis. Exercise 6.1.2 is one simple example; Cauchy's theorem, Section 6.3, is another.

Multiplication of complex numbers is defined as

$$\boxed{z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).} \tag{6.2b}$$

[1] This is precisely how a computer does complex arithmetic.

Using Eq. (6.2b) we verify that $i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$, so we can also identify $i = \sqrt{-1}$, as usual and further rewrite Eq. (6.1) as

$$\boxed{z = (x, y) = (x, 0) + (0, y) = x + (0, 1) \cdot (y, 0) = x + iy.} \tag{6.2c}$$

Clearly, the i is not necessary here but it is convenient. It serves to keep pairs in order—somewhat like the unit vectors of Chapter 1.[2]

Permanence of Algebraic Form

All our elementary functions, e^z, $\sin z$, and so on, can be extended into the complex plane (compare Exercise 6.1.9). For instance, they can be defined by power-series expansions, such as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{6.3}$$

for the exponential. Such definitions agree with the real variable definitions along the real x-axis and extend the corresponding real functions into the complex plane. This result is often called **permanence of the algebraic form**.

It is convenient to employ a graphical representation of the complex variable. By plotting x—the real part of z—as the abscissa and y—the imaginary part of z—as the ordinate, we have the complex plane, or Argand plane, shown in Fig. 6.1. If we assign specific values to x and y, then z corresponds to a point (x, y) in the plane. In terms of the ordering mentioned before, it is obvious that the point (x, y) does not coincide with the point (y, x) except for the special case of $x = y$. Further, from Fig. 6.1 we may write

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{6.4a}$$

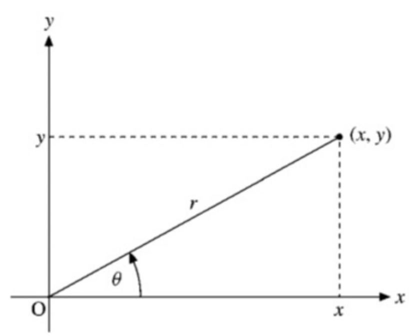

FIGURE 6.1 Complex plane—Argand diagram.

[2]The algebra of complex numbers, (a, b), is isomorphic with that of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(compare Exercise 3.2.4).

and

$$z = r(\cos\theta + i\sin\theta). \tag{6.4b}$$

Using a result that is suggested (but not rigorously proved)[3] by Section 5.6 and Exercise 5.6.1, we have the useful polar representation

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta}.} \tag{6.4c}$$

In order to prove this identity, we use $i^3 = -i,\ i^4 = 1, \ldots$ in the Taylor expansion of the exponential and trigonometric functions and separate even and odd powers in

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu+1}}{(2\nu+1)!} \\ &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu}}{(2\nu)!} + i \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu+1}}{(2\nu+1)!} = \cos\theta + i\sin\theta. \end{aligned}$$

For the special values $\theta = \pi/2$ and $\theta = \pi$, we obtain

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i, \qquad e^{i\pi} = \cos(\pi) = -1,$$

intriguing connections between e, i, and π. Moreover, the exponential function $e^{i\theta}$ is periodic with period 2π, just like $\sin\theta$ and $\cos\theta$.

In this representation r is called the **modulus** or **magnitude** of z ($r = |z| = (x^2 + y^2)^{1/2}$) and the angle θ ($= \tan^{-1}(y/x)$) is labeled the argument or **phase** of z. (Note that the arctan function $\tan^{-1}(y/x)$ has infinitely many branches.)

The choice of polar representation, Eq. (6.4c), or Cartesian representation, Eqs. (6.1) and (6.2c), is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation, Eq. (6.2a). Multiplication, division, powers, and roots are easier to handle in polar form, Eq. (6.4c).

Analytically or graphically, using the vector analogy, we may show that the modulus of the sum of two complex numbers is no greater than the sum of the moduli and no less than the difference, Exercise 6.1.3,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|. \tag{6.5}$$

Because of the vector analogy, these are called the **triangle** inequalities.

Using the polar form, Eq. (6.4c), we find that the magnitude of a product is the product of the magnitudes:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|. \tag{6.6}$$

Also,

$$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2. \tag{6.7}$$

[3]Strictly speaking, Chapter 5 was limited to real variables. The development of power-series expansions for complex functions is taken up in Section 6.5 (Laurent expansion).

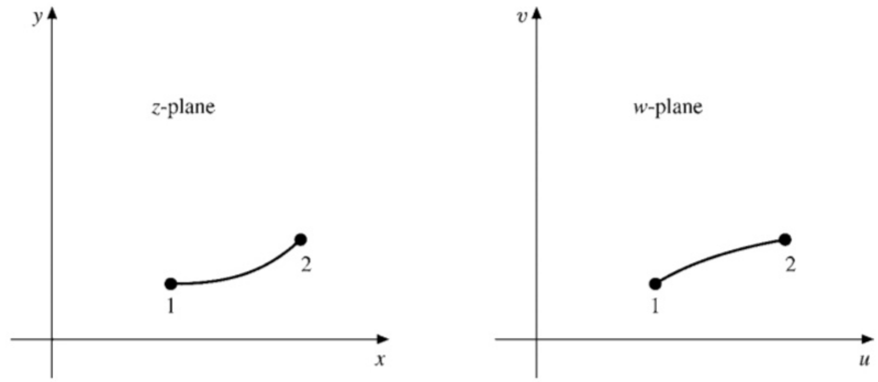

FIGURE 6.2 The function $w(z) = u(x, y) + iv(x, y)$ maps points in the xy-plane into points in the uv-plane.

From our complex variable z complex functions $f(z)$ or $w(z)$ may be constructed. These complex functions may then be resolved into real and imaginary parts,

$$w(z) = u(x, y) + iv(x, y), \tag{6.8}$$

in which the separate functions $u(x, y)$ and $v(x, y)$ are pure real. For example, if $f(z) = z^2$, we have

$$f(z) = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

The **real part** of a function $f(z)$ will be labeled $\Re f(z)$, whereas the **imaginary part** will be labeled $\Im f(z)$. In Eq. (6.8)

$$\Re w(z) = \mathrm{Re}(w) = u(x, y), \qquad \Im w(z) = \mathrm{Im}(w) = v(x, y).$$

The relationship between the independent variable z and the dependent variable w is perhaps best pictured as a mapping operation. A given $z = x + iy$ means a given point in the z-plane. The complex value of $w(z)$ is then a point in the w-plane. Points in the z-plane map into points in the w-plane and curves in the z-plane map into curves in the w-plane, as indicated in Fig. 6.2.

Complex Conjugation

In all these steps, complex number, variable, and function, the operation of replacing i by $-i$ is called "taking the complex conjugate." The complex conjugate of z is denoted by z^*, where[4]

$$\boxed{z^* = x - iy.} \tag{6.9}$$

[4]The complex conjugate is often denoted by $\bar{z}$ in the mathematical literature.

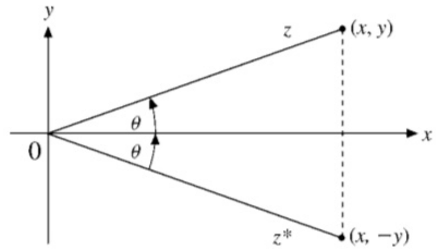

FIGURE 6.3 Complex conjugate points.

The complex variable z and its complex conjugate z^* are mirror images of each other reflected in the x-axis, that is, inversion of the y-axis (compare Fig. 6.3). The product zz^* leads to

$$\boxed{zz^* = (x+iy)(x-iy) = x^2 + y^2 = r^2.} \tag{6.10}$$

Hence

$$(zz^*)^{1/2} = |z|,$$

the **magnitude** of z.

Functions of a Complex Variable

All the elementary functions of real variables may be extended into the complex plane—replacing the real variable x by the complex variable z. This is an example of the analytic continuation mentioned in Section 6.5. The extremely important relation of Eq. (6.4c) is an illustration. Moving into the complex plane opens up new opportunities for analysis.

Example 6.1.1 DE MOIVRE'S FORMULA

If Eq. (6.4c) (setting $r = 1$) is raised to the nth power, we have

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n. \tag{6.11}$$

Expanding the exponential now with argument $n\theta$, we obtain

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n. \tag{6.12}$$

De Moivre's formula is generated if the right-hand side of Eq. (6.12) is expanded by the binomial theorem; we obtain $\cos n\theta$ as a series of powers of $\cos\theta$ and $\sin\theta$, Exercise 6.1.6. ■

Numerous other examples of relations among the exponential, hyperbolic, and trigonometric functions in the complex plane appear in the exercises.

Occasionally there are complications. The logarithm of a complex variable may be expanded using the polar representation

$$\ln z = \ln re^{i\theta} = \ln r + i\theta. \tag{6.13a}$$

This is not complete. To the phase angle, θ, we may add any integral multiple of 2π without changing z. Hence Eq. (6.13a) should read

$$\ln z = \ln re^{i(\theta+2n\pi)} = \ln r + i(\theta + 2n\pi). \tag{6.13b}$$

The parameter n may be any integer. This means that $\ln z$ is a **multivalued** function having an infinite number of values for a single pair of real values r and θ. To avoid ambiguity, the simplest choice is $n = 0$ and limitation of the phase to an interval of length 2π, such as $(-\pi, \pi)$.[5] The line in the z-plane that is not crossed, the negative real axis in this case, is labeled a **cut line** or **branch cut**. The value of $\ln z$ with $n = 0$ is called the **principal value** of $\ln z$. Further discussion of these functions, including the logarithm, appears in Section 6.7.

Exercises

6.1.1 (a) Find the reciprocal of $x + iy$, working entirely in the Cartesian representation.
(b) Repeat part (a), working in polar form but expressing the final result in Cartesian form.

6.1.2 The complex quantities $a = u + iv$ and $b = x + iy$ may also be represented as two-dimensional vectors $\mathbf{a} = \hat{\mathbf{x}}u + \hat{\mathbf{y}}v$, $\mathbf{b} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. Show that

$$a^*b = \mathbf{a} \cdot \mathbf{b} + i\hat{\mathbf{z}} \cdot \mathbf{a} \times \mathbf{b}.$$

6.1.3 Prove algebraically that for complex numbers,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of two-dimensional vectors. Prove that

$$|z - 1| < \left|\sqrt{z^2 - 1}\right| < |z + 1|, \qquad \text{for} \quad \Re(z) > 0.$$

6.1.4 We may define a complex conjugation operator K such that $Kz = z^*$. Show that K is not a linear operator.

6.1.5 Show that complex numbers have square roots and that the square roots are contained in the complex plane. What are the square roots of i?

6.1.6 Show that

(a) $\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \cdots$.
(b) $\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \cdots$.

Note. The quantities $\binom{n}{m}$ are binomial coefficients: $\binom{n}{m} = n!/[(n-m)!m!]$.

(b) $$\sum_{n=0}^{N-1} \sin nx = \frac{\sin(Nx/2)}{\sin x/2} \sin(N-1)\frac{x}{2}.$$

These series occur in the analysis of the multiple-slit diffraction pattern. Another application is the analysis of the Gibbs phenomenon, Section 14.5.
Hint. Parts (a) and (b) may be combined to form a geometric series (compare Section 5.1).

6.1.8 For $-1 < p < 1$ prove that

(a) $$\sum_{n=0}^{\infty} p^n \cos nx = \frac{1 - p\cos x}{1 - 2p\cos x + p^2},$$

(b) $$\sum_{n=0}^{\infty} p^n \sin nx = \frac{p\sin x}{1 - 2p\cos x + p^2}.$$

These series occur in the theory of the Fabry–Perot interferometer.

6.1.9 Assume that the trigonometric functions and the hyperbolic functions are defined for complex argument by the appropriate power series

$$\sin z = \sum_{n=1,\text{odd}}^{\infty} (-1)^{(n-1)/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!},$$

$$\cos z = \sum_{n=0,\text{even}}^{\infty} (-1)^{n/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!},$$

$$\sinh z = \sum_{n=1,\text{odd}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!},$$

$$\cosh z = \sum_{n=0,\text{even}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!}.$$

(a) Show that

$$i\sin z = \sinh iz, \qquad \sin iz = i\sinh z,$$
$$\cos z = \cosh iz, \qquad \cos iz = \cosh z.$$

(b) Verify that familiar functional relations such as

$$\cosh z = \tfrac{e^z + e^{-z}}{2},$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1,$$

still hold in the complex plane.

6.1.10 Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

established from comparison of power series, show that

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$

$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$

This demonstrates that we may have $|\sin z|, |\cos z| > 1$ in the complex plane.

6.1.11 From the identities in Exercises 6.1.9 and 6.1.10 show that

(a) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$

$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$

(b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y, \qquad |\cosh z|^2 = \cosh^2 x + \sin^2 y.$

6.1.12 Prove that

(a) $|\sin z| \geq |\sin x|$ (b) $|\cos z| \geq |\cos x|$.

6.1.13 Show that the exponential function e^z is periodic with a pure imaginary period of $2\pi i$.

6.1.14 Show that

(a) $\tanh \frac{z}{2} = \frac{\sinh x + i \sin y}{\cosh x + \cos y}$, (b) $\coth \frac{z}{2} = \frac{\sinh x - i \sin y}{\cosh x - \cos y}$.

6.1.15 Find all the zeros of

(a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

6.1.16 Show that

(a) $\sin^{-1} z = -i \ln\left(iz \pm \sqrt{1 - z^2}\right)$, (d) $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$,

(b) $\cos^{-1} z = -i \ln\left(z \pm \sqrt{z^2 - 1}\right)$, (e) $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

(c) $\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)$, (f) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$.

Hint. 1. Express the trigonometric and hyperbolic functions in terms of exponentials. 2. Solve for the exponential and then for the exponent.

6.1.17 In the quantum theory of the photoionization we encounter the identity

$$\left(\frac{ia - 1}{ia + 1}\right)^{ib} = \exp\left(-2b \cot^{-1} a\right),$$

in which a and b are real. Verify this identity.

6.1.18 A plane wave of light of angular frequency ω is represented by

$$e^{i\omega(t-nx/c)}.$$

In a certain substance the simple real index of refraction n is replaced by the complex quantity $n - ik$. What is the effect of k on the wave? What does k correspond to physically? The generalization of a quantity from real to complex form occurs frequently in physics. Examples range from the complex Young's modulus of viscoelastic materials to the complex (optical) potential of the "cloudy crystal ball" model of the atomic nucleus.

6.1.19 We see that for the angular momentum components defined in Exercise 2.5.14,

$$L_x - iL_y \neq (L_x + iL_y)^*.$$

Explain why this occurs.

6.1.20 Show that the **phase** of $f(z) = u + iv$ is equal to the imaginary part of the logarithm of $f(z)$. Exercise 8.2.13 depends on this result.

6.1.21 (a) Show that $e^{\ln z}$ always equals z.
(b) Show that $\ln e^z$ does not always equal z.

6.1.22 The infinite product representations of Section 5.11 hold when the real variable x is replaced by the complex variable z. From this, develop infinite product representations for
(a) $\sinh z$, (b) $\cosh z$.

6.1.23 The equation of motion of a mass m **relative to a rotating coordinate system** is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right) - m\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right).$$

Consider the case $\mathbf{F} = 0$, $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, with ω constant. Show that the replacement of $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ by $z = x + iy$ leads to

$$\frac{d^2z}{dt^2} + i2\omega\frac{dz}{dt} - \omega^2 z = 0.$$

Note. This ODE may be solved by the substitution $z = fe^{-i\omega t}$.

6.1.24 Using the complex arithmetic available in FORTRAN, write a program that will calculate the complex exponential e^z from its series expansion (definition). Calculate e^z for $z = e^{in\pi/6}$, $n = 0, 1, 2, \ldots, 12$. Tabulate the phase angle $(\theta = n\pi/6)$, $\Re z$, $\Im z$, $\Re(e^z)$, $\Im(e^z)$, $|e^z|$, and the phase of e^z.

Check value. $n = 5$, $\theta = 2.61799$, $\Re(z) = -0.86602$,
$\Im z = 0.50000$, $\Re(e^z) = 0.36913$, $\Im(e^z) = 0.20166$,
$|e^z| = 0.42062$, phase$(e^z) = 0.50000$.

6.1.25 Using the complex arithmetic available in FORTRAN, calculate and tabulate $\Re(\sinh z)$, $\Im(\sinh z)$, $|\sinh z|$, and phase $(\sinh z)$ for $x = 0.0(0.1)1.0$ and $y = 0.0(0.1)1.0$.