

# Topic 4 Notes

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## 4 Cauchy's integral formula

### 4.1 Introduction

Cauchy's theorem is a big theorem which we will use almost daily from here on out. Right away it will reveal a number of interesting and useful properties of analytic functions. More will follow as the course progresses.

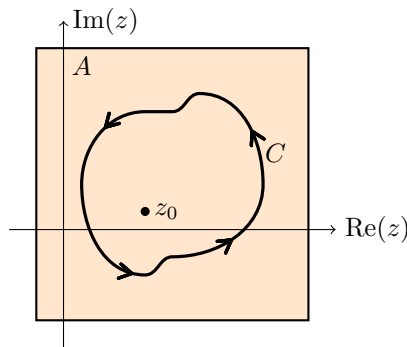
If you learn just one theorem this week it should be Cauchy's integral formula!

We start with a statement of the theorem for functions. After some examples, we'll give a generalization to all derivatives of a function. After some more examples we will prove the theorems. After that we will see some remarkable consequences that follow fairly directly from the Cauchy's formula.

### 4.2 Cauchy's integral for functions

**Theorem 4.1.** (Cauchy's integral formula) Suppose  $C$  is a simple closed curve and the function  $f(z)$  is analytic on a region containing  $C$  and its interior. We assume  $C$  is oriented counterclockwise. Then for any  $z_0$  inside  $C$ :

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \quad (1)$$



Cauchy's integral formula: simple closed curve  $C$ ,  $f(z)$  analytic on and inside  $C$ .

This is remarkable: it says that knowing the values of  $f$  on the boundary curve  $C$  means we know **everything** about  $f$  inside  $C$ !! This is probably unlike anything you've encountered with functions of real variables.

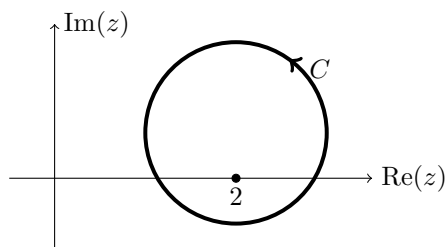
**Aside 1.** With a slight change of notation ( $z$  becomes  $w$  and  $z_0$  becomes  $z$ ) we often write the formula as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \quad (2)$$

**Aside 2.** We're not being entirely fair to functions of real variables. We will see that for  $f = u + iv$  the real and imaginary parts  $u$  and  $v$  have many similar remarkable properties.  $u$  and  $v$  are called conjugate harmonic functions.

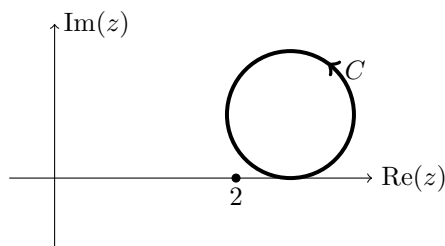
### 4.2.1 Examples

**Example 4.2.** Compute  $\int_C \frac{e^{z^2}}{z-2} dz$ , where  $C$  is the curve shown.



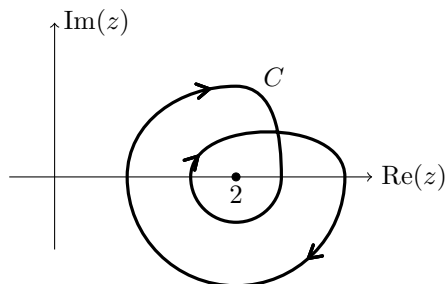
*Solution:* Let  $f(z) = e^{z^2}$ .  $f(z)$  is entire. Since  $C$  is a simple closed curve (counterclockwise) and  $z = 2$  is inside  $C$ , Cauchy's integral formula says that the integral is  $2\pi i f(2) = 2\pi i e^4$ .

**Example 4.3.** Do the same integral as the previous example with  $C$  the curve shown.

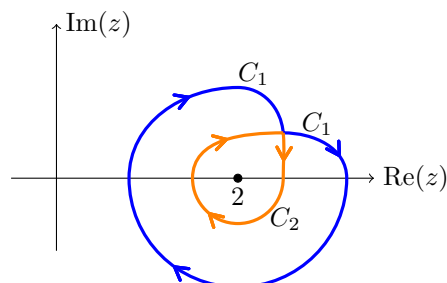


*Solution:* Since  $f(z) = e^{z^2}/(z-2)$  is analytic on and inside  $C$ , Cauchy's theorem says that the integral is 0.

**Example 4.4.** Do the same integral as the previous examples with  $C$  the curve shown.



*Solution:* This one is trickier. Let  $f(z) = e^{z^2}$ . The curve  $C$  goes around 2 twice in the clockwise direction, so we break  $C$  into  $C_1 + C_2$  as shown in the next figure.



These are both simple closed curves, so we can apply the Cauchy integral formula to each separately. (The negative signs are because they go clockwise around  $z = 2$ .)

$$\int_C \frac{f(z)}{z-2} dz = \int_{C_1} \frac{f(z)}{z-2} dz + \int_{C_2} \frac{f(z)}{z-2} dz = -2\pi i f(2) - 2\pi i f(2) = -4\pi i f(2).$$

### 4.3 Cauchy's integral formula for derivatives

Cauchy's integral formula is worth repeating several times. So, now we give it for all derivatives  $f^{(n)}(z)$  of  $f$ . This will include the formula for functions as a special case.

**Theorem 4.5. Cauchy's integral formula for derivatives.** If  $f(z)$  and  $C$  satisfy the same hypotheses as for Cauchy's integral formula then, for all  $z$  inside  $C$  we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw, \quad n = 0, 1, 2, \dots \quad (3)$$

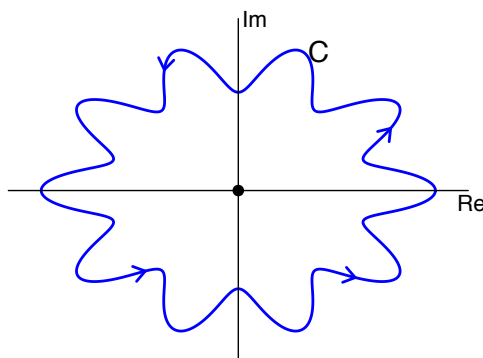
where,  $C$  is a simple closed curve, oriented counterclockwise,  $z$  is inside  $C$  and  $f(w)$  is analytic on and inside  $C$ .

**Example 4.6.** Evaluate  $I = \int_C \frac{e^{2z}}{z^4} dz$  where  $C : |z| = 1$ .

*Solution:* With Cauchy's formula for derivatives this is easy. Let  $f(z) = e^{2z}$ . Then,

$$I = \int_C \frac{f(z)}{z^4} dz = \frac{2\pi i}{3!} f'''(0) = \frac{8}{3}\pi i.$$

**Example 4.7.** Now Let  $C$  be the contour shown below and evaluate the same integral as in the previous example.



*Solution:* Again this is easy: the integral is the same as the previous example, i.e.  $I = \frac{8}{3}\pi i$ .

### 4.3.1 Another approach to some basic examples

Suppose  $C$  is a simple closed curve around 0. We have seen that

$$\int_C \frac{1}{z} dz = 2\pi i.$$

The Cauchy integral formula gives the same result. That is, let  $f(z) = 1$ , then the formula says

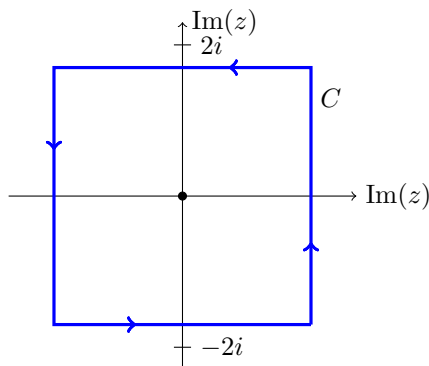
$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-0} dz = f(0) = 1.$$

Likewise Cauchy's formula for derivatives shows

$$\int_C \frac{1}{(z)^n} dz = \int_C \frac{f(z)}{z^{n+1}} dz = f^{(n)}(0) = 0, \quad \text{for integers } n > 1.$$

### 4.3.2 More examples

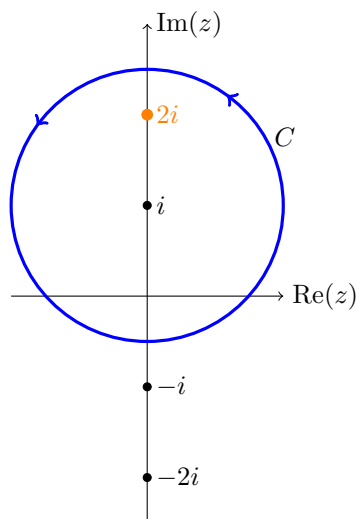
**Example 4.8.** Compute  $\int_C \frac{\cos(z)}{z(z^2+8)} dz$  over the contour shown.



*Solution:* Let  $f(z) = \cos(z)/(z^2+8)$ .  $f(z)$  is analytic on and inside the curve  $C$ . That is, the roots of  $z^2+8$  are outside the curve. So, we rewrite the integral as

$$\int_C \frac{\cos(z)/(z^2+8)}{z} dz = \int_C \frac{f(z)}{z} dz = 2\pi i f(0) = 2\pi i \frac{1}{8} = \frac{\pi i}{4}.$$

**Example 4.9.** Compute  $\int_C \frac{1}{(z^2+4)^2} dz$  over the contour shown.



*Solution:* We factor the denominator as

$$\frac{1}{(z^2 + 4)^2} = \frac{1}{(z - 2i)^2(z + 2i)^2}.$$

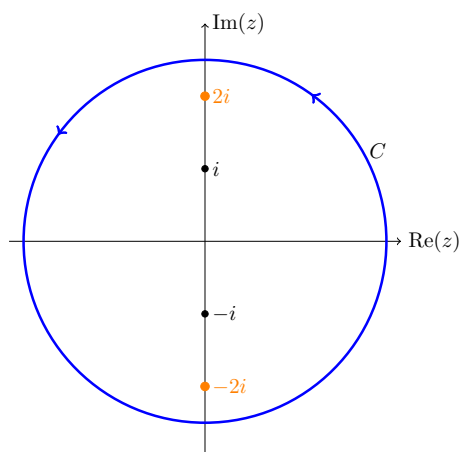
Let

$$f(z) = \frac{1}{(z + 2i)^2}$$

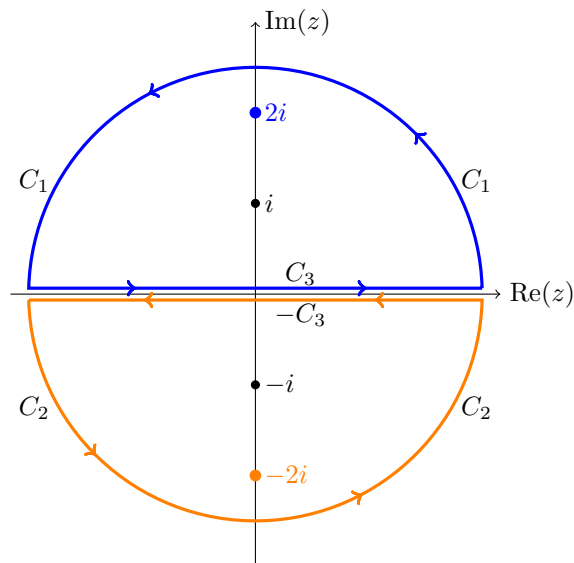
. Clearly  $f(z)$  is analytic inside  $C$ . So, by Cauchy's formula for derivatives:

$$\int_C \frac{1}{(z^2 + 4)^2} dz = \int_C \frac{f(z)}{(z - 2i)^2} dz = 2\pi i f'(2i) = 2\pi i \left[ \frac{-2}{(z + 2i)^3} \right]_{z=2i} = \frac{4\pi i}{64i} = \frac{\pi}{16}$$

**Example 4.10.** Compute  $\int_C \frac{z}{z^2 + 4} dz$  over the curve  $C$  shown below.



*Solution:* The integrand has singularities at  $\pm 2i$  and the curve  $C$  encloses them both. The solution to the previous solution won't work because we can't find an appropriate  $f(z)$  that is analytic on the whole interior of  $C$ . Our solution is to split the curve into two pieces. Notice that  $C_3$  is traversed both forward and backward.



Split the original curve  $C$  into 2 pieces that each surround just one singularity.

We have

$$\frac{z}{z^2 + 4} = \frac{z}{(z - 2i)(z + 2i)}.$$

We let

$$f_1(z) = \frac{z}{z + 2i} \quad \text{and} \quad f_2(z) = \frac{z}{z - 2i}.$$

So,

$$\frac{z}{z^2 + 4} = \frac{f_1(z)}{z - 2i} = \frac{f_2(z)}{z + 2i}.$$

The integral, can be written out as

$$\int_C \frac{z}{z^2 + 4} dz = \int_{C_1 + C_3 - C_3 + C_2} \frac{z}{z^2 + 4} dz = \int_{C_1 + C_3} \frac{f_1(z)}{z - 2i} dz + \int_{C_2 - C_3} \frac{f_2(z)}{z + 2i} dz$$

Since  $f_1$  is analytic inside the simple closed curve  $C_1 + C_3$  and  $f_2$  is analytic inside the simple closed curve  $C_2 - C_3$ , Cauchy's formula applies to both integrals. The total integral equals

$$2\pi i(f_1(2i) + f_2(-2i)) = 2\pi i(1/2 + 1/2) = 2\pi i.$$

**Remarks.** 1. We could also have done this problem using partial fractions:

$$\frac{z}{(z - 2i)(z + 2i)} = \frac{A}{z - 2i} + \frac{B}{z + 2i}.$$

It will turn out that  $A = f_1(2i)$  and  $B = f_2(-2i)$ . It is easy to apply the Cauchy integral formula to both terms.

2. **Important note.** In an upcoming topic we will formulate the Cauchy residue theorem. This will allow us to compute the integrals in Examples 4.8-4.10 in an easier and less ad hoc manner.

### 4.3.3 The triangle inequality for integrals

We discussed the triangle inequality in the Topic 1 notes. It says that

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad (4a)$$

with equality if and only if  $z_1$  and  $z_2$  lie on the same ray from the origin.

A useful variant of this statement is

$$|z_1| - |z_2| \leq |z_1 - z_2|. \quad (4b)$$

This follows because Equation 4a implies

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|.$$

Now subtracting  $z_2$  from both sides give Equation 4b

Since an integral is basically a sum, this translates to the triangle inequality for integrals. We'll state it in two ways that will be useful to us.

**Theorem 4.11.** (Triangle inequality for integrals) Suppose  $g(t)$  is a complex valued function of a real variable, defined on  $a \leq t \leq b$ . Then

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| dt,$$

with equality if and only if the values of  $g(t)$  all lie on the same ray from the origin.

*Proof.* This follows by approximating the integral as a Riemann sum.

$$\left| \int_a^b g(t) dt \right| \approx \left| \sum g(t_k) \Delta t \right| \leq \sum |g(t_k)| \Delta t \approx \int_a^b |g(t)| dt.$$

The middle inequality is just the standard triangle inequality for sums of complex numbers.  $\square$

**Theorem 4.12.** (Triangle inequality for integrals II) For any function  $f(z)$  and any curve  $\gamma$ , we have

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz|.$$

Here  $dz = \gamma'(t) dt$  and  $|dz| = |\gamma'(t)| dt$ .

*Proof.* This follows immediately from the previous theorem:

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt = \int_{\gamma} |f(z)| |dz|.$$

**Corollary.** If  $|f(z)| < M$  on  $C$  then

$$\left| \int_C f(z) dz \right| \leq M \cdot (\text{length of } C).$$

*Proof.* Let  $\gamma(t)$ , with  $a \leq t \leq b$ , be a parametrization of  $C$ . Using the triangle inequality

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)| |dz| = \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq \int_a^b M |\gamma'(t)| dt = M \cdot (\text{length of } C).$$

Here we have used that

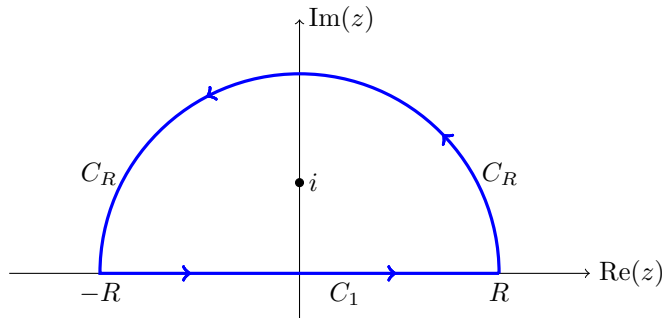
$$|\gamma'(t)| dt = \sqrt{(x')^2 + (y')^2} dt = ds,$$

the arclength element.  $\square$

**Example 4.13.** Compute the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} dx$$

*Solution:* The trick is to integrate  $f(z) = 1/(z^2 + 1)^2$  over the closed contour  $C_1 + C_R$  shown, and then show that the contribution of  $C_R$  to this integral vanishes as  $R$  goes to  $\infty$ .



The only singularity of

$$f(z) = \frac{1}{(z+i)^2(z-i)^2}$$

inside the contour is at  $z = i$ . Let

$$g(z) = \frac{1}{(z+i)^2}.$$

Since  $g$  is analytic on and inside the contour, Cauchy's formula gives

$$\int_{C_1+C_R} f(z) dz = \int_{C_1+C_R} \frac{g(z)}{(z-i)^2} dz = 2\pi i g'(i) = 2\pi i \frac{-2}{(2i)^3} = \frac{\pi}{2}.$$

We parametrize  $C_1$  by

$$\gamma(x) = x, \quad \text{with } -R \leq x \leq R.$$

So,

$$\int_{C_1} f(z) dz = \int_{-R}^R \frac{1}{(x^2 + 1)^2} dx.$$

This goes to  $I$  (the value we want to compute) as  $R \rightarrow \infty$ .

Next, we parametrize  $C_R$  by

$$\gamma(\theta) = Re^{i\theta}, \quad \text{with } 0 \leq \theta \leq \pi.$$



So,

$$\int_{C_R} f(z) dz = \int_0^\pi \frac{1}{(R^2 e^{2i\theta} + 1)^2} iR e^{i\theta} d\theta$$

By the triangle inequality for integrals, if  $R > 1$

$$\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi \left| \frac{1}{(R^2 e^{2i\theta} + 1)^2} iR e^{i\theta} \right| d\theta. \quad (5)$$

From the triangle equality in the form Equation 4b we know that

$$|R^2 e^{2i\theta} + 1| \geq |R^2 e^{2i\theta}| - |1| = R^2 - 1.$$

Thus,

$$\frac{1}{|R^2 e^{2i\theta} + 1|} \leq \frac{1}{R^2 - 1} \quad \Rightarrow \quad \frac{1}{|R^2 e^{2i\theta} + 1|^2} \leq \frac{1}{(R^2 - 1)^2}.$$

Using Equation 5, we then have

$$\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi \left| \frac{1}{(R^2 e^{2i\theta} + 1)^2} iR e^{i\theta} \right| d\theta \leq \int_0^\pi \frac{R}{(R^2 - 1)^2} d\theta = \frac{\pi R}{(R^2 - 1)^2}$$

Clearly this goes to 0 as  $R$  goes to infinity. Thus, the integral over the contour  $C_1 + C_R$  goes to  $I$  as  $R$  gets large. But

$$\int_{C_1 + C_R} f(z) dz = \pi/2$$

for all  $R > 1$ . We can therefore conclude that  $I = \pi/2$ .

As a sanity check, we note that our answer is real and positive as it needs to be.

## 4.4 Proof of Cauchy's integral formula

### 4.4.1 A useful theorem

Before proving the theorem we'll need a theorem that will be useful in its own right.

**Theorem 4.14.** (A second extension of Cauchy's theorem) Suppose that  $A$  is a simply connected region containing the point  $z_0$ . Suppose  $g$  is a function which is

1. Analytic on  $A - \{z_0\}$
2. Continuous on  $A$ . (In particular,  $g$  does not blow up at  $z_0$ .)

Then,

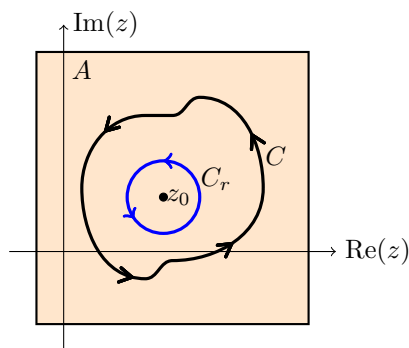
$$\int_C g(z) dz = 0$$

for all closed curves  $C$  in  $A$ .

*Proof.* The extended version of Cauchy's theorem in the Topic 3 notes tells us that

$$\int_C g(z) dz = \int_{C_r} g(z) dz,$$

where  $C_r$  is a circle of radius  $r$  around  $z_0$ .



Since  $g(z)$  is continuous we know that  $|g(z)|$  is bounded inside  $C_r$ . Say,  $|g(z)| < M$ . The corollary to the triangle inequality says that

$$\left| \int_{C_r} g(z) dz \right| \leq M (\text{length of } C_r) = M 2\pi r.$$

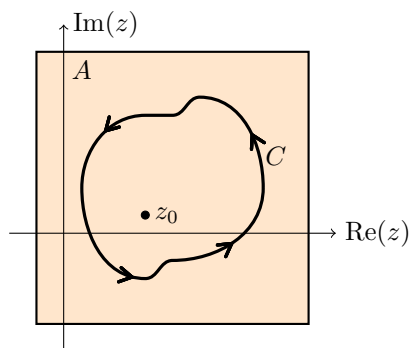
Since  $r$  can be as small as we want, this implies that

$$\int_{C_r} g(z) dz = 0.$$

**Note.** Using this, we can show that  $g(z)$  is, in fact, analytic at  $z_0$ . The proof will be the same as in our proof of Cauchy's theorem that  $g(z)$  has an antiderivative.

#### 4.4.2 Proof of Cauchy's integral formula

We reiterate Cauchy's integral formula from Equation 1:  $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$ .



*Proof.* (of Cauchy's integral formula) We use a trick that is useful enough to be worth remembering. Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

Since  $f(z)$  is analytic on  $A$ , we know that  $g(z)$  is analytic on  $A - \{z_0\}$ . Since the derivative of  $f$  exists at  $z_0$ , we know that

$$\lim_{z \rightarrow z_0} g(z) = f'(z_0).$$

That is, if we define  $g(z_0) = f'(z_0)$  then  $g$  is continuous at  $z_0$ . From the extension of Cauchy's theorem just above, we have

$$\int_C g(z) dz = 0, \quad \text{i.e.} \quad \int_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

Thus

$$\int_C \frac{f(z)}{z - z_0} dz = \int_C \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_C \frac{1}{z - z_0} dz = 2\pi i f(z_0).$$

The last equality follows from our, by now, well known integral of  $1/(z - z_0)$  on a loop around  $z_0$ .

#### 4.5 Proof of Cauchy's integral formula for derivatives

Recall that Cauchy's integral formula in Equation 3 says

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z)^{n+1}} dw, \quad n = 0, 1, 2, \dots$$

First we'll offer a quick proof which captures the reason behind the formula, and then a formal proof.

*Quick proof:* We have an integral representation for  $f(z)$ ,  $z \in A$ , we use that to find an integral representation for  $f'(z)$ ,  $z \in A$ .

$$f'(z) = \frac{d}{dz} \left[ \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw \right] = \frac{1}{2\pi i} \int_C \frac{d}{dz} \left( \frac{f(w)}{w - z} \right) dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z)^2} dw$$

(Note, since  $z \in A$  and  $w \in C$ , we know that  $w - z \neq 0$ ) Thus,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z)^2} dw$$

Now, by iterating this process, i.e. by mathematical induction, we can show the formula for higher order derivatives.

*Formal proof:* We do this by taking the limit of

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

using the integral representation of both terms:

$$f(z + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z - \Delta z} dw, \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

Now, using a little algebraic manipulation we get

$$\begin{aligned} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i \Delta z} \int_C \frac{f(w)}{w - z - \Delta z} - \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i \Delta z} \int_C \frac{f(w) \Delta z}{(w - z - \Delta z)(w - z)} dw \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z)^2 - \Delta z(w - z)} dw \end{aligned}$$

Letting  $\Delta z$  go to 0, we get Cauchy's formula for  $f'(z)$ :

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw$$

There is no problem taking the limit under the integral sign because everything is continuous and the denominator is never 0.  $\square$

## 4.6 Amazing consequence of Cauchy's integral formula

### 4.6.1 Existence of derivatives

**Theorem.** Suppose  $f(z)$  is analytic on a region  $A$ . Then,  $f$  has derivatives of all order.

*Proof.* This follows from Cauchy's integral formula for derivatives. That is, we have a formula for all the derivatives, so in particular the derivatives all exist.

A little more precisely: for any point  $z$  in  $A$  we can put a small disk around  $z_0$  that is entirely contained in  $A$ . Let  $C$  be the boundary of the disk, then Cauchy's formula gives a formula for all the derivatives  $f^{(n)}(z_0)$  in terms of integrals over  $C$ . In particular, those derivatives exist.  $\square$

**Remark.** If you look at the proof of Cauchy's formula for derivatives you'll see that  $f$  having derivatives of all orders boils down to  $1/(w-z)$  having derivatives of all orders for  $w$  on a curve not containing  $z$ .

**Important remark.** We have at times assumed that for  $f = u + iv$  analytic,  $u$  and  $v$  have continuous higher order partial derivatives. This theorem confirms that fact. In particular,  $u_{xy} = u_{yx}$ , etc.

### 4.6.2 Cauchy's inequality

**Theorem 4.15. (Cauchy's inequality)** Let  $C_R$  be the circle  $|z - z_0| = R$ . Assume that  $f(z)$  is analytic on  $C_R$  and its interior, i.e. on the disk  $|z - z_0| \leq R$ . Finally let  $M_R = \max |f(z)|$  over  $z$  on  $C_R$ . Then

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}, \quad n = 1, 2, 3, \dots \quad (6)$$

*Proof.* Using Cauchy's integral formula for derivatives (Equation 3) we have

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_{C_R} \frac{|f(w)|}{|w - z_0|^{n+1}} |dw| \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \int_{C_R} |dw| = \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \cdot 2\pi R$$

### 4.6.3 Liouville's theorem

**Theorem 4.16. (Liouville's theorem)** Assume  $f(z)$  is entire and suppose it is bounded in the complex plane, namely  $|f(z)| < M$  for all  $z \in \mathbf{C}$  then  $f(z)$  is constant.

*Proof.* For any circle of radius  $R$  around  $z_0$  the Cauchy inequality says  $|f'(z_0)| \leq \frac{M}{R}$ . But,  $R$  can be as large as we like so we conclude that  $|f'(z_0)| = 0$  for every  $z_0 \in \mathbf{C}$ . Since the derivative is 0, the function itself is constant.

In short:

If  $f$  is entire and bounded then  $f$  is constant.

**Note.**  $P(z) = a_n z^n + \dots + a_0$ ,  $\sin(z)$ ,  $e^z$  are all entire but **not bounded**.

Now, practically for free, we get the fundamental theorem of algebra.

**Corollary.** (**Fundamental theorem of algebra**) Any polynomial  $P$  of degree  $n \geq 1$ , i.e.

$$P(z) = a_0 + a_1 z + \dots + a_n z^n, \quad a_n \neq 0,$$

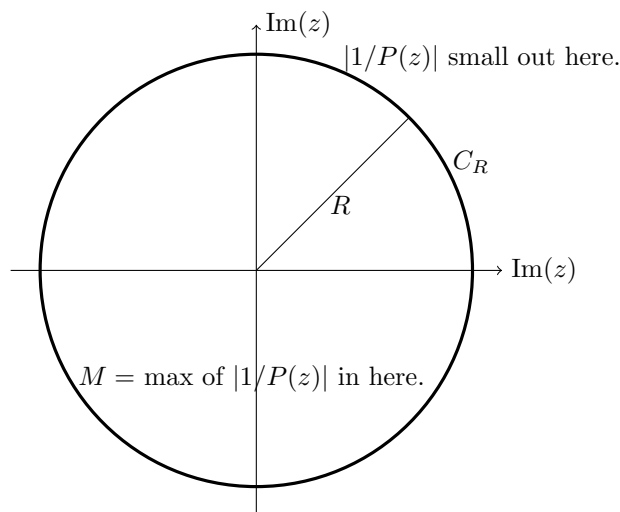
has exactly  $n$  roots.

*Proof.* There are two parts to the proof.

**Hard part:** Show that  $P$  has at least one root.

This is done by contradiction, together with Liouville's theorem. Suppose  $P(z)$  **does not** have a zero. Then

1.  $f(z) = 1/P(z)$  is entire. This is obvious because (by assumption)  $P(z)$  has no zeros.
2.  $f(z)$  is bounded. This follows because  $1/P(z)$  goes to 0 as  $|z|$  goes to  $\infty$ .



(It is clear that  $|1/P(z)|$  goes to 0 as  $z$  goes to infinity, i.e.  $|1/P(z)|$  is small outside a large circle. So  $|1/P(z)|$  is bounded by  $M$ .)

So, by Liouville's theorem  $f(z)$  is constant, and therefore  $P(z)$  must be constant as well. But this is a contradiction, so the hypothesis of "No zeros" must be wrong, i.e.  $P$  must have a zero.

**Easy part:**  $P$  has exactly  $n$  zeros. Let  $z_0$  be one zero. We can factor  $P(z) = (z - z_0)Q(z)$ .  $Q(z)$  has degree  $n - 1$ . If  $n - 1 > 0$ , then we can apply the result to  $Q(z)$ . We can continue this process until the degree of  $Q$  is 0.

#### 4.6.4 Maximum modulus principle

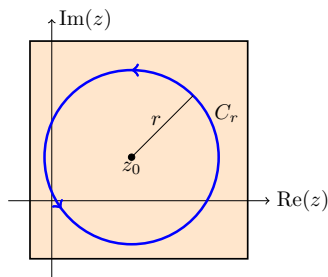
Briefly, the maximum modulus principle states that if  $f$  is analytic and not constant in a domain  $A$  then  $|f(z)|$  has no relative maximum in  $A$  and the absolute maximum of  $|f|$  occurs on the boundary of  $A$ .

In order to prove the maximum modulus principle we will first prove the mean value property. This will give you a good feel for the maximum modulus principle. It is also important and interesting in its own right.

**Theorem 4.17. (Mean value property)** Suppose  $f(z)$  is analytic on the closed disk of radius  $r$  centered at  $z_0$ , i.e. the set  $|z - z_0| \leq r$ . Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \quad (7)$$

*Proof.* This is an application of Cauchy's integral formula on the disk  $D_r = |z - z_0| \leq r$ .



We can parametrize  $C_r$ , the boundary of  $D_r$ , as

$$\gamma(t) = z_0 + re^{i\theta}, \text{ with } 0 \leq \theta \leq 2\pi, \text{ so } \gamma'(\theta) = ire^{i\theta}.$$

By Cauchy's formula we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

This proves the property.  $\square$

In words, the mean value property says  $f(z_0)$  is the arithmetic mean of the values on the circle.

Now we can state and prove the maximum modulus principle. We state the assumptions carefully. When applying this theorem, it is important to verify that the assumptions are satisfied.

**Theorem 4.18. (Maximum modulus principle)** Suppose  $f(z)$  is analytic in a connected region  $A$  and  $z_0$  is a point in  $A$ .

1. If  $|f|$  has a relative maximum at  $z_0$  then  $f(z)$  is constant in a neighborhood of  $z_0$ .
2. If  $A$  is bounded and connected, and  $f$  is continuous on  $A$  and its boundary, then either  $f$  is constant or the absolute maximum of  $|f|$  occurs only on the boundary of  $A$ .

*Proof. Part (1):* The argument for part (1) is a little fussy. We will use the mean value property and the triangle inequality from Theorem 4.11.

Since  $z_0$  is a relative maximum of  $|f|$ , for every small enough circle  $C : |z - z_0| = r$  around  $z_0$  we have  $|f(z)| \leq |f(z_0)|$  for  $z$  on  $C$ . Therefore, by the mean value property and the

triangle inequality

$$\begin{aligned}
 |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \right| && \text{(mean value property)} \\
 &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta && \text{(triangle inequality)} \\
 &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| d\theta && (|f(z_0 + re^{i\theta})| \leq |f(z_0)|) \\
 &= |f(z_0)|
 \end{aligned}$$

Since the beginning and end of the above are both  $|f(z_0)|$  all the inequalities in the chain must be equalities.

The first inequality can only be an equality if for all  $\theta$ ,  $f(z_0 + re^{i\theta})$  lie on the same ray from the origin, i.e. have the same argument or are 0.

The second inequality can only be an equality if all  $|f(z_0 + re^{i\theta})| = |f(z_0)|$ . So we have all  $f(z_0 + re^{i\theta})$  have the same magnitude and the same argument. This implies they are all the same.

Finally, if  $f(z)$  is constant along the circle and  $f(z_0)$  is the average of  $f(z)$  over the circle then  $f(z) = f(z_0)$ , i.e.  $f$  is constant on a small disk around  $z_0$ .

**Part (2):** The assumptions that  $A$  is bounded and  $f$  is continuous on  $A$  and its boundary serve to guarantee that  $|f|$  has an absolute maximum (on  $A$  combined with its boundary). Part (1) guarantees that the absolute maximum can not lie in the interior of the region  $A$  unless  $f$  is constant. (This requires a bit more argument. Do you see why?) If the absolute maximum is not in the interior it must be on the boundary.  $\square$

**Example 4.19.** Find the maximum modulus of  $e^z$  on the unit square with  $0 \leq x, y \leq 1$ .

*Solution:*

$$|e^{x+iy}| = e^x,$$

so the maximum is when  $x = 1$ ,  $0 \leq y \leq 1$  is arbitrary. This is indeed on the boundary of the unit square

**Example 4.20.** Find the maximum modulus for  $\sin(z)$  on the square  $[0, 2\pi] \times [0, 2\pi]$ .

*Solution:* We use the formula

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y.$$

So,

$$\begin{aligned}
 |\sin(z)|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
 &= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
 &= \sin^2 x + \sinh^2 y.
 \end{aligned}$$

We know the maximum over  $x$  of  $\sin^2(x)$  is at  $x = \pi/2$  and  $x = 3\pi/2$ . The maximum of  $\sinh^2 y$  is at  $y = 2\pi$ . So maximum modulus is

$$\sqrt{1 + \sinh^2(2\pi)} = \sqrt{\cosh^2(2\pi)} = \cosh(2\pi).$$

This occurs at the points

$$z = x + iy = \frac{\pi}{2} + 2\pi i, \quad \text{and} \quad z = \frac{3\pi}{2} + 2\pi i.$$

Both these points are on the boundary of the region.

**Example 4.21.** Suppose  $f(z)$  is entire. Show that if  $\lim_{z \rightarrow \infty} f(z) = 0$  then  $f(z) \equiv 0$ .

*Solution:* This is a standard use of the maximum modulus principle. The strategy is to show that the maximum of  $|f(z)|$  is not on the boundary (of the appropriately chosen region), so  $f(z)$  must be constant.

Fix  $z_0$ . For  $R > |z_0|$  let  $M_R$  be the maximum of  $|f(z)|$  on the circle  $|z| = R$ . The maximum modulus theorem says that  $|f(z_0)| < M_R$ . Since  $f(z)$  goes to 0, as  $R$  goes to infinity, we must have  $M_R$  also goes to 0. This means  $|f(z_0)| = 0$ . Since this is true for any  $z_0$ , we have  $f(z) \equiv 0$ .

**Example 4.22.** Here is an example of why you need  $A$  to be bounded in the maximum modulus theorem. Let  $A$  be the upper half-plane

$$\text{Im}(z) > 0.$$

So the boundary of  $A$  is the real axis.

Let  $f(z) = e^{-iz}$ . We have

$$|f(x)| = |e^{-ix}| = 1$$

for  $x$  along the real axis. Since  $|f(2i)| = |e^2| > 1$ , we see  $|f|$  cannot take its maximum along the boundary of  $A$ .

Of course, it can't take its maximum in the interior of  $A$  either. What happens here is that  $f(z)$  doesn't have a maximum modulus. Indeed  $|f(z)|$  goes to infinity along the positive imaginary axis.

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