6.3 CAUCHY'S INTEGRAL THEOREM

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a contour in the complex plane may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour from z_0 to z_0' into n intervals by picking $n-1$ intermediate points $z_1, z_2, \ldots$ on the contour (Fig. 6.5). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}), \tag{6.26}$$

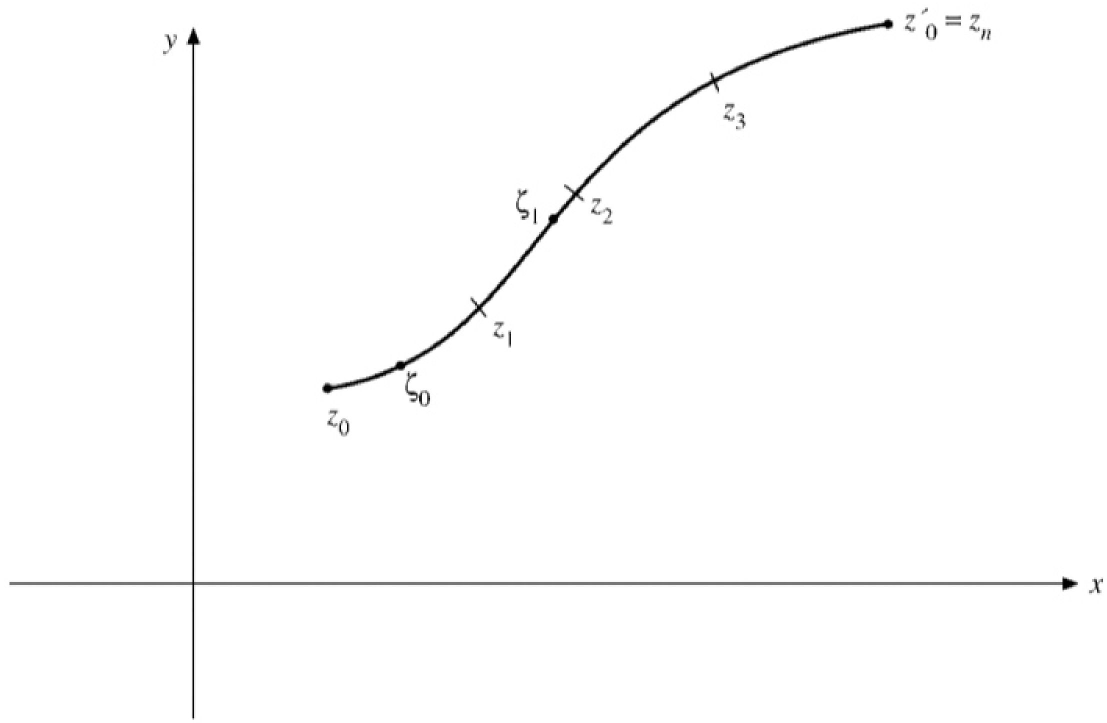

FIGURE 6.5 Integration path.

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If the $\lim_{n\to\infty} S_n$ exists and is independent of the details of choosing the points z_j and ζ_j, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\,dz. \tag{6.27}$$

The right-hand side of Eq. (6.27) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$).

The preceding development of the contour integral is closely analogous to the Riemann integral of a real function of a real variable. As an alternative, the contour integral may be defined by

$$\begin{aligned}\int_{z_1}^{z_2} f(z)dz &= \int_{x_1,y_1}^{x_2,y_2} \big[u(x,y) + iv(x,y)\big][dx + idy] \\ &= \int_{x_1,y_1}^{x_2,y_2} \big[u(x,y)\,dx - v(x,y)\,dy\big] + i\int_{x_1,y_1}^{x_2,y_2} \big[v(x,y)\,dx + u(x,y)\,dy\big]\end{aligned}$$

with the path joining (x_1, y_1) and (x_2, y_2) specified. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals, Section 1.10.

An important example is the contour integral $\int_C z^n\,dz$, where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates of Eq. (6.4c) we parameterize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For $n \neq -1$, n an integer, we then obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_C z^n\,dz &= \frac{r^{n+1}}{2\pi}\int_0^{2\pi} \exp\big[i(n+1)\theta\big]\,d\theta \\ &= \big[2\pi i(n+1)\big]^{-1} r^{n+1}\big[e^{i(n+1)\theta}\big]\big|_0^{2\pi} = 0\end{aligned} \tag{6.27a}$$

because 2π is a period of $e^{i(n+1)\theta}$, while for $n = -1$

$$\frac{1}{2\pi i}\int_C \frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1, \tag{6.27b}$$

again independent of r.

Alternatively, we can **integrate around a rectangle** with the corners z_1, z_2, z_3, z_4 to obtain for $n \neq -1$

$$\int z^n\,dz = \frac{z^{n+1}}{n+1}\bigg|_{z_1}^{z_2} + \frac{z^{n+1}}{n+1}\bigg|_{z_2}^{z_3} + \frac{z^{n+1}}{n+1}\bigg|_{z_3}^{z_4} + \frac{z^{n+1}}{n+1}\bigg|_{z_4}^{z_1} = 0,$$

because each corner point appears once as an upper and a lower limit that cancel. For $n = -1$ the corresponding real parts of the logarithms cancel similarly, but their imaginary parts involve the increasing arguments of the points from z_1 to z_4 and, when we come back to the first corner z_1, its argument has increased by 2π due to the multivaluedness of the logarithm, so $2\pi i$ is left over as the value of the integral. Thus, **the value of the integral involving a multivalued function must be that which is reached in a continuous fashion on the path being taken**. These integrals are examples of Cauchy's integral theorem, which we consider in the next section.

Stokes' Theorem Proof

Cauchy's integral theorem is the first of two basic theorems in the theory of the behavior of functions of a complex variable. First, we offer a proof under relatively restrictive conditions — conditions that are intolerable to the mathematician developing a beautiful abstract theory but that are usually satisfied in physical problems.

If a function $f(z)$ is analytic, that is, if its partial derivatives are continuous throughout some **simply connected region** R,[7] for every closed path C (Fig. 6.6) in R, and if it is single-valued (assumed for simplicity here), the line integral of $f(z)$ around C is zero, or

$$\int_C f(z)\,dz = \oint_C f(z)\,dz = 0. \tag{6.27c}$$

Recall that in Section 1.13 such a function $f(z)$, identified as a force, was labeled conservative. The symbol $\oint$ is used to emphasize that the path is closed. Note that the interior of the simply connected region bounded by a contour is that region lying to the left when moving in the direction implied by the contour; as a rule, a simply connected region is bounded by a single closed curve.

In this form the Cauchy integral theorem may be proved by direct application of Stokes' theorem (Section 1.12). With $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + i dy$,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u + iv)(dx + i dy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy).\end{aligned} \tag{6.28}$$

These two line integrals may be converted to surface integrals by Stokes' theorem, a procedure that is justified if the partial derivatives are continuous within C. In applying Stokes' theorem, note that the final two integrals of Eq. (6.28) are real. Using

$$\mathbf{V} = \hat{\mathbf{x}}V_x + \hat{\mathbf{y}}V_y,$$

Stokes' theorem says that

$$\oint_C (V_x\,dx + V_y\,dy) = \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy. \tag{6.29}$$

For the first integral in the last part of Eq. (6.28) let $u = V_x$ and $v = -V_y$.[8] Then

[7] Any closed simple curve (one that does not intersect itself) inside a simply connected region or domain may be contracted to a single point that still belongs to the region. If a region is not simply connected, it is called multiply connected. As an example of a multiply connected region, consider the z-plane with the interior of the unit circle **excluded**.

[8] In the proof of Stokes' theorem, Section 1.12, V_x and V_y are any two functions (with continuous partial derivatives).

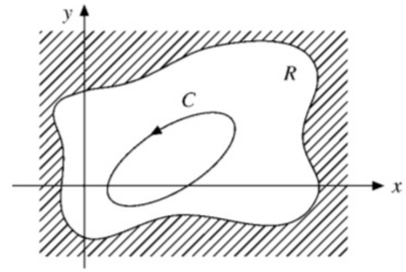

FIGURE 6.6 A closed contour C within a simply connected region R.

$$\oint_C (u\,dx - v\,dy) = \oint_C (V_x\,dx + V_y\,dy)$$
$$= \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dx\,dy = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\,dy. \quad (6.30)$$

For the second integral on the right side of Eq. (6.28) we let $u = V_y$ and $v = V_x$. Using Stokes' theorem again, we obtain

$$\oint (v\,dx + u\,dy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy. \quad (6.31)$$

On application of the Cauchy–Riemann conditions, which must hold, since $f(z)$ is assumed analytic, each integrand vanishes and

$$\oint f(z)\,dz = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\,dy + i \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy = 0. \quad (6.32)$$

Cauchy–Goursat Proof

This completes the proof of Cauchy's integral theorem. However, the proof is marred from a theoretical point of view by the need for continuity of the first partial derivatives. Actually, as shown by Goursat, this condition is not necessary. An outline of the Goursat proof is as follows. We subdivide the region inside the contour C into a network of small squares, as indicated in Fig. 6.7. Then

$$\oint_C f(z)\,dz = \sum_j \oint_{C_j} f(z)\,dz, \quad (6.33)$$

all integrals along interior lines canceling out. To estimate the $\oint_{C_j} f(z)\,dz$, we construct the function

$$\delta_j(z, z_j) = \frac{f(z) - f(z_j)}{z - z_j} - \left. \frac{df(z)}{dz} \right|_{z=z_j}, \quad (6.34)$$

with z_j an interior point of the jth subregion. Note that $[f(z) - f(z_j)]/(z - z_j)$ is an approximation to the derivative at $z = z_j$. Equivalently, we may note that if $f(z)$ had

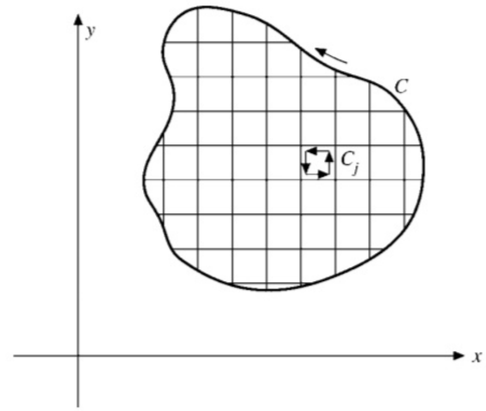

FIGURE 6.7 Cauchy–Goursat contours.

a Taylor expansion (which we have not yet proved), then $\delta_j(z, z_j)$ would be of order $z - z_j$, approaching zero as the network was made finer. But since $f'(z_j)$ exists, that is, is finite, we may make

$$\left|\delta_j(z, z_j)\right| < \varepsilon, \tag{6.35}$$

where ε is an arbitrarily chosen small positive quantity. Solving Eq. (6.34) for $f(z)$ and integrating around C_j, we obtain

$$\oint_{C_j} f(z)\,dz = \oint_{C_j} (z - z_j)\delta_j(z, z_j)\,dz, \tag{6.36}$$

the integrals of the other terms vanishing.[9] When Eqs. (6.35) and (6.36) are combined, one shows that

$$\left|\sum_j \oint_{C_j} f(z)\,dz\right| < A\varepsilon, \tag{6.37}$$

where A is a term of the order of the area of the enclosed region. Since ε is arbitrary, we let $\varepsilon \to 0$ and conclude that if a function $f(z)$ is analytic on and within a closed path C,

$$\oint_C f(z)\,dz = 0. \tag{6.38}$$

Details of the proof of this significantly more general and more powerful form can be found in Churchill in the Additional Readings. Actually we can still prove the theorem for $f(z)$ analytic within the interior of C and only continuous on C.

The consequence of the Cauchy integral theorem is that for analytic functions the line integral is a function only of its endpoints, independent of the path of integration,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)\,dz, \tag{6.39}$$

again exactly like the case of a conservative force, Section 1.13.

[9] $\oint dz$ and $\oint z\,dz = 0$ by Eq. (6.27a).

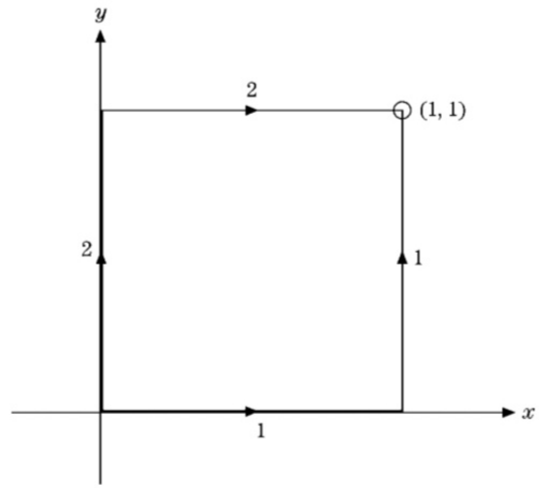

FIGURE 6.10 Contour.

6.4 CAUCHY'S INTEGRAL FORMULA

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. We seek to prove that

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0),} \tag{6.43}$$

in which z_0 is any point in the interior region bounded by C. This is the second of the two basic theorems mentioned in Section 6.3. Note that since z is on the contour C while z_0 is in the interior, $z - z_0 \neq 0$ and the integral Eq. (6.43) is well defined. Although $f(z)$ is assumed analytic, the integrand is $f(z)/(z-z_0)$ and is not analytic at $z = z_0$ unless $f(z_0) = 0$. If the contour is deformed as shown in Fig. 6.11 (or Fig. 6.9, Section 6.3), Cauchy's integral theorem applies. By Eq. (6.42),

$$\oint_C \frac{f(z)}{z-z_0}dz - \oint_{C_2} \frac{f(z)}{z-z_0}dz = 0, \tag{6.44}$$

where C is the original outer contour and C_2 is the circle surrounding the point z_0 traversed in a **counterclockwise** direction. Let $z = z_0 + re^{i\theta}$, using the polar representation because of the circular shape of the path around z_0. Here r is small and will eventually be made to approach zero. We have (with $dz = ire^{ir\theta}d\theta$ from Eq. (6.27a))

$$\oint_{C_2} \frac{f(z)}{z-z_0}dz = \oint_{C_2} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} rie^{i\theta}\,d\theta.$$

Taking the limit as $r \to 0$, we obtain

$$\oint_{C_2} \frac{f(z)}{z-z_0}dz = if(z_0)\int_{C_2} d\theta = 2\pi i f(z_0), \tag{6.45}$$

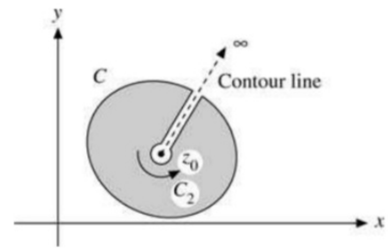

FIGURE 6.11 Exclusion of a singular point.

since $f(z)$ is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an interior point $z = z_0$ once the values on the boundary C are specified. This is closely analogous to a two-dimensional form of Gauss' law (Section 1.14) in which the magnitude of an interior line charge would be given in terms of the cylindrical surface integral of the electric field **E**.

A further analogy is the determination of a function in real space by an integral of the function and the corresponding Green's function (and their derivatives) over the bounding surface. Kirchhoff diffraction theory is an example of this.

It has been emphasized that z_0 is an interior point. What happens if z_0 is exterior to C? In this case the entire integrand is analytic on and within C. Cauchy's integral theorem, Section 6.3, applies and the integral vanishes. We have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \quad \text{interior} \\ 0, & z_0 \quad \text{exterior.} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. From Eq. (6.43), with $f(z)$ analytic,

$$\frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} = \frac{1}{2\pi i\,\delta z_0}\left(\oint \frac{f(z)}{z - z_0 - \delta z_0}dz - \oint \frac{f(z)}{z - z_0}dz\right).$$

Then, by definition of derivative (Eq. (6.14)),

$$\begin{aligned} f'(z_0) &= \lim_{\delta z_0 \to 0} \frac{1}{2\pi i\,\delta z_0}\oint \frac{\delta z_0\, f(z)}{(z - z_0 - \delta z_0)(z - z_0)}dz \\ &= \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^2}dz. \end{aligned} \tag{6.46}$$

This result could have been obtained by differentiating Eq. (6.43) under the integral sign with respect to z_0. This formal, or turning-the-crank, approach is valid, but the justification for it is contained in the preceding analysis.

This technique for constructing derivatives may be repeated. We write $f'(z_0 + \delta z_0)$ and $f'(z_0)$, using Eq. (6.46). Subtracting, dividing by δz_0, and finally taking the limit as $\delta z_0 \to 0$, we have

$$f^{(2)}(z_0) = \frac{2}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^3}.$$

Note that $f^{(2)}(z_0)$ is independent of the direction of δz_0, as it must be. Continuing, we get[10]

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^{n+1}};} \tag{6.47}$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative but derivatives of **all** orders as well! The derivatives of $f(z)$ are automatically analytic. Notice that this statement assumes the Goursat version of the Cauchy integral theorem. This is also why Goursat's contribution is so significant in the development of the theory of complex variables.

Morera's Theorem

A further application of Cauchy's integral formula is in the proof of Morera's **theorem**, which is the converse of Cauchy's integral theorem. The theorem states the following:

> *If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)\,dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.*

Let us integrate $f(z)$ from z_1 to z_2. Since every closed-path integral of $f(z)$ vanishes, the integral is independent of path and depends only on its endpoints. We label the result of the integration $F(z)$, with

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)\,dz. \tag{6.48}$$

As an identity,

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]\,dt}{z_2 - z_1}, \tag{6.49}$$

using t as another complex variable. Now we take the limit as $z_2 \to z_1$:

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]\,dt}{z_2 - z_1} = 0, \tag{6.50}$$

[10]This expression is the starting point for defining derivatives of **fractional order**. See A. Erdelyi (ed.), *Tables of Integral Transforms*, Vol. 2. New York: McGraw-Hill (1954). For recent applications to mathematical analysis, see T. J. Osler, An integral analogue of Taylor's series and its use in computing Fourier transforms. *Math. Comput.* **26**: 449 (1972), and references therein.

Exercises

6.4.1 Show that

$$\oint_C (z - z_0)^n \, dz = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1, \end{cases}$$

where the contour C encircles the point $z = z_0$ in a positive (counterclockwise) sense. The exponent n is an integer. See also Eq. (6.27a). The calculus of residues, Chapter 7, is based on this result.

6.4.2 Show that

$$\frac{1}{2\pi i} \oint z^{m-n-1} \, dz, \qquad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once counterclockwise) is a representation of the Kronecker δ_{mn}.

6.4.3 Solve Exercise 6.3.4 by separating the integrand into partial fractions and then applying Cauchy's integral theorem for multiply connected regions.
Note. Partial fractions are explained in Section 15.8 in connection with Laplace transforms.

6.4.4 Evaluate

$$\oint_C \frac{dz}{z^2 - 1},$$

where C is the circle $|z| = 2$.

6.4.5 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z - z_0} \, dz = \oint_C \frac{f(z)}{(z - z_0)^2} \, dz.$$

6.4.6 You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

Using mathematical induction, prove that this expression is correct.

6.4.7 (a) A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z) \neq 0$ within C and $|f(z)| \leq M$ on C, show that

$$|f(z)| \leq M$$

for all points within C.
Hint. Consider $w(z) = 1/f(z)$.

(b) If $f(z) = 0$ within the contour C, show that the foregoing result does not hold and that it is possible to have $|f(z)| = 0$ at one or more points in the interior with $|f(z)| > 0$ over the entire bounding contour. Cite a specific example of an analytic function that behaves this way.

6.4.8 Using the Cauchy integral formula for the nth derivative, convert the following Rodrigues formulas into the corresponding so-called Schlaefli integrals.

(a) Legendre:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$$\text{ANS. } \frac{(-1)^n}{2^n}\cdot\frac{1}{2\pi i}\oint\frac{(1-z^2)^n}{(z-x)^{n+1}}dz.$$

(b) Hermite:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

(c) Laguerre:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}).$$

Note. From the Schlaefli integral representations one can develop generating functions for these special functions. Compare Sections 12.4, 13.1, and 13.2.

6.5 LAURENT EXPANSION

Taylor Expansion

The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series (Section 5.6), but this time for functions of a complex variable. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius less than $|z_1 - z_0|$ (Fig. 6.12). Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic, $f(z)$ is necessarily analytic on and within C.

From Eq. (6.43), the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}. \end{aligned} \tag{6.53}$$

Here z' is a point on the contour C and z is any point interior to C. It is not legal yet to expand the denominator of the integrand in Eq. (6.53) by the binomial theorem, for we have not yet proved the binomial theorem for complex variables. Instead, we note the identity

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n, \tag{6.54}$$

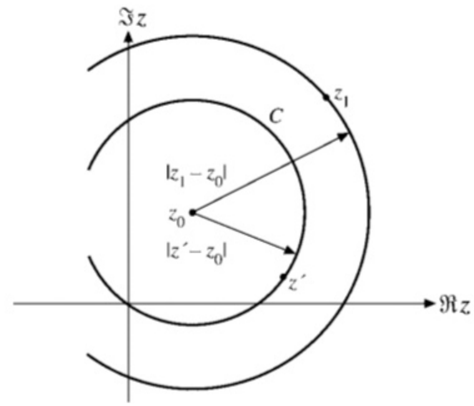

FIGURE 6.12 Circular domain for Taylor expansion.

which may easily be verified by multiplying both sides by $1 - t$. The infinite series, following the methods of Section 5.2, is convergent for $|t| < 1$.

Now, for a point z interior to C, $|z - z_0| < |z' - z_0|$, and, using Eq. (6.54), Eq. (6.53) becomes

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z')\,dz'}{(z' - z_0)^{n+1}}. \tag{6.55}$$

Interchanging the order of integration and summation (valid because Eq. (6.54) is uniformly convergent for $|t| < 1$), we obtain

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_C \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}. \tag{6.56}$$

Referring to Eq. (6.47), we get

$$f(z) = \sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!}, \tag{6.57}$$

which is our desired Taylor expansion. Note that it is based only on the assumption that $f(z)$ is analytic for $|z - z_0| < |z_1 - z_0|$. Just as for real variable power series (Section 5.7), this expansion is unique for a given z_0.

From the Taylor expansion for $f(z)$ a binomial theorem may be derived (Exercise 6.5.2).

Schwarz Reflection Principle

From the binomial expansion of $g(z) = (z - x_0)^n$ for integral n it is easy to see that the complex conjugate of the function g is the function of the complex conjugate for real x_0:

$$g^*(z) = \left[(z - x_0)^n\right]^* = (z^* - x_0)^n = g(z^*). \tag{6.58}$$

Laurent Series

We frequently encounter functions that are analytic and single-valued in an annular region, say, of inner radius r and outer radius R, as shown in Fig. 6.15. Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula, and for two circles C_2 and C_1 centered at $z = z_0$ and with radii r_2 and r_1, respectively, where $r < r_2 < r_1 < R$, we have[13]

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\,dz'}{z' - z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')\,dz'}{z' - z}. \tag{6.64}$$

Note that in Eq. (6.64) an explicit minus sign has been introduced so that the contour C_2 (like C_1) is to be traversed in the positive (counterclockwise) sense. The treatment of Eq. (6.64) now proceeds exactly like that of Eq. (6.53) in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$ and expanded by the binomial theorem, which now follows from the Taylor series (Eq. (6.57)).

Noting that for C_1, $|z' - z_0| > |z - z_0|$ while for C_2, $|z' - z_0| < |z - z_0|$, we find

$$\begin{aligned} f(z) = {} & \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} \\ & + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz'. \end{aligned} \tag{6.65}$$

The minus sign of Eq. (6.64) has been absorbed by the binomial expansion. Labeling the first series S_1 and the second S_2 we have

$$S_1 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}, \tag{6.66}$$

which is the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$, that is, for all z **interior** to the larger circle, C_1. For the second series in Eq. (6.65) we have

$$S_2 = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz', \tag{6.67}$$

convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z **exterior** to the smaller circle, C_2. Remember, C_2 now goes counterclockwise.

These two series are combined into one series[14] (a Laurent series) by

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,} \tag{6.68}$$

[13] We may take r_2 arbitrarily close to r and r_1 arbitrarily close to R, maximizing the area enclosed between C_1 and C_2.
[14] Replace n by $-n$ in S_2 and add.

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}. \tag{6.69}$$

Since, in Eq. (6.69), convergence of a binomial expansion is no longer a problem, C may be any contour within the annular region $r < |z - z_0| < R$ encircling z_0 once in a counterclockwise sense. If we assume that such an annular region of convergence does exist, then Eq. (6.68) is the Laurent series, or Laurent expansion, of $f(z)$.

The use of the contour line (Fig. 6.15) is convenient in converting the annular region into a simply connected region. Since our function is analytic in this annular region (and single-valued), the contour line is not essential and, indeed, does not appear in the final result, Eq. (6.69).

Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques, such as ordinary series expansions, may provide the coefficients.

Numerous examples of Laurent series appear in Chapter 7. We limit ourselves here to one simple example to illustrate the application of Eq. (6.68).

Example 6.5.1 LAURENT EXPANSION

Let $f(z) = [z(z-1)]^{-1}$. If we choose $z_0 = 0$, then $r = 0$ and $R = 1$, $f(z)$ diverging at $z = 1$. A partial fraction expansion yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots = -\sum_{n=-1}^{\infty} z^n. \tag{6.70}$$

From Eqs. (6.70), (6.68), and (6.69) we then have

$$a_n = \frac{1}{2\pi i} \oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \ge -1, \\ 0 & \text{for } n < -1. \end{cases} \tag{6.71}$$

The integrals in Eq. (6.71) can also be directly evaluated by substituting the geometric-series expansion of $(1-z')^{-1}$ used already in Eq. (6.70) for $(1-z)^{-1}$:

$$a_n = \frac{-1}{2\pi i} \oint \sum_{m=0}^{\infty} (z')^m \frac{dz'}{(z')^{n+2}}. \tag{6.72}$$

Upon interchanging the order of summation and integration (uniformly convergent series), we have

$$a_n = -\frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint \frac{dz'}{(z')^{n+2-m}}. \tag{6.73}$$

6.6 SINGULARITIES

The Laurent expansion represents a generalization of the Taylor series in the presence of singularities. We define the point z_0 as an **isolated singular point** of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighboring points.

Poles

In the Laurent expansion of $f(z)$ about z_0,

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m, \tag{6.75}$$

if $a_m = 0$ for $m < -n < 0$ and $a_{-n} \neq 0$, we say that z_0 is a pole of order n. For instance, if $n = 1$, that is, if $a_{-1}/(z - z_0)$ is the first nonvanishing term in the Laurent series, we have a pole of order 1, often called a **simple** pole.

If, on the other hand, the summation continues to $m = -\infty$, then z_0 is a pole of infinite order and is called an **essential singularity**. These essential singularities have many pathological features. For instance, we can show that in any small neighborhood of an essential singularity of $f(z)$ the function $f(z)$ comes arbitrarily close to any (and therefore every) preselected complex quantity w_0.[15] Here, the entire w-plane is mapped by f into the neighborhood of the point z_0. One point of fundamental difference between a pole of finite order n and an essential singularity is that by multiplying $f(z)$ by $(z - z_0)^n$, $f(z)(z - z_0)^n$ is no longer singular at z_0. This obviously cannot be done for an essential singularity.

The behavior of $f(z)$ as $z \to \infty$ is defined in terms of the behavior of $f(1/t)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}. \tag{6.76}$$

As $z \to \infty$, we replace the z by $1/t$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,t^{2n+1}}. \tag{6.77}$$

From the definition, $\sin z$ has an essential singularity at infinity. This result could be anticipated from Exercise 6.1.9 since

$$\sin z = \sin iy = i \sinh y, \qquad \text{when } x = 0,$$

which approaches infinity exponentially as $y \to \infty$. Thus, although the absolute value of $\sin x$ for real x is equal to or less than unity, the absolute value of $\sin z$ is not bounded.

A function that is analytic throughout the finite complex plane **except** for isolated poles is called **meromorphic**, such as ratios of two polynomials or $\tan z$, $\cot z$. Examples are also **entire** functions that have no singularities in the finite complex plane, such as $\exp(z)$, $\sin z$, $\cos z$ (see Sections 5.9, 5.11).

[15] This theorem is due to Picard. A proof is given by E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. New York: Oxford University Press (1939).