Hint. Beware of dividing by zero when calculating an angle as an arc tangent.

Check value. $z = 0.2 + 0.1i$, $\Re(\sinh z) = 0.20033$,
$\Im(\sinh z) = 0.10184$, $|\sinh z| = 0.22473$,
$\text{phase}(\sinh z) = 0.47030$.

6.1.26 Repeat Exercise 6.1.25 for $\cosh z$.

6.2 CAUCHY–RIEMANN CONDITIONS

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{z+\delta z - z} = \lim_{\delta z\to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z), \tag{6.14}$$

provided that the limit is **independent** of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive.

Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y. \tag{6.15}$$

Also,

$$\delta f = \delta u + i\delta v, \tag{6.16}$$

so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}. \tag{6.17}$$

Let us take the limit indicated by Eq. (6.14) by two different approaches, as shown in Fig. 6.4. First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (6.14) yields

$$\lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{6.18}$$

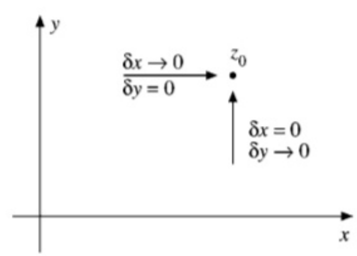

FIGURE 6.4 Alternate approaches to z_0.

assuming the partial derivatives exist. For a second approach, we set $\delta x = 0$ and then let $\delta y \to 0$. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{6.19}$$

If we are to have a derivative df/dz, Eqs. (6.18) and (6.19) must be identical. Equating real parts to real parts and imaginary parts to imaginary parts (like components of vectors), we obtain

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.} \tag{6.20}$$

These are the famous **Cauchy–Riemann** conditions. They were discovered by Cauchy and used extensively by Riemann in his theory of analytic functions. These Cauchy–Riemann conditions are necessary for the existence of a derivative of $f(z)$; that is, if df/dz exists, the Cauchy–Riemann conditions must hold.

Conversely, if the Cauchy–Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative df/dz exists. This may be shown by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y. \tag{6.21}$$

The justification for this expression depends on the continuity of the partial derivatives of u and v. Dividing by δz, we have

$$\begin{aligned} \frac{\delta f}{\delta z} &= \frac{(\partial u/\partial x + i(\partial v/\partial x))\delta x + (\partial u/\partial y + i(\partial v/\partial y))\delta y}{\delta x + i\delta y} \\ &= \frac{(\partial u/\partial x + i(\partial v/\partial x)) + (\partial u/\partial y + i(\partial v/\partial y))\delta y/\delta x}{1 + i(\delta y/\delta x)}. \end{aligned} \tag{6.22}$$

If $\delta f/\delta z$ is to have a unique value, the dependence on $\delta y/\delta x$ must be eliminated. Applying the Cauchy–Riemann conditions to the y derivatives, we obtain

$$\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x}. \tag{6.23}$$

Substituting Eq. (6.23) into Eq. (6.22), we may cancel out the $\delta y/\delta x$ dependence and

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \tag{6.24}$$

which shows that $\lim \delta f/\delta z$ is independent of the direction of approach in the complex plane as long as the partial derivatives are continuous. Thus, $\frac{df}{dz}$ exists and f is analytic at z.

It is worthwhile noting that the Cauchy–Riemann conditions guarantee that the curves $u = c_1$ will be orthogonal to the curves $v = c_2$ (compare Section 2.1). This is fundamental in application to potential problems in a variety of areas of physics. If $u = c_1$ is a line of electric force, then $v = c_2$ is an equipotential line (surface), and vice versa. To see this, let us write the Cauchy–Riemann conditions as a product of ratios of partial derivatives,

$$\frac{u_x}{u_y} \cdot \frac{v_x}{v_y} = -1, \tag{6.25}$$

with the abbreviations

$$\frac{\partial u}{\partial x} \equiv u_x, \qquad \frac{\partial u}{\partial y} \equiv u_y, \qquad \frac{\partial v}{\partial x} \equiv v_x, \qquad \frac{\partial v}{\partial y} \equiv v_y.$$

Now recall the geometric meaning of $-u_x/u_y$ as the slope of the tangent of each curve $u(x, y) = \text{const.}$ and similarly for $v(x, y) = \text{const.}$ This means that the $u = \text{const.}$ and $v = \text{const.}$ curves are mutually orthogonal at each intersection. Alternatively,

$$u_x\, dx + u_y\, dy = 0 = v_y\, dx - v_x\, dy$$

says that, if (dx, dy) is tangent to the u-curve, then the orthogonal $(-dy, dx)$ is tangent to the v-curve at the intersection point, $z = (x, y)$. Or equivalently, $u_x v_x + u_y v_y = 0$ implies that the **gradient vectors** (u_x, u_y) and (v_x, v_y) **are perpendicular**. A further implication for potential theory is developed in Exercise 6.2.1.

Analytic Functions

Finally, if $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, we say that $f(z)$ is **analytic**[6] at $z = z_0$. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. Our theory of complex variables here is one of analytic functions of a complex variable, which points up the crucial importance of the Cauchy–Riemann conditions. The concept of analyticity carried on in advanced theories of modern physics plays a crucial role in dispersion theory (of elementary particles). If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a singular point and consideration of it is postponed until Section 6.6.

To illustrate the Cauchy–Riemann conditions, consider two very simple examples.

Example 6.2.1 z^2 Is Analytic

Let $f(z) = z^2$. Then the real part $u(x, y) = x^2 - y^2$ and the imaginary part $v(x, y) = 2xy$. Following Eq. (6.20),

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

We see that $f(z) = z^2$ satisfies the Cauchy–Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic. ■

[6] Some writers use the term **holomorphic** or **regular**.

Example 6.2.2 z^* IS NOT ANALYTIC

Let $f(z) = z^*$. Now $u = x$ and $v = -y$. Applying the Cauchy–Riemann conditions, we obtain

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy–Riemann conditions are not satisfied and $f(z) = z^*$ is not an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane.

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighborhood — for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far-reaching implications. The real and imaginary parts of our analytic function must separately satisfy Laplace's equation. This is Exercise 6.2.1. Further, our analytic function is guaranteed derivatives of all orders, Section 6.4. In this sense the derivative not only governs the local behavior of the complex function, but controls the distant behavior as well. ■

Exercises

6.2.1 The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.

(a) Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's equation such as $u(x, y)$ and $v(x, y)$ are called **harmonic** functions.

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0,$$

and give a geometric interpretation.
Hint. The technique of Section 1.6 allows you to construct vectors normal to the curves $u(x, y) = c_i$ and $v(x, y) = c_j$.

6.2.2 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

6.2.3 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$ and $v(x, y)$ **cannot both have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have saddle points only.)

6.2.4 Let $A = \partial^2 w/\partial x^2$, $B = \partial^2 w/\partial x \partial y$, $C = \partial^2 w/\partial y^2$. From the calculus of functions of two variables, $w(x, y)$, we have a **saddle point** if

$$B^2 - AC > 0.$$

With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy–Riemann conditions and show that **neither** $u(x, y)$ nor $v(x, y)$ **has a maximum or a minimum** in a finite region of the complex plane. (See also Section 7.3.)

6.2.5 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

if (a) $u(x, y) = x^3 - 3xy^2$, (b) $v(x, y) = e^{-y} \sin x$.

6.2.6 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

6.2.7 The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.

6.2.8 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Phi(r,\theta)}$, in which $R(r, \theta)$ and $\Phi(r, \theta)$ are differentiable real functions of r and θ, show that the Cauchy–Riemann conditions in polar coordinates become

(a) $\dfrac{\partial R}{\partial r} = \dfrac{R}{r}\dfrac{\partial \Theta}{\partial \theta}$, (b) $\dfrac{1}{r}\dfrac{\partial R}{\partial \theta} = -R\dfrac{\partial \Theta}{\partial r}$.

Hint. Set up the derivative first with δz radial and then with δz tangential.

6.2.9 As an extension of Exercise 6.2.8 show that $\Theta(r, \theta)$ satisfies Laplace's equation in polar coordinates. Equation (2.35) (without the final term and set to zero) is the Laplacian in polar coordinates.

6.2.10 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

(a) Show that $df/dz = V_x - iV_y$;
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks);
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

6.2.11 A proof of the Schwarz inequality (Section 10.4) involves minimizing an expression,

$$f = \psi_{aa} + \lambda\psi_{ab} + \lambda^*\psi_{ab}^* + \lambda\lambda^*\psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex and λ is a complex parameter.

(a) Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f/\partial\lambda^*$ equal to zero yields

$$\lambda = -\frac{\psi_{ab}^*}{\psi_{bb}}.$$