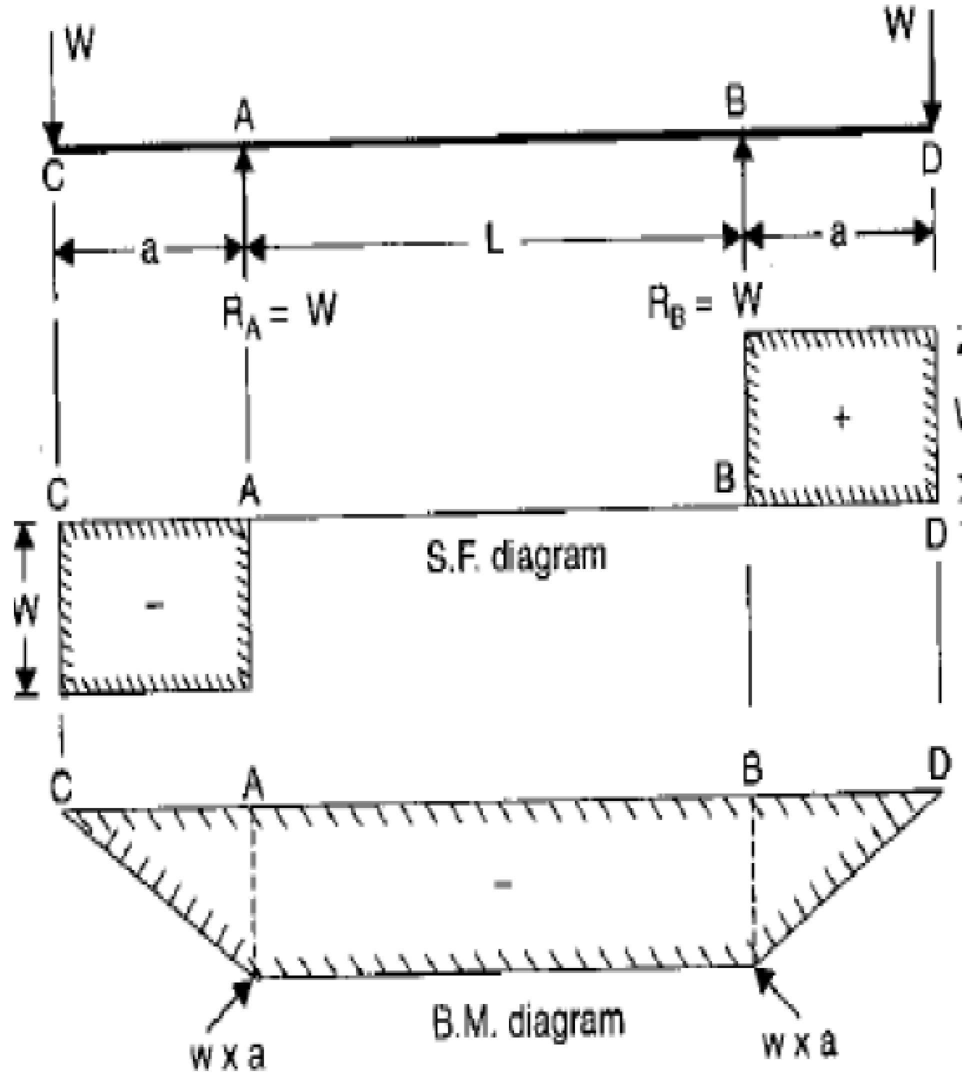
Stresses in Beams – Bending and Shear

LEARNING OUTCOMES:

- Explain the theory of simple or pure bending (L2).
- List the assumptions underlying bending theory (L1).
- Derive relations between bending stress and radius of curvature, and bending moment and radius of curvature (L3).
- Apply above expressions to calculate bending stresses and other beam parameters (L3).

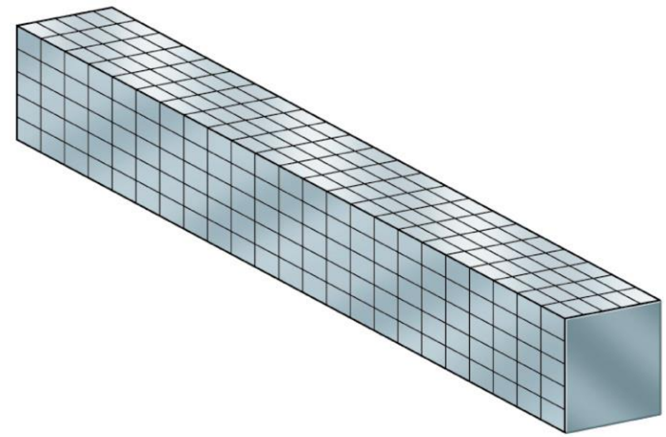
Stresses in Beams – Bending and Shear

- If a specific length of a beam is subjected to a constant bending moment & shear force is zero, then the stresses set up in that length of the beam are known as bending stresses. The length of the beam under constant bending moment is said to be in ***pure bending.***

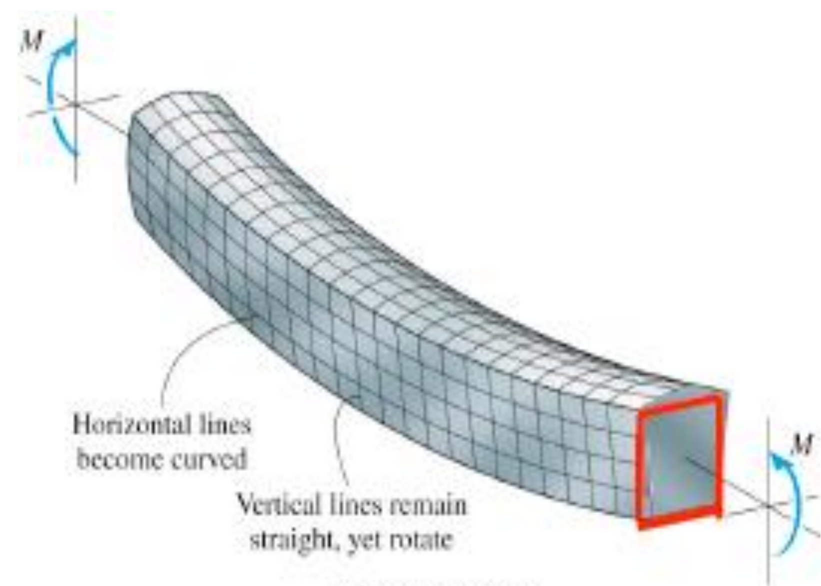


Stresses in Beams – Bending and Shear

- Internal bending moment causes beam to deform.
- Top fibers in compression, bottom in tension.
- **Neutral surface** – no change in length.
- **Neutral Axis** – Line of intersection of neutral surface with the transverse section.
- All cross-sections remain plane and perpendicular to longitudinal axis.

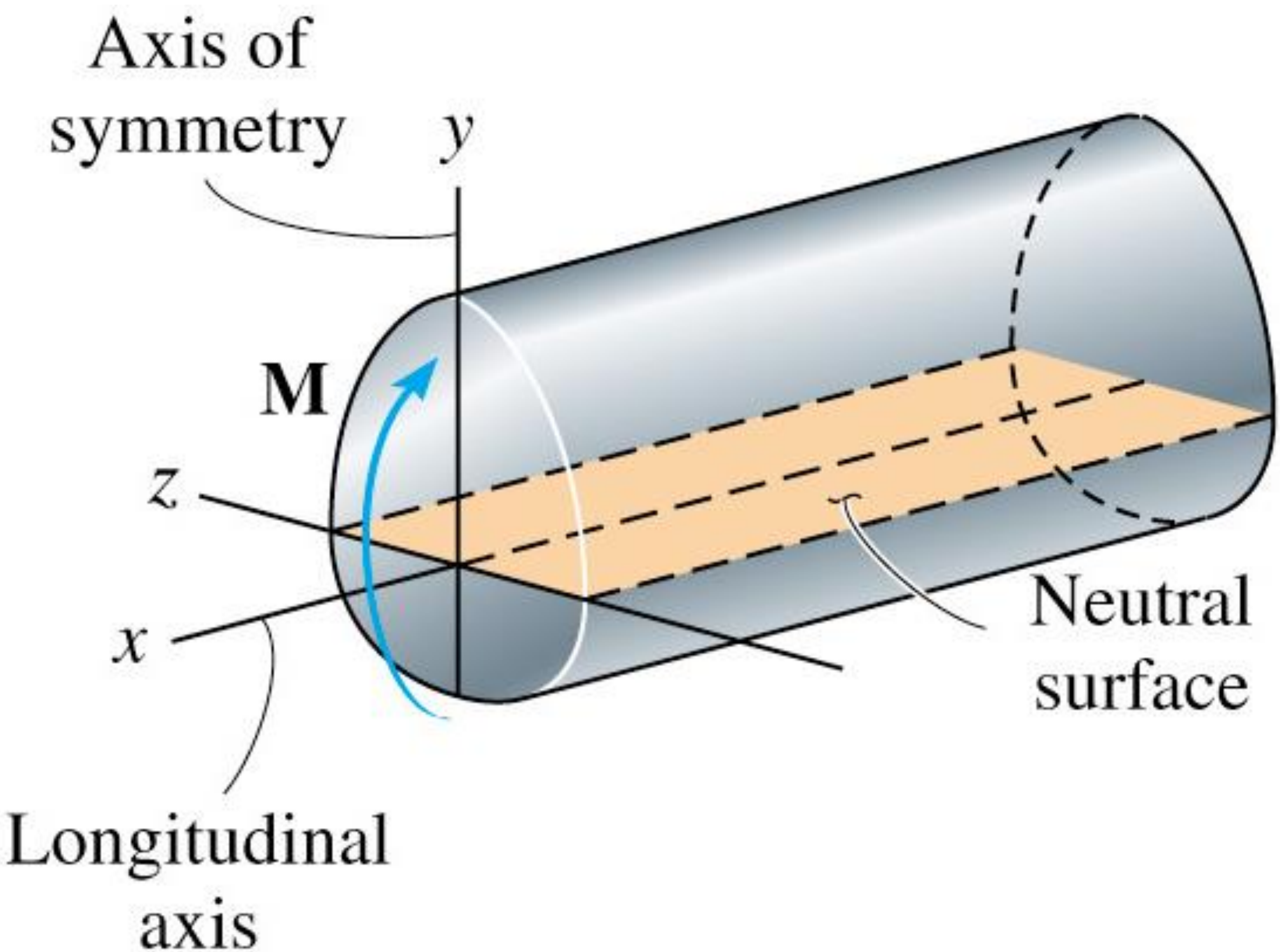


Before deformation



After deformation

Stresses in Beams – Bending and Shear

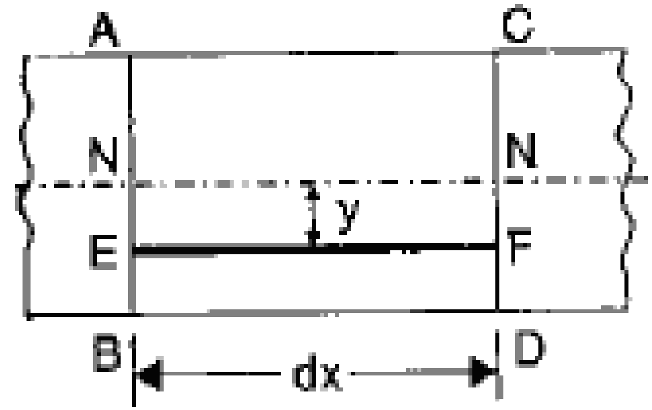


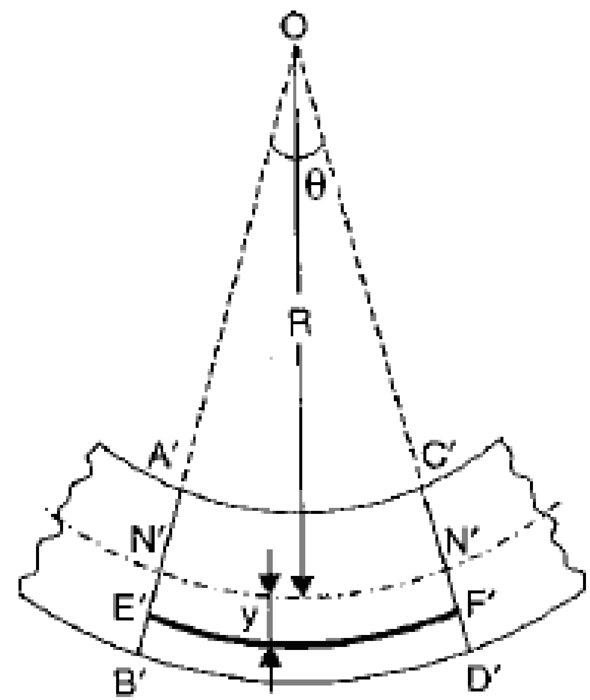
Stresses in Beams – Bending and Shear

- Assumptions in simple (pure) bending theory:
  - Material of beam is homogenous and isotropic (same composition & constant E in all directions).
  - Young's modulus is constant in compression and tension.
  - Transverse section which are plane before bending remain plain after bending (Eliminate effects of strains in other direction).
  - Beam is initially straight and all longitudinal filaments bend in circular arcs.
  - Radius of curvature is large compared with dimension of cross sections.
  - Each layer of the beam is free to expand or contract.

Derivation of Relationship Between Bending Stress and Radius of Curvature

- Consider a small length δx of a beam subjected to a simple bending as shown in the figure (a).
- Due to action of bending, the length δx will be deformed as shown in the figure (b).





Derivation of Relationship Between Bending Stress and Radius of Curvature

- Due to the decrease in length of the layers above N-N, these layers will be subjected to compressive stresses.
- Due to the increase in length of the layers above N-N, these layers will be subjected to tensile stresses.
- The amount by which a layer increases or decreases in length, depends upon the position of the layer w.r.t. N-N. This theory of bending is known as theory of simple bending.

Derivation of Relationship Between Bending Stress and Radius of Curvature

Let, R = Radius of curvature of neutral layer N'-N'.

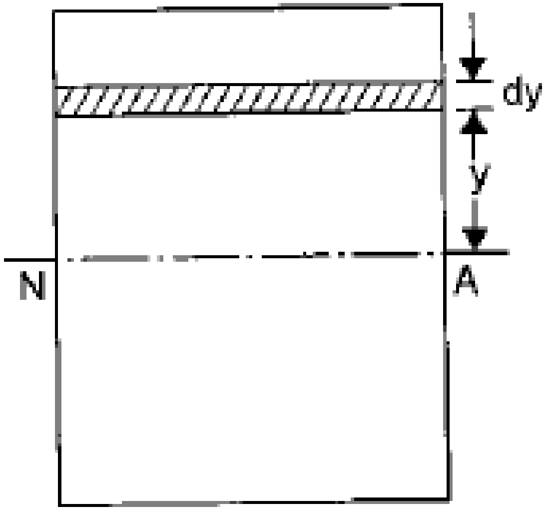
- θ = Angle subjected at O by A'B' and C'D' produced.
- y = Distance from the neutral layer.
- Original length of the layer = EF = δx = NN = N'N'
- From figure (b), N'N' = **R θ**
- Change (Increase) in length of the EF = E'F' –EF = **(R + y) θ – R θ**

  = y θ
- Strain in the layer EF = Increase in length / original length

  = **y θ / R θ = y / R**
- According to linear elasticity, σ α ϵ,.That is, ϵ = σ / E.
- $\frac{y}{R} = \frac{\sigma}{E}$ $\sigma = \frac{E}{R} y$ σ α y

Derivation of Relationship Between Bending Stress and Radius of Curvature (Moment of Resistance of a Section)

- The stresses induced in the layers of the beam create compressive and tensile forces.
- These forces will have moment about NA.
- The total moment of these forces about NA for a section is known as moment of resistance of that section.
- Consider a cross section of a beam as shown:



Derivation of Relationship Between Bending Stress and Radius of Curvature (Moment of Resistance of a Section)

Force on layer $= \frac{E}{R} \times y \times dA$

Moment of this force about N.A.

$$= \text{Force on layer} \times y$$

$$= \frac{E}{R} \times y \times dA \times y$$

$$= \frac{E}{R} \times y^2 \times dA$$

Total moment of the forces on the section of the beam (or moment of resistance)

$$\therefore \quad M = \int \frac{E}{R} \times y^2 \times dA = \frac{E}{R} \int y^2 \times dA$$

$$\therefore \quad M = \frac{E}{R} \times I \quad \text{or} \quad \frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Euler – Bernoulli Bending Equation

Section Modulus (Z)

- It is the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis.

$$Z = \frac{I}{y_{max}}$$

I = Moment of Inertia about neutral axis.

y_{max} = Distance of the outermost layer from the neutral axis.

Hence, $M = \sigma_{max} . Z$

- Thus, moment of resistance offered by the section is maximum when Z is maximum. Hence, Z represents the strength of the section.

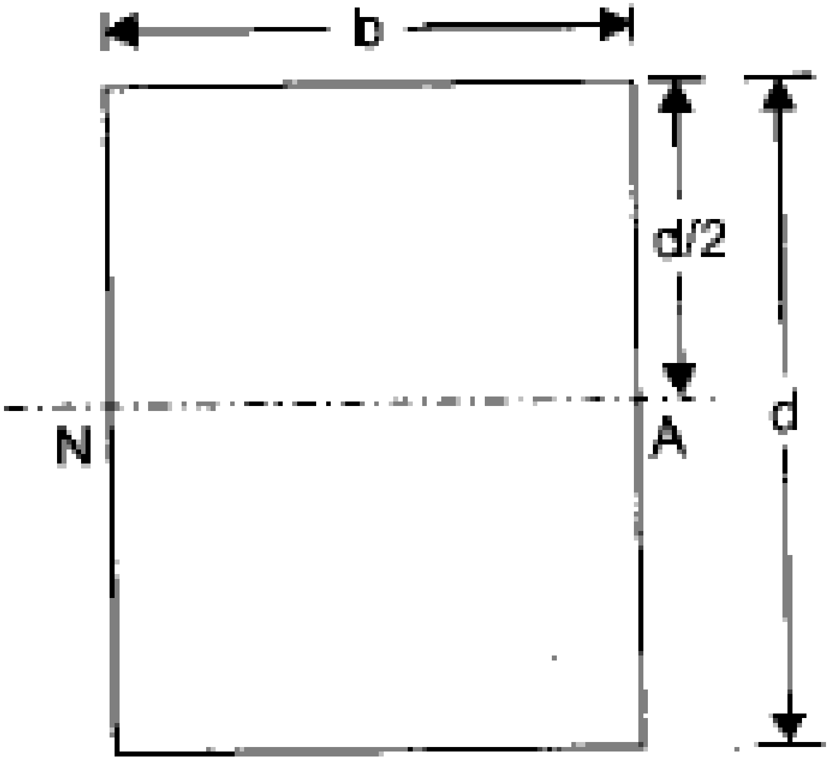
Section Modulus (Z)

1. Rectangular Section

$$I = \frac{bd^3}{12}$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{bd^2}{6}$$



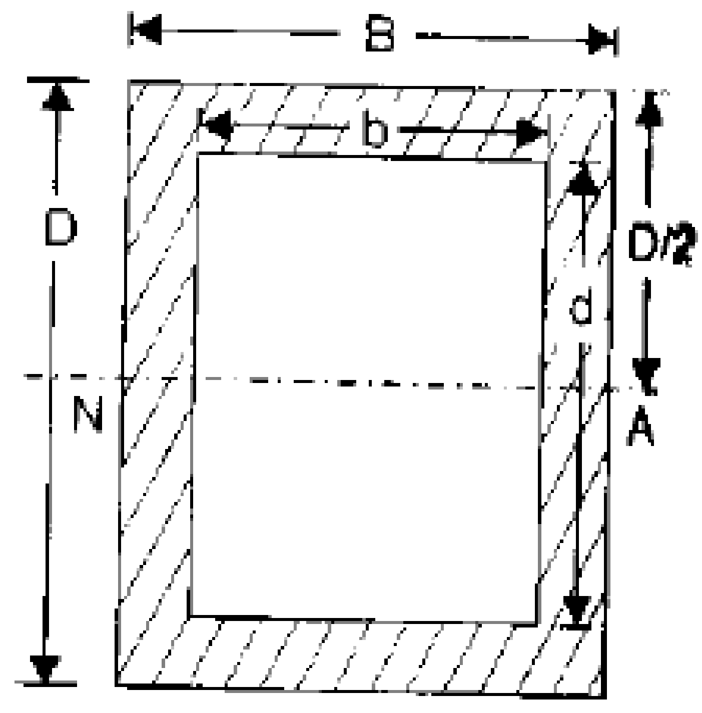
Section Modulus (Z)

2. Rectangular Hollow Section

$$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$y_{max} = \left(\frac{D}{2}\right)$$

$$Z = \frac{1}{6D}\left[BD^3 - bd^3\right]$$



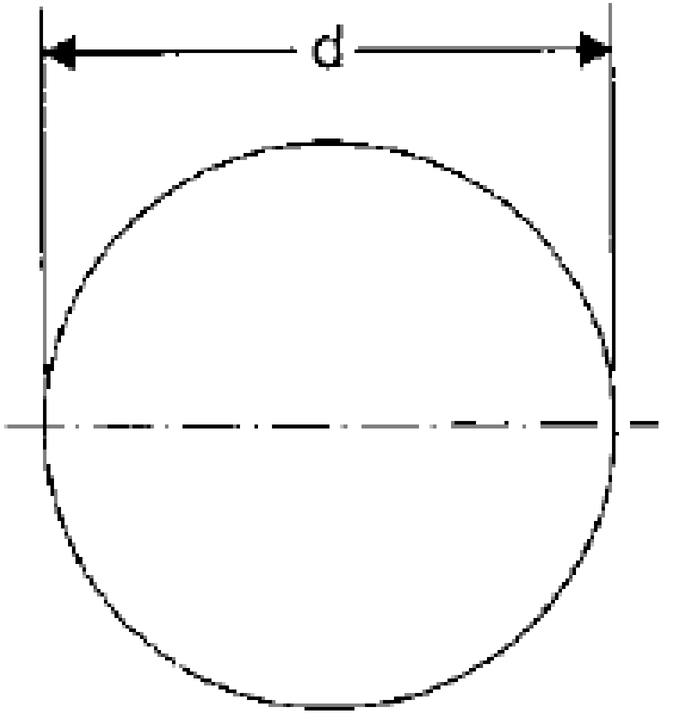
Section Modulus (Z)

3. Circular Section

$$I = \frac{\pi}{64} d^4$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{\pi}{32} d^3$$



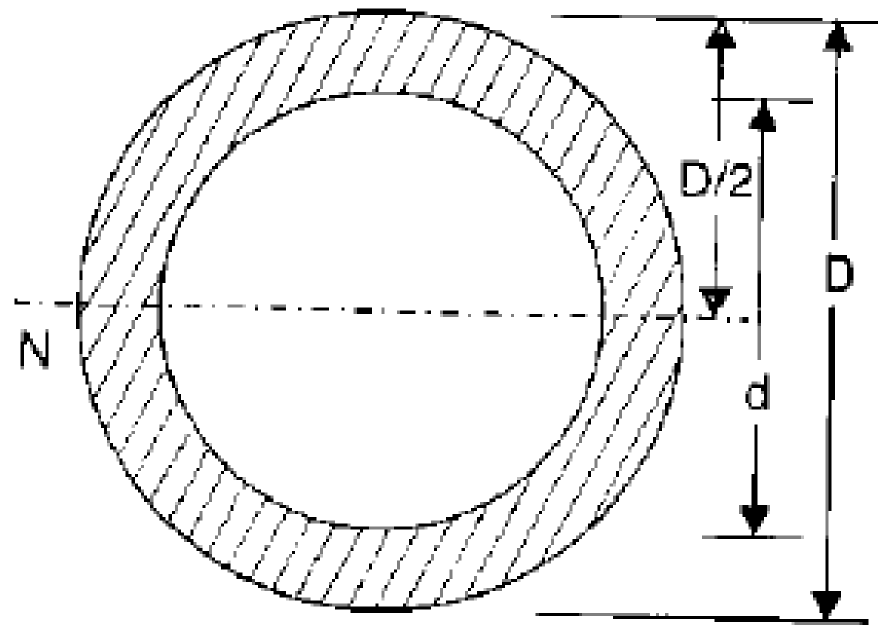
Section Modulus (Z)

4. Circular Hollow Section

$$I = \frac{\pi}{64}\,[D^4 - d^4]$$

$$y_{max} = \frac{D}{2}$$

$$Z = \frac{\pi}{32D}\,[D^4 - d^4]$$

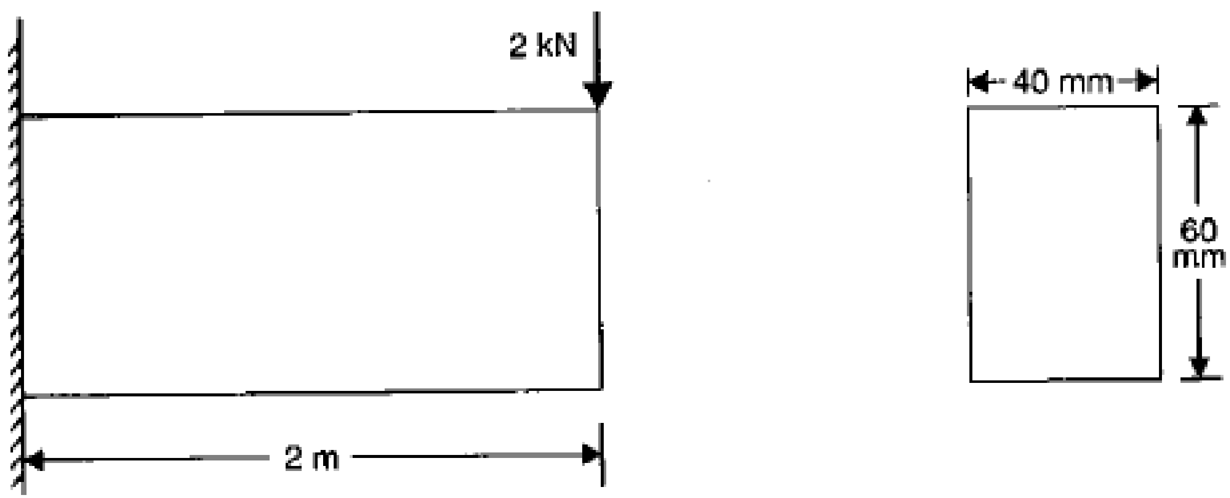


Problems

A cantilever of length 2m fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm X 60 mm, find the stress at the failure.

Solution:

Problem Sketch:



- Let, σ_{max} = Stress at failure (Maximum stress)
- Since R is not given, we cannot use $\sigma = \frac{E}{R} y$
- Instead, use $M = \sigma_{max} \times I/y_{max}$ OR
- $M = \sigma_{max} \times Z$, where

$$Z = \frac{bd^2}{6} = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

and, $M = W \times L = 2000 \times 2 \times 10^3 = 4 \times 10^6 \text{ Nmm}$

Hence,

$$\sigma_{max} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = \mathbf{166.67 \ N/mm^2}.$$

Problem

A square beam 20mm X 20mm in section and 2m long is simply supported at the ends. The beam fails when a point load of 400N is applied at the centre of the beam. What uniformly distributed load per meter length will break a cantilever of the same material 40mm wide, 60mm deep and 3m long?

Solution:

Square c/s.: 20 mm x 20 mm; L = 2m; W = 400 N

Rectangular c/s.: 40 mm x 60 mm; L = 3m; w = ?

Equate the maximum stress in the two cases.

Maximum stress in beam of square c/s.:

$$M = \sigma_{max} \times Z, \text{ where}$$

$$Z = \frac{bd^2}{6} = \frac{20 \times 20^2}{6} = \frac{4000}{3} \text{ mm}^3$$

$$M = \frac{w \times L}{4} = \frac{400 \times 2}{4} = 200 \text{ Nm}$$

$$= 200 \times 1000 = 200000 \text{ Nmm}$$

$$\sigma_{max} = \frac{200000 \times 3}{4000} = 150 \text{ N/mm}^2$$

Maximum stress in beam of rectangular c/s.:

$$M = \sigma_{max} \times Z, \text{ where}$$

$$Z = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

Maximum bending moment for a cantilever beam loaded with UDL for the entire span is given by

$$M_{max} = wL^2 / 2$$

Hence, $M_{max} = \dfrac{w \times 3^2}{2} = 4.5w \text{ Nm}$

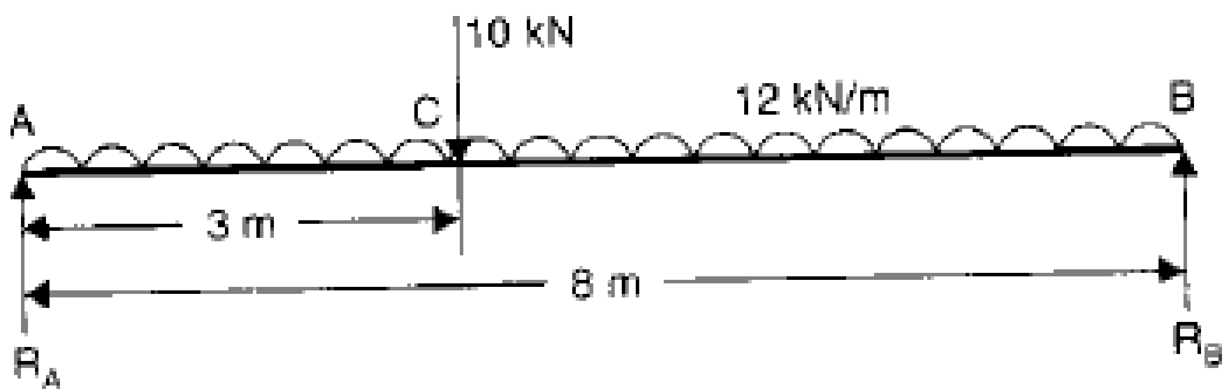
$$M_{max} = 4500 \text{ w N-mm}$$

$$
\begin{aligned}
M &= \sigma_{max} \,.\, Z \\
4.5 \times 1000w &= 150 \times 24000 \\
w &= \frac{150 \times 24000}{4.5 \times 1000} = \mathbf{800\ N/m.}
\end{aligned}
$$

Problem

A timber beam of rectangular section of length 8m is simply supported. The beam carries a U.D.L. of 12 kN/m run over the entire length and a point load of 10 kN at 3m from the left support. If the depth is two times the width and the stress in the timber is not to exceed 8 N/mm^2, find the suitable dimensions of the section.

Solution: Problem sketch



Given Data:

Length,	$L = 8$ m
U.D.L.,	$w = 12$ kN/m = 12000 N/m
Point load,	$W = 10$ kN = 10000 N
Depth of beam	= 2 × Width of beam
∴	$d = 2b$
Stress,	$\sigma_{max} = 8$ N/mm^2

Find the maximum bending moment in the beam and use $M = \sigma_{max} \times Z$ to find ***b*** and ***d***.

- Find the reaction forces at the supports
- Find the shear force at all the points of interest
- Find the maximum bending moment where SF is zero.

$$R_B \times 8 = 12000 \times 8 \times 4 + 10000 \times 3$$

$$R_B = \frac{12000 \times 32 + 30000}{8} = 51750 \text{ N}$$

$$R_A = \text{Total load} - R_B$$

$$= (12000 \times 8 + 10000) - 51750 = 54250 \text{ N}$$

Shear Forces:

$F_B = - R_B = - 51750$ N; $F_C = +18250$ N; $F_A = +54250$ N

Shear force changes its sign between B and C.

Let D be the point where SF = 0. Let ***x*** be the distance (in meters) of this point on the beam from B.

By calculation, x = 4.3125 m

- Find the BM at D.
- Find the Moment of Inertia (I) or section modulus (Z) of the beam's cross section
- Use the equation, M = σ_{max} x Z

$$M = R_B \times 4.3125 - 12000 \times 4.3125 \times \frac{4.3125}{2}$$

$$51750 \times 4.3125 - 111585.9375$$

$$111585.9375 \text{ Nm} = 111585.9375 \times 1000 \text{ Nmm}$$

$$Z = \frac{bd^2}{6} = \frac{b \times (2b)^2}{6} = \frac{2b^3}{3}$$

$$M = \sigma_{max} \cdot Z$$

$$111585.9375 \times 1000 = 8 \times \frac{2b^3}{3}$$

Hence,
b = 275.5 mm
d = 551 mm