Example: The series $\sum \frac{1}{n^a}$ is cg if $a>1$ & dvgs if $a \le 1$

Solution:-

Let $S_n = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{n^a}$

if $a > 1$, then

$S_n < S_{2n}$ &

Let $n = 3$, $a = 2$

$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$

$= 1 + \frac{1}{4} + \frac{1}{9} = \text{---}$

$S_{2n} = S_{2(3)} = S_6 =$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2}$

$\Rightarrow S_3 < S_6$

$\Rightarrow S_n < S_{2n}$

$\Rightarrow n^a > (n-1)^a$

$\Rightarrow \frac{1}{n^a} < \frac{1}{(n-1)^a}$

Now

$S_{2n} = \left[1 + \frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} + \cdots + \frac{1}{(2n)^a}\right]$

$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + \cdots + \frac{1}{(2n-1)^a}\right] + \left[\frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{6^a} + \cdots + \frac{1}{(2n)^a}\right]$

$= \left[1 + \frac{1}{3^a} + \frac{1}{5^a} + \cdots + \frac{1}{(2n-1)^a}\right] + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + \frac{1}{3^a} + \cdots + \frac{1}{(n)^a}\right]$

$< \left[1 + \frac{1}{2^a} + \frac{1}{4^a} + \cdots + \frac{1}{(2n-2)^a}\right] + \frac{1}{2^a} S_n$

$= 1 + \frac{1}{2^a}\left[1 + \frac{1}{2^a} + \cdots + \frac{1}{(n-1)^a}\right] + \frac{1}{2^a} S_n$

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$= 1 + \frac{1}{2^q} S_{n-1} + \frac{1}{2^q} S_n$

$\leq 1 + \frac{1}{2^q} S_{2n} + \frac{1}{2^q} S_{2n} \quad \because S_{n-1} < S_n < S_{2n}$

$= 1 + \frac{2}{2^q} S_{2n}$

$\Rightarrow S_{2n} < 1 + \frac{1}{2^{q-1}} S_{2n}$

$\Rightarrow S_{2n} - \frac{1}{2^{q-1}} S_{2n} < 1$

$\Rightarrow S_{2n}\left(1 - \frac{1}{2^{q-1}}\right) < 1$

$\Rightarrow \left(\frac{2^{q-1} - 1}{2^{q-1}}\right) S_{2n} < 1$

$\Rightarrow S_{2n} < \frac{2^{q-1}}{2^{q-1} - 1}$

i.e

$S_n < S_{2n} < \frac{2^{q-1}}{2^{q-1} - 1}$

$\Rightarrow \{S_n\}$ is bounded & also monotonic.

We conclude that $\sum \frac{1}{n^q}$ is cgt if $q > 1$

If $q \leq 1$ Then

3/

$\Rightarrow n^q \leq n \quad \forall n \geq 1$

$\Rightarrow \frac{1}{n^q} \geq \frac{1}{n} \quad \forall n \geq 1$

$\because \sum \frac{1}{n}$ is dvgT $\Rightarrow \sum \frac{1}{n^q}$ is dgT

if $q \leq 1$

Example:-

$\sum \frac{1}{n} \sin^2 \frac{x}{n}$, check dvgs or cgs.

Sol:-

$a_n = \frac{1}{n} \frac{\sin^2 x}{n}$, Take $b_n = \frac{1}{n^3}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{n} \sin^2 \frac{x}{n}}{\frac{1}{n^3}} = n^2 \sin^2 x/n$$

$$= \frac{\sin^2 x/n}{1/n^2} \quad \text{—①}$$

multiplying ① by x^2; as

$$① \Rightarrow x^2 \frac{\sin^2 x/n}{x^2/n^2} = x^2 \left[\frac{\sin \frac{x}{n}}{x/n}\right]^2$$

applying $\lim_{n\to\infty}$; as

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} x^2 \left(\frac{\sin x/n}{x/n}\right)^2$$

$$= x^2 \lim_{n\to\infty} \left(\frac{\sin x/n}{x/n}\right)^2$$

$= x^2 (1) = x^2$ 4

$\Rightarrow \sum a_n$ & $\sum b_n$ behave similar

$\because \lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$

$\because \sum \frac{1}{n^3}$ is cgt series then given series is cgt $\forall$ values of x except $x=0$

Cauchy Condensation Test:-

Let $a_n \geq 0$, $a_n > a_{n+1}$ $\forall n \geq 1$ then then series $\sum a_n$ & $\sum 2^{n-1} a_{2^{n-1}}$ cgs / dgs together.

$a_n > a_{n+1}$

$a_1 > a_2$, $1 < 2$

$a_1 > a_2$, $n < n+1$

Proof:-

Let us suppose that

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$t_n = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^{n-1} a_{2^{n-1}}$

$\because a_n \geq 0$ & $n < 2^{n-1} < 2^n - 1$

$\therefore S_n < S_{2^{n-1}} < S_{2^n - 1}$ for $n > 2$

Then

$S_{2^n-1} = a_1 + a_2 + a_3 + \cdots + a_{2^n - 1}$

$= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots +$

$(a_{2^{n-1}} + a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n - 1})$

$$2 \to 2^1, \quad 3 \to 2^2 - 1$$

$$4 - 1 = 3$$

$$4 \to 2^2, \quad 7 \to 2^3 - 1 \quad \to a_{2^n}, \; a_{2^{n+1}-1}$$

$$(a_4 + a_5 + a_6 + a_7) = (a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1})$$

if $n < n+1$ then $n-1 < n$

$\Rightarrow a_{2^{n-1}}$ & a_{2^n-1}

~~$= a_1 + (a_2 + a_3) +$~~

~~$= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}+1} + a_{2^{n-1}+2} + a_{2^{n-1}+3} + \cdots + a_{2^n})$~~

Cauchy Condensation Test:-

Proof:-

$$= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n-1})$$

$$< a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^{n-1}} + a_{2^{n-1}} + a_{2^{n-1}} + \cdots + a_{2^{n-1}})$$

$$< a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}}$$

$$= a_1 + 2a_2 + 2^2 a_4 + \cdots + 2^{n-1} a_{2^{n-1}} = t_n$$

$$\Rightarrow S_{2^n-1} < t_n$$

$$\Rightarrow S_n < t_n \qquad \because S_n < S_{2^n-1}$$

$\Rightarrow S_n < T_n < 2S_{2^n}$ —— 1

Now consider

$S_{2^n} = a_1 + a_2 + a_3 + \cdots + a_{2^n}$

$= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots +$

$(a_{2^{n-1}+1} + a_{2^{n-1}+2} + a_{2^{n-1}+3} + \cdots + a_{2^n})$

$> \frac{1}{2}a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots +$

$(a_{2^n} + a_{2^n} + a_{2^n} + \cdots + a_{2^n})$

$= \frac{1}{2}a_1 + a_2 + 2a_4 + 2^2 a_8 + \cdots + 2^{n-1} a_{2^n}$

$= \frac{1}{2}(a_1 + 2a_2 + 2^2 a_4 + 2^3 a_8 + \cdots + 2^n a_{2^n})$

$\Rightarrow S_{2^n} > \frac{1}{2} T_n$ —— 2

$\Rightarrow 2S_{2^n} > T_n$

from 1 & 2, S_n & T_n are either both cgs / dgs as both bounded or unbounded.