lecture-11

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Infinite series:-

Given a seq $\{S_n\}$, we use $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$ & its called infinite series / series

→ $S_n = \sum_{k=1}^{n} a_k$ are called partial sums.

→ if $\{S_n\}$ cgs to 's' ⇒ series cgs &

→ $\sum_{n=1}^{\infty} a_n = s$, 's' is sum of series.

Here, 's' is lim of seq of sums & is not obtained simply by addition.

Theorem: If

$$\sum_{n=1}^{\infty} a_n \text{ cgs then } \lim_{n\to\infty} a_n = 0$$

proof:-

let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

∵ if $\{S_n\}$ cgs to 's' then we write

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} S_n$$

∴ $a_n = S_n - S_{n-1}$ —(1)

~~$\because S_n = a_1 + a_2 + \cdots + a_n$~~
~~$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$~~
~~$S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_n) - (a -$~~

∵ $S_n = a_1 + a_2 + \cdots + a_n$

$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$

$S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$

$S_n - S_{n-1} = a_n$, so, now

(1) ⇒ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= s - s = 0$

If we talk about its converse, this will be something like this:-

$\Rightarrow$ If $\lim_{n\to\infty} a_n = 0$ Then $\sum_{n=1}^{\infty} a_n$ cgs.
But converse of theorem is false, why, let's check it by example.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$S_n = \{\tfrac{1}{n} ; n \in N\}$$

$$= \{1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{n}\}$$

$\because$ we know that $\{S_n\}$ is dvgt

But $\sum_{n=1}^{\infty} \frac{1}{n} = 0$, so if this holds, Then it must cgs. but here is opposition, so This implies

$\Rightarrow$ if $\lim_{n\to\infty} a_n \neq 0$ Then $\sum a_n$ is dvgt.

$\Rightarrow$ Basic dvgt Test.

Example:

$$\sum_{n=1}^{\infty} n/2n+1$$

$$\lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{2+\frac{1}{n}}\right)$$

$$= \frac{1}{2} \neq 0$$

$\Rightarrow S_n = \frac{n}{2n+1}$ is dvgt.

Theorem: Let $\sum a_n$ be an infinite series of non-negative terms & let $\{S_n\}$ be seq of its partial sums. Then $\sum a_n$ is cgt if $\{S_n\}$ is bounded & is dvgs if $\{S_n\}$ is unbounded.

Proof:-

$\because \sum a_n$ is infinite series of non-negative terms that is

$a_n \geq 0 \quad \forall n \geq 0$

we can write;

$S_n = S_{n-1} + a_n \quad \because a_n = S_n - S_{n-1}$

$\geq S_{n-1} \quad \forall n \geq 0$

$\Rightarrow \{S_n\}$ is monotonically ↑ & hence it cgs by using the statement that {if $\{S_n\}$ is bound & monotonically ↑ seq then it cgs to its sup}.

so $\{S_n\}$ is cgs if bounded & dvgs if unbounded.

$\Rightarrow \sum a_n$ is cgt if $\{S_n\}$ bounded & dgt if $\{S_n\}$ is unbounded.

proved.

Theorem (Comparison Test):-

Suppose $\sum a_n$ & $\sum b_n$ are infinite series s.t $a_n \geq 0$, $b_n \geq 0$ $\forall n$. Also suppose that for a fixed +ve number I & +ve integer k, $a_n \leq I b_n$ $\forall n \geq k$, Then

i) $\sum a_n$ cgs if $\sum b_n$ cgs

ii) $\sum b_n$ cgs dvgs if $\sum a_n$ dvgs

proof:-

(i) Suppose $\sum b_n$ is cgt &

$$a_n < I b_n \quad \forall n \geq k \text{ --- } (1)$$

Then by def of general principle of cgs,

"A series $\sum a_n$ is cgt iff for any real number $\epsilon > 0$, $\exists$ a +ve int n_0 s.t

$$\left| \sum_{i=m+1}^{\infty} a_i \right| < \epsilon \quad \forall n > m > n_0"$$

Here $\because$ $\sum b_n$ is cgt, so $\exists$ a +ve int n_0 for any $\epsilon > 0$ s.t

$$\sum_{i=m+1}^{\infty n} b_i < \epsilon \text{ --- } (2)$$

applying $\sum_{i=m+1}^{\infty}$ in eq (1); as

$$(1) \Rightarrow \sum_{i=m+1}^{n \infty} a_i < I \sum_{i=m+1}^{n} b_i$$

$$< I(\epsilon) \quad (\text{frm } (2))$$

$\Rightarrow \sum_{i=m+1}^{n} a_i < \epsilon \quad \forall\, n > m \geq n_0$

$\Rightarrow \sum a_n$ is cgt

(ii)

Now suppose $\sum a_n$ is dgt, then $\{S_n\}$ is unbounded

$\Rightarrow \exists$ a real number $b > 0$ s.t

$\sum_{i=m+1}^{n} b_i \geq l b \;,\; n > m$

① $\Rightarrow \sum_{i=m+1}^{n} b_i > \frac{1}{l} \sum_{i=m+1}^{n} a_i > b \,,\quad n > m$

$\Rightarrow \sum b_n$ is cgt

Example: Show that $\sum \frac{1}{n}$ is dgt

$\because n \geq \sqrt{n} \quad \forall\, n \geq 1$

$\Rightarrow \frac{1}{n} \leq \frac{1}{\sqrt{n}}$

$\Rightarrow \sum \frac{1}{\sqrt{n}}$ is dgt as we know that $\sum \frac{1}{n}$ is dgt.

Theorem:

Let $a_n > 0$, $b_n > 0$ & $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = l \neq 0$, then the series $\sum a_n$ & $\sum b_n$ behave alike.

proof:-

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = I$$

$$\Rightarrow \left|\frac{a_n}{b_n} - I\right| < \epsilon \quad \forall\, n \geq n_0$$

Use $\epsilon = I/2$

$$\Rightarrow \left|\frac{a_n}{b_n} - I\right| < I/2 \quad \forall\, n \geq n_0$$

$$\Rightarrow I - \frac{I}{2} < \frac{a_n}{b_n} < I + \frac{I}{2}$$

$$\Rightarrow \frac{I}{2} < \frac{a_n}{b_n} < 3I/2$$

$$\Rightarrow a_n < \frac{3I}{2} b_n \text{ \& } b_n < \frac{2}{I} a_n$$

$\Rightarrow \Sigma a_n$ & Σb_n both behave alike.