again chose $n_2 > n_1$ s.t

$|S_{n_2} - s| < 1/2$

continuing in this manner, we find a sequence $\{S_n\}$ for each +ve int 'k' s.t

$n_k < n_{k+1}$ &

$|S_{n_k} - s| < 1/k \quad \forall\ k = 1, 2, 3, \ldots$

$\Rightarrow$ There $\exists$ a sub-seq which cgs to 's'.

Limit Inferior of the sequence:-

Suppose $\{S_n\}$ is bounded, then

~~$\lim_{n\to\infty}(\inf S_n) = \lim_{n\to\infty} U_k$ where $U_k =$~~

$$\lim_{n\to\infty} \inf S_n = \lim_{n\to\infty}\left(\inf_{m\geq n} S_m\right)$$

or

$$\lim_{n\to\infty} \inf S_n = \sup_{n\geq 0} \inf_{m\geq n} S_n$$

$$\boxed{\lim_{n\to\infty} \inf S_n = \sup\{\inf\{S_m, S_{m+1}, S_{m+2}, \ldots\}\}}$$

Limit Superior:-

$$\lim_{n\to\infty} \sup S_n = \inf_{n\geq 0} \sup_{m\geq n} S_n$$

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If $\{S_n\}$ is not bounded below, then
$\lim_{n\to\infty}(\inf S_n) = -\infty$

If $\{S_n\}$ is not bounded above, then
$\lim_{n\to\infty}\{\sup S_n\} = +\infty$

points:-

i) A bounded seq has unique inf & sup Limit

ii) if $\{S_n\}$ contains all the rationals then every real No is sub-seq Limit then
$\lim_{n\to\infty}(\inf S_n) = -\infty$ & $\lim_{n\to\infty}\{\sup S_n\} = +\infty$

iii) $\{S_n\} = (-1)^n\left(1+\frac{1}{n}\right)$

Solution :-

$\lim_{n\to\infty}\inf\{S_n\}$:-

$\lim_{n\to\infty}\inf S_n = \sup\{\inf(S_m, S_{m+1}, \dots)\}$

$\because$

$$(-1)^n\left(1+\frac{1}{n}\right) = \begin{cases} \left(1+\frac{1}{n}\right); & \text{if } n \text{ is even} \\ -\left(1+\frac{1}{n}\right); & \text{if } n \text{ is odd} \end{cases}$$

$= \sup\{\inf\left(\left(1+\frac{1}{m}\right), \left(1+\frac{1}{m+1}\right) \dots\right)\}$

$= \sup\{\inf\left(\sup\{-1\}\right.$

$= -1$

$\lim_{n\to\infty}\inf S_n = -1$

$\lim_{n\to\infty} \sup S_n$:-

$= \inf\{\sup\{(1+\frac{1}{m}), (1+\frac{1}{m+1}), \dots\}\}$

$= \inf\{1\} = 1$

$\lim_{n\to\infty} \sup S_n = 1$

$(1+\frac{1}{n})$:-

$(1+1) = 2$

$(1+\frac{1}{2}) = 3/2$

$(1+\frac{1}{3}) = 4/3$

-1 0 1

$S_n \le 1 \to$ (0

$S_n \ge -1 \to$ S

Assignment:-

(1) Find the limits of following

i) $\lim((2+\frac{1}{n})^2)$

ii) $\lim\left(\frac{(-1)^n}{n+2}\right)$, iii) $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$

(By using 'ϵ' def)

(2)

Find convergence & divergence:-

(i) $T_n = \frac{2n^2+3}{n^2+1}$

(ii) $S_n = \frac{(-1)^n n}{n+1}$

(iii) $P_n = \frac{n^2}{n+1}$

Theorem: If $\{S_n\}$ is cgt seq. then
$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sup\{S_n\} = \lim_{n\to\infty} \inf\{S_n\} = s$.

Proof:-

Let $\lim_{n\to\infty} S_n = s$, Then for a real no. $\epsilon > 0$,
$\exists$ a +ve integer n_0 s.t
$|S_n - s| < \epsilon \quad \forall n \geq n_0$ — (1)
$\Rightarrow s - \epsilon < S_n < s + \epsilon \quad \forall n \geq n_0$ — (2)

Firstly we will prove That:-
$$\lim_{n\to\infty} (\inf\{S_n\}) = s$$
$\therefore$
$$\lim_{n\to\infty} \inf\{S_n\} = \sup_{n\geq 0} \left(\inf_{m\geq n} S_m \right)$$
$$= \lim_{n\to\infty} \left(\inf_{m\geq n} S_m \right)$$
Let $U_n = \inf\{S_m : m \geq n\}$
$\Rightarrow \lim_{n\to\infty} U_n$
$\Rightarrow s - \epsilon < U_n < s + \epsilon$
$\Rightarrow s - \epsilon < \lim_{n\to\infty} U_n < s + \epsilon$ — (3)
from (1), (2) & (3)
$$\lim_{n\to\infty} U_n = s$$
Similarly, you have to prove it for $\lim_{n\to\infty} \sup\{S_n\} = s$