Nested Intervals Theorem:- 1

Suppose that $\{I_n\}$ is a seq of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \; \forall \; n \geq 1$ & $(b_n - a_n) \to 0$ as $n \to \infty$, Then $\cap I_n$ contains only & only one point.

proof:-

$I_n = [a_n, b_n]$, $n \in N$

$I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $I_3 = [a_3, b_3]$ --- $I_n = [a_n, b_n]$ ---- $I_{n+1} = [a_{n+1}, b_{n+1}]$

so $I_n = \{[a_1, b_1], [a_2, b_2], ---\}$

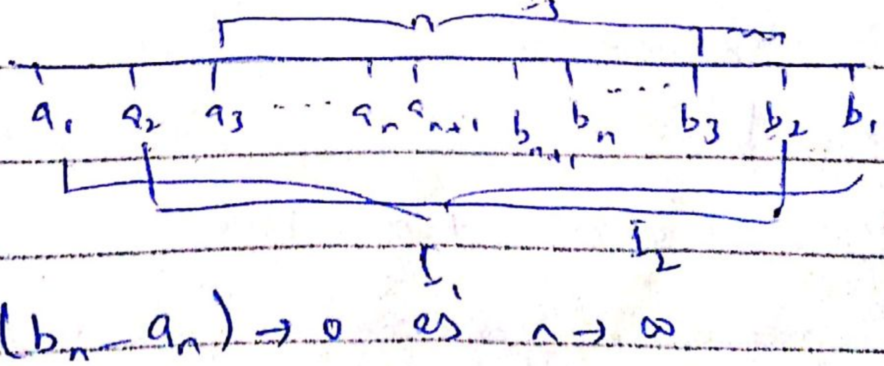
$\because I_{n+1} \subset I_n$, let for $n=1 \to I_2 \subset I_1$

for $n=2 \to I_3 \subset I_2$

if we take $I_2 \subset I_1$

$I_2 = [a_2, b_2]$, $I_1 = [a_1, b_1]$

2) $I_2 \subset I_1 = [a_2, b_2] \subset [a_1, b_1]$



And $(b_n - a_n) \to 0$ as $n \to \infty$

$\lim_{n \to \infty} b_n - a_n = 0$

Then, $\exists$ one & only one pt. let
$k \in I_1, k \in I_2, \dots, k \in I_n$

Now let's make the proof:-

$\because I_{n+1} \subset I_n$

$\therefore a_1 \le a_2 \le a_3 \le \dots \le a_{n-1} \le a_n$ — (1)

$b_n \le b_{n-1} \le \dots \le b_3 \le b_2 \le b_1$ — (2)

where $a_n \le b_n$.

from (1) $\Rightarrow \{a_n\}$ is $\uparrow$ seq, & bounded above by b_1 & bounded below by a_1.
$(a_n \le b_1)$ $(a_1 \le a_n)$

& from (2)

$\{b_n\}$ is $\downarrow$ seq & upper & lower bounds are same

$\Rightarrow \{a_n\}$ & $\{b_n\}$ both cgs.

suppose $\{a_n\}$ cgs to a & $\{b_n\}$ cgs to b.

$$|a-b| = |a - a_n + a_n - b_n + b_n - b|$$
$$\le |a_n - a| + |a_n - b_n| + |b_n - b| \to 0$$
as $n \to \infty$

$\Rightarrow a = b$

$\Rightarrow a_n \le a \le b_n \quad \forall n \ge 1$

proved.

Bolzano-Weierstrass Theorem:-

Every bounded sequence has cgt sub-seq

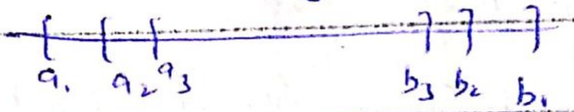
proof:-

let $\{S_n\}$ be a bounded sequence.

Take $a_1 = \inf S_n$ & $b_1 = \sup S_n$

$\Rightarrow a_1 \le S_n \le b_1 \quad \forall n \ge 1$

Now bisect interval $[a_1, b_1]$ s.t atleast one of the two sub-intervals contains infinite number of terms of the seq.



Denote this sub-interval by $[a_2, b_2]$

If both the sub-intervals contain infinite number of terms of the seq then choose the one on the right hand.

$\Rightarrow a_1 \le a_2 \le b_2 \le b_1$

Suppose there exist a sub-interval $[a_k, b_k]$ s.t

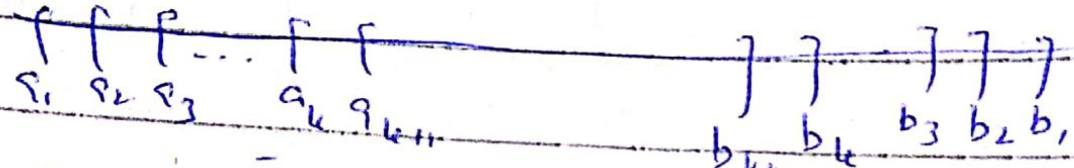
$a_1 \le a_2 \le \ldots \le a_k \le b_k \le \ldots \le b_2 \le b_1$

$\Rightarrow (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1)$

Bisect the interval $[a_k, b_k]$ in the same manner & choose $[a_{k+1}, b_{k+1}]$ to have

$$a_1 \le a_2 \le \cdots \le a_k \le a_{k+1} < b_{k+1} \le b_k \le \cdots < b_2 < b_1$$

& $b_{k+1} - a_{k+1} = \frac{1}{2^{k+1}}(b_1 - a_1)$



$\Rightarrow$ we obtain a seq of interval $[a_n, b_n]$ s.t

$$b_n - a_n = \frac{1}{2^n}(b_1 - a_1) \quad \text{—— (1)}$$

By using Nested Interval theorem

(1) $\Rightarrow 0$ as $n \to \infty$ &

$\Rightarrow$ unique pt 's' s.t

$$s = \bigcap [a_n, b_n] \text{ (again N.I.T)}$$

$\Rightarrow$ There are infinitely many terms of seq whose length $\epsilon > 0$ that contains 's'. For $\epsilon = 1$, there are infinitely many values of n s.t

$$|s_n - s| < 1$$

let n_1 be one of such value

$$|s_{n_1} - s| < 1$$

again chose $n_2 > n_1$ s.t

$|S_{n_2} - s| < 1/2$

continuing in This manner, we find a sequence $\{S_n\}$ for each +ve int 'k' s.t

$n_k < n_{k+1}$ &

$|S_{n_k} - s| < 1/k \quad \forall \; k = 1, 2, 3, \dots$

$\Rightarrow$ There $\exists$ a sub-seq which cgs to 's'.

Limit Inferior of the sequence:-

Suppose $\{S_n\}$ is bounded, Then

$\cancel{\lim_{n\to\infty} (\inf S_n) = \lim_{n\to\infty} U_k}$ ~~where~~

$\cancel{U_k =}$

$$\lim_{n\to\infty} \inf S_n = \lim_{n\to\infty} \left(\inf_{m \geq n} S_m \right)$$

or

$$\lim_{n\to\infty} \inf S_n = \sup_{n \geq 0} \inf_{m \geq n} S_n$$

$$\boxed{\lim_{n\to\infty} \inf S_n = \sup \{ \inf \{ S_m, S_{m+1}, S_{m+2}, \dots \} \}}$$

Limit Superior:-

$$\lim_{n\to\infty} \sup S_n = \inf_{n \geq 0} \sup_{m \geq n} S_n$$

not

If $\{S_n\}$ is not bounded below, then $\lim_{n\to\infty}(\inf S_n) = -\infty$

If $\{S_n\}$ is not bounded above, then $\lim_{n\to\infty}\{\sup S_n\} = +\infty$

points:-

i) A bounded seq has unique inf & sup Limit

ii) if $\{S_n\}$ contains all the rationals then every real No is sub-seq Limit then

$\lim_{n\to\infty}(\inf S_n) = -\infty$ & $\lim_{n\to\infty}\{\sup S_n\} = +\infty$

iii) $\{S_n\} = (-1)^n(1+\frac{1}{n})$

solution :-

$\lim_{n\to\infty}\inf\{S_n\}$:-

$\lim_{n\to\infty}\inf S_n = \sup\{\inf(S_m, S_{m+1}, \ldots)\}$

$\because$

$(-1)^n(1+\frac{1}{n}) = \begin{cases} (1+\frac{1}{n}) ; & \text{if } n \text{ is even} \\ -(1+\frac{1}{n}) ; & \text{if } n \text{ is odd} \end{cases}$

$= \sup\{\inf((1+\frac{1}{m}), (1+\frac{1}{m+1}) \ldots)\}$

$= \sup\{\inf(\sup\{-1\}$

$= -1$

$\lim_{n\to\infty}\inf S_n = -1$

$\lim\limits_{n\to\infty} \sup S_n$:-

$= \inf \{ \sup \{ (1+\frac{1}{m}), (1+\frac{1}{m+1}), \ldots \} \}$

$= \inf \{1\} = 1$

$\lim\limits_{n\to\infty} \sup S_n = 1$

$(1+\frac{1}{n})$:-

$(1+1) = 2$

$(1+\frac{1}{2}) = 3/2$

$(1+\frac{1}{3}) = 4/3$

$-1 \quad 0 \quad 1$

$S_n \leq 1 \to$

$S_n \geq -1 \to$

Assignment:-

1. Find the limits of following

i) $\lim \left(\left(2+\frac{1}{n}\right)^2 \right)$

ii) $\lim \left(\frac{(-1)^n}{n+2} \right)$, iii) $\lim \left(\frac{\sqrt{n}-1}{\sqrt{n}+1} \right)$

(By using 'ϵ' def)

2. Find convergence or divergence:-

(i) $T_n = \frac{2n^2+3}{n^2+1}$

(ii) $S_n = \frac{(-1)^n n}{n+1}$

(iii) $P_n = \frac{n^2}{n+1}$