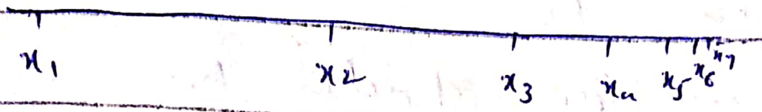
Cauchy sequence:

A sequence $\{x_n\}$ of real Numbers is said to be a Cauchy sequence if for given +ve real number ε, $\exists$ a +ve integer $n_0(\epsilon)$ s.t

$$|x_n - x_m| < \epsilon \quad \forall\ m, n > n_0$$



Theorem: A Cauchy seq of reals is bounded.

proof:-

Let $\{S_n\}$ be a Cauchy sequence.

Take $\epsilon = 1$, $\exists$ a +ve int n_0 s.t

$$|S_n - S_m| < 1 \quad \forall\ m, n > n_0$$

Let $m = n_0 + 1$, then

$$|S_n| = |S_n - S_m + S_m| = |S_n - S_{n_0+1} + S_{n_0+1}|$$

$$\leq |S_n - S_{n_0+1}| + |S_{n_0+1}|$$

$$< 1 + |S_{n_0+1}| \quad \forall\ n > n_0$$

$$< d$$

Here $1 + |S_{n_0+1}| = d$

$\Rightarrow |S_n| < d \Rightarrow \{S_n\}$ is bounded.

→ **Every bounded seq is not cgt.**

Consider the seq $\{S_n\}$, where

$S_n = (-1)^n, n \geq 1$

→ It's bounded seq; as

$|S_n| = |(-1)^n| = 1 < 2 \quad \forall n \geq 1$

But it's not a cauchy seq, as

if its then let

$\epsilon = 1$, Then we should be able

to find a +ve int n_0 s.t

$|S_n - S_m| < 1 \quad \forall m, n \geq n_0$

let $m = 2k+1$, $n = 2k+2$, when $2k+1 \geq n_0$

$\Rightarrow |S_n - S_m| = |(-1)^{2k+2} - (-1)^{2k+1}|$

$= |1+1| = 2 < \epsilon$

$= 2 < 1$

Not valid

→ Not cauchy.

Divergent Seq:

If seq is not cgt or its unbounded

i) $\{n^2\}$

ii) $\{(-1)^n\}$

iii) $\{(-1)^n n\}$

Theorem: If $S_n < U_n < T_n \ \forall \ n \geq n_0$ & if both the $\{S_n\}$ & $\{T_n\}$ cgs to same limit 's', then the seq $\{U_n\}$ also cgs to 's'.

Proof:-

$\because$ The seq $\{S_n\}$ & $\{T_n\}$ cgs to the same limit 's', therefore according to def of cgs, for $\epsilon > 0$, $\exists$ two tue int $n_1, n_2 > n_0$ s.t

$|S_n - s| < \epsilon \quad \forall n > n_1$ — (1)

$|t_n - s| < \epsilon \quad \forall n > n_2$ — (2)

(1) $\Rightarrow$ $s - \epsilon < S_n < s + \epsilon \quad \forall \ n > n_1$ — (3)

(2) $\Rightarrow$ $s - \epsilon < T_n < s + \epsilon \quad \forall \ n > n_2$ — (4)

according to given condition

$S_n < U_n < T_n \quad \forall \ n > n_0$

from (3) & (4), we conclude; as

$\Rightarrow s - \epsilon < S_n < U_n < T_n < s + \epsilon$

$\Rightarrow s - \epsilon < U_n < s + \epsilon \quad \forall \ n > \max(n_0, n_1, n_2)$

i.e $|U_n - s| < \epsilon \quad \forall \ n > \max(n_0, n_1, n_2)$

$\Rightarrow \lim\limits_{n \to \infty} U_n = s$

proved.

Scanned by CamScanner

$$\lim_{n\to\infty} n^{1/n} = 1$$

Sol:-

Using Bernoulli's Inequality

$$(1+p)^n \geq (1+np)$$

$$\left(1+\frac{1}{\sqrt{n}}\right)^n \geq \left(1+\frac{n}{\sqrt{n}}\right) \qquad \because p = \frac{1}{\sqrt{n}}$$

Also

$$\left(1+\frac{1}{\sqrt{n}}\right)^n \geq 1+\frac{n}{\sqrt{n}} \geq \sqrt{n} \geq 1 \quad \forall\ n$$

Also

$$\left(1+\frac{1}{\sqrt{n}}\right)^2 = \left[\left(1+\frac{1}{\sqrt{n}}\right)^n\right]^{2/n} \geq (\sqrt{n})^{2/n} \geq n^{1/n} \geq 1$$

$$\Rightarrow 1 \leq n^{1/n} < \left(1+\frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow \lim_{n\to\infty} 1 \leq \lim_{n\to\infty} n^{1/n} < \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^2$$

$$\Rightarrow 1 \leq \lim_{n\to\infty} n^{1/n} < 1$$

$$\lim_{n\to\infty} n^{1/n} = 1$$

solved.

$$\lim_{n\to\infty}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right) = 0$$

Sol:-

$$S_n = \left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right)$$

$$\because (2n)^2 > n^2$$

$$\frac{1}{2n^2} < \frac{1}{n^2} \Rightarrow \frac{n}{(2n)^2} < \frac{n}{n^2}$$

$$\Rightarrow \frac{n}{(2n)^2} < S_n < \frac{n}{n^2}$$

$$\Rightarrow \frac{1}{4n^2} < S_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{4n} < \lim_{n\to\infty} S_n < \lim_{n\to\infty} \frac{1}{n}$$

$$0 < \lim_{n\to\infty} S_n < 0$$

$$\Rightarrow \lim_{n\to\infty} S_n = 0$$

$$\Rightarrow \lim_{n\to\infty} \left\{ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2} \right\} = 0$$

✓ **Theorem:** If $\{S_n\}$ cgs to 's' $\exists$ +ve int n s.t $|S_n| > \frac{1}{2}s$ H.W

Theorem

Let a and b be fixed real numbers if $\{S_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(*i*) $\{aS_n + bt_n\}$ converges to $as + bt$.

(*ii*) $\{S_n t_n\}$ converges to st.

(*iii*) $\left\{\dfrac{S_n}{t_n}\right\}$ converges to $\dfrac{s}{t}$, provided $t_n \neq 0 \ \forall \ n$ and $t \neq 0$.

Proof

Since $\{S_n\}$ and $\{t_n\}$ converge to s and t respectively,

$$\therefore \ |S_n - s| < e \qquad \forall \ n > n_1 \in \mathbb{N}$$

$$|t_n - t| < e \qquad \forall \ n > n_2 \in \mathbb{N}$$

Also $\exists \ \ell > 0$ such that $|S_n| < \ell \quad \forall \ n > 1 \qquad (\because \{S_n\}$ is bounded $)$

(*i*) We have

$$\begin{aligned} |(aS_n + bt_n) - (as + bt)| &= |a(S_n - s) + b(t_n - t)| \\ &\leq |a(S_n - s)| + |b(t_n - t)| \\ &< |a|e + |b|e \qquad \forall \ n > \max(n_1, n_2) \\ &= e_1 \qquad \text{Where } e_1 = |a|e + |b|e \text{ a certain} \end{aligned}$$

This implies $\{aS_n + bt_n\}$ converges to $as + bt$.

(*ii*)

$$\begin{aligned} |S_n t_n - st| &= |S_n t_n - S_n t + S_n t - st| \\ &= |S_n(t_n - t) + t(S_n - s)| \ \leq |S_n| \cdot |(t_n - t)| + |t| \cdot |(S_n - s)| \\ &< \ell e + |t|e \qquad \forall \ n > \max(n_1, n_2) \\ &= e_2 \qquad \text{where } e_2 = \ell e + |t|e \text{ a certain number.} \end{aligned}$$

This implies $\{S_n t_n\}$ converges to st.

(*iii*)

$$\begin{aligned} \left|\frac{1}{t_n} - \frac{1}{t}\right| &= \left|\frac{t - t_n}{t_n t}\right| \\ &= \frac{|t_n - t|}{|t_n||t|} < \frac{e}{\frac{1}{2}|t||t|} \qquad \forall \ n > \max(n_1, n_2) \qquad \because |t_n| > \\ &= \frac{e}{\frac{1}{2}|t|^2} = e_3 \qquad \text{where } e_3 = \frac{e}{\frac{1}{2}|t|^2} \text{ a certain numbe} \end{aligned}$$

This implies $\left\{\dfrac{1}{t_n}\right\}$ converges to $\dfrac{1}{t}$.

Hence $\left\{\dfrac{S_n}{t_n}\right\} = \left\{S_n \cdot \dfrac{1}{t_n}\right\}$ converges to $s \cdot \dfrac{1}{t} = \dfrac{s}{t}$. (from (ii))

Theorem

For each irrational number x, there exists a sequence $\{r_n\}$ of distinct ratio

numbers such that $\lim r_n = x$.