

55. A function f is called **homogeneous of degree n** if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t , where n is a positive integer and f has continuous second-order partial derivatives.

- (a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.
 (b) Show that if f is homogeneous of degree n , then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$$

[Hint: Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t .]

56. If f is homogeneous of degree n , show that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$$

57. If f is homogeneous of degree n , show that

$$f_x(tx, ty) = t^{n-1} f_x(x, y)$$

58. Suppose that the equation $F(x, y, z) = 0$ implicitly defines the three variables $x, y,$ and z as functions of the others: $z = f(x, y), y = g(x, z), x = h(y, z)$. If F is differentiable, $F_x, F_y,$ and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$$

59. Equation 6 is a formula for the derivative dy/dx of a function defined implicitly by an equation $F(x, y) = 0$, provided F is differentiable and $F_y \neq 0$. Prove that if F has continuous second-order partial derivatives, then a formula for the second derivative

$$\frac{d^2 y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_xF_yF_{xy} + F_x^2F_{yy}}{F_y^3}$$

14.6 Directional Derivatives and the Gradient Vector

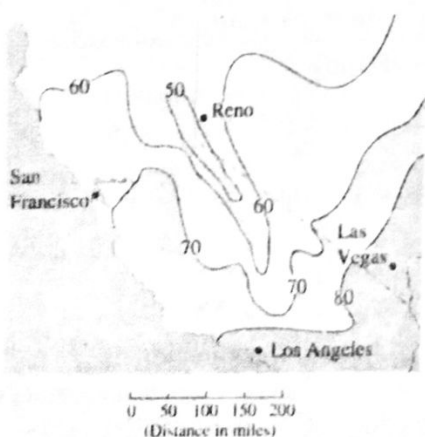


FIGURE 1

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ in the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isotherms, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

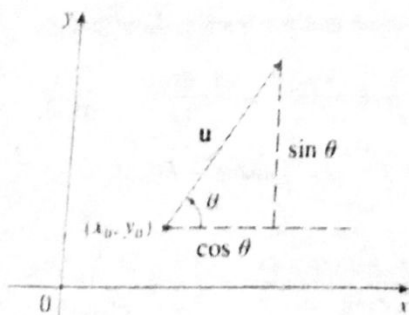


FIGURE 2

A unit vector

$$\mathbf{u} = \langle a, b \rangle = \langle \cos \theta, \sin \theta \rangle$$

and represent the rates of change of z in the x - and y -directions, that is, in the directions of the unit vectors \mathbf{i} and \mathbf{j} .

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S . The vertical plane that passes through P in the direction of

sects S in a curve C . (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of \mathbf{u} .

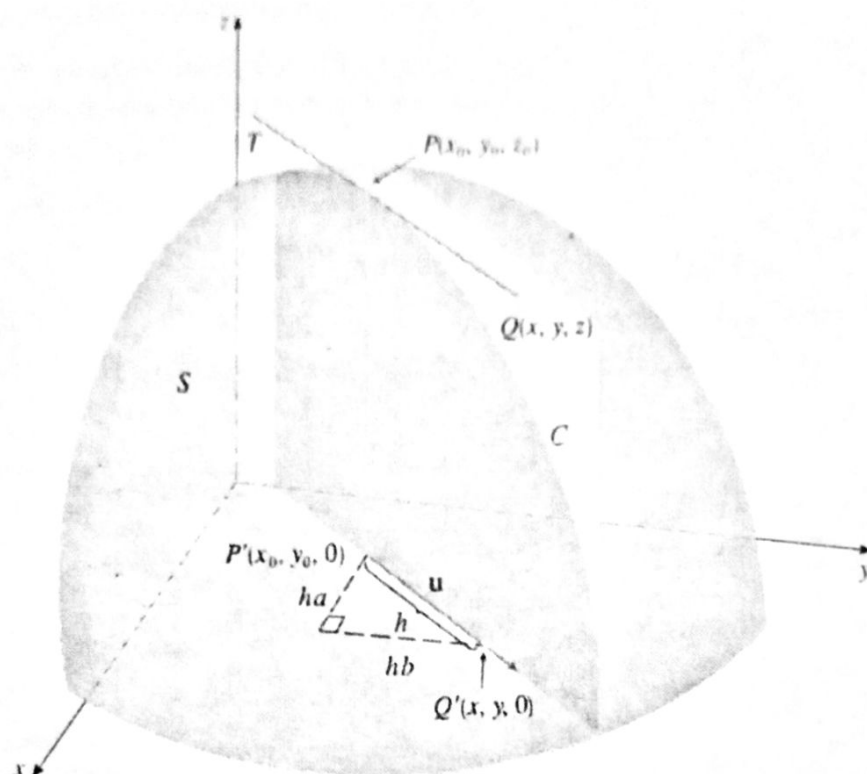


FIGURE 3

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy -plane, then the vector $\overrightarrow{P'Q'}$ is parallel to \mathbf{u} and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h . Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \rightarrow 0$, we obtain the rate of change of z (with respect to distance) in the direction of \mathbf{u} , which is called the directional derivative of f in the direction of \mathbf{u} .

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \quad g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

If the unit vector \mathbf{u} makes an angle θ with the positive x -axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \quad D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and \mathbf{u} is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\ &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2} \\ &= \frac{1}{2} [3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y] \end{aligned}$$

Therefore

$$D_{\mathbf{u}}f(1, 2) = \frac{1}{2} [3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2}$$

■ The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} \boxed{7} \quad D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation ($\text{grad } f$ or ∇f , which is read "del f ").

directional derivative $D_{\mathbf{u}}f(1, 2)$
Example 2 represents the rate of
change of f in the direction of \mathbf{u} . This
slope of the tangent line to the
graph of the surface
 $f(x, y) = x^3 - 3xy + 4y^2$ and the vertical
line through $(1, 2, 0)$ in the direction
shown in Figure 5.

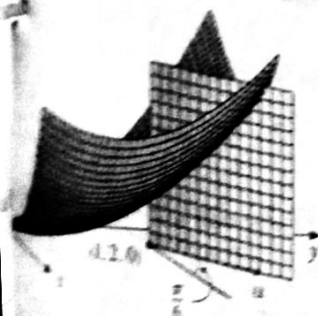


FIGURE 5

8 Definition If f is a function of two variables x and y , then the gradient is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and

$$\nabla f(0, 1) = \langle 2, 0 \rangle$$

With this notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of a unit vector \mathbf{u} as the projection of the gradient vector onto \mathbf{u} .

EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3$ at point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that \mathbf{v} is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of \mathbf{v} is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation 9, we have

$$\begin{aligned} D_{\mathbf{u}}f(2, -1) &= \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j} \right) \\ &= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}} \end{aligned}$$

The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 6 with initial point $(2, -1)$. Also shown is the vector \mathbf{v} that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f .

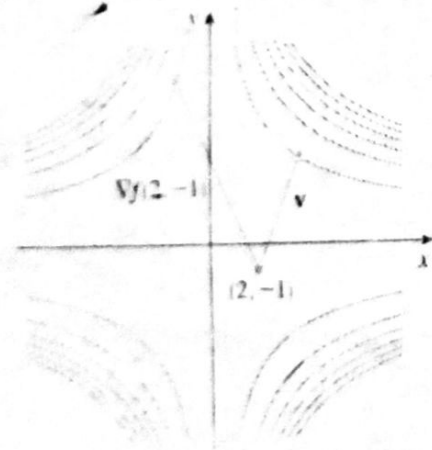


FIGURE 6

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = (0, 0, 3)$. The unit vector in the direction $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}}f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 3 \left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider the directional derivatives of f at a given point. These give the rates of change in various possible directions. We can then ask the questions: in which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when \mathbf{u} has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Visual 14.6B provides visual confirmation of Theorem 15.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

where θ is the angle between ∇f and \mathbf{u} . The maximum value of $\cos \theta$ is 1 and it occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when \mathbf{u} has the same direction as ∇f .

EXAMPLE 6

- (a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
 (b) In what direction does f have the maximum rate of change? What is the maximum rate of change?

SOLUTION

- (a) We first compute the gradient vector:

$$\nabla f(x, y) = (f_x, f_y) = (e^y, xe^y)$$

$$\nabla f(2, 0) = (1, 2)$$

The unit vector in the direction of $\vec{PQ} = \langle -\frac{1}{2}, 2 \rangle$ is $\mathbf{u} = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}}f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} = (1, 2) \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \\ &= 1\left(-\frac{1}{\sqrt{5}}\right) + 2\left(\frac{2}{\sqrt{5}}\right) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = (1, 2)$. The maximum rate of change is

$$|\nabla f(2, 0)| = |(1, 2)| = \sqrt{5}$$

FIGURE 7 is the function in Example 6. The level curves in the direction of the gradient vector $\nabla f(2, 0) = (1, 2)$. Note from Figure 7 that this vector is perpendicular to the level curve through $(2, 0)$. Figure 8 shows the surface $f(x, y)$ and the gradient vector.

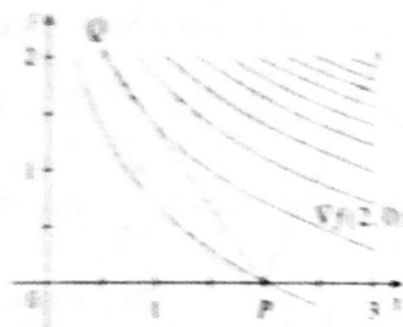


FIGURE 7

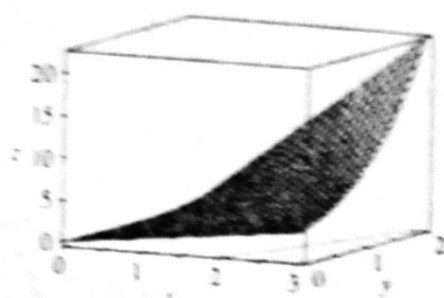


FIGURE 8

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2} (-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k}) \end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \frac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \left| \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) \right| = \frac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4^\circ\text{C/m}$.