

67–70 Describe the level surfaces of the function.

67. $f(x, y, z) = z + 3y + 5x$

68. $f(x, y, z) = z^2 + 3y^2 + 5x^2$

69. $f(x, y, z) = y^2 + z^2$

70. $f(x, y, z) = x^2 - y^2 - z^2$

71–72 Describe how the graph of g is obtained from the graph of f .

71. (a) $g(x, y) = f(x, y) + 2$

(b) $g(x, y) = 2f(x, y)$

(c) $g(x, y) = -f(x, y)$

(d) $g(x, y) = 2 - f(x, y)$

72. (a) $g(x, y) = f(x - 2, y)$

(b) $g(x, y) = f(x, y + 2)$

(c) $g(x, y) = f(x + 3, y - 4)$

73–74 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the “peaks and valleys.” Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be “local maximum points”? What about “local minimum points”?

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

74. $f(x, y) = xye^{-x^2-y^2}$

75–76 Graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

75. $f(x, y) = \frac{x + y}{x^2 + y^2}$

76. $f(x, y) = \frac{xy}{x^2 + y^2}$

77. Investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c ?

78. Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b ?

79. Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

80. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2 + y^2}}$$

$$f(x, y) = \ln \sqrt{x^2 + y^2}$$

$$f(x, y) = \sin(\sqrt{x^2 + y^2})$$

and
$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g(\sqrt{x^2 + y^2})$$

obtained from the graph of g ?

81. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^\alpha K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 3) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

4.2 Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 Values of $f(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1, whereas the values of $g(x, y)$ aren't approaching any number. It turns out that our guesses based on numerical evidence are correct, and we write

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist}$$

In general, we use the notation

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f . In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) close to the point (a, b) , but not equal to (a, b) . A more precise definition for

1 Definition Let f be a function of two variables whose domain D contains points arbitrarily close to (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = L \quad \text{and} \quad f(x, y) \rightarrow L \text{ as } (x, y) \rightarrow (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L , and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b) . Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by taking (x, y) close enough to (a, b) .

making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \epsilon, L + \epsilon)$ is given around L , then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \epsilon, L + \epsilon)$.

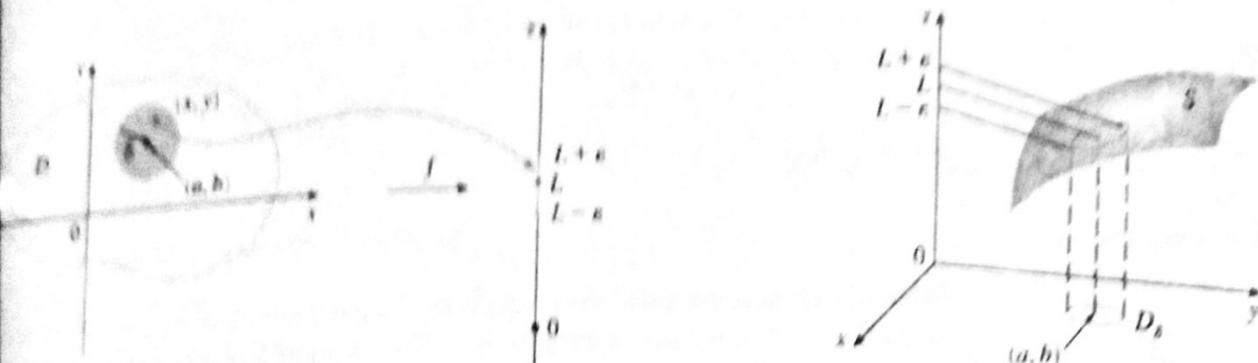


FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f . If $\epsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \epsilon$ and $z = L + \epsilon$.

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right. We recall from Chapter 2 that if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f .

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b) . It does not refer to the *direction* of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b) . Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x -axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \rightarrow 1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y -axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \rightarrow -1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits along two different lines, the given limit

FIGURE 1

FIGURE 3

does not exist. (This confirms the conjecture we made on the basis of evidence at the beginning of this section.)

EXAMPLE 2 If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not mean the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$.

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along $y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the limit does not exist.

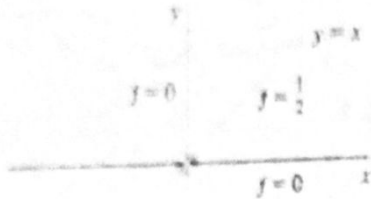


FIGURE 5

Figure 6 sheds some light on Example 2. The ridge that occurs above the origin corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line.

TEC In Visual 14.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

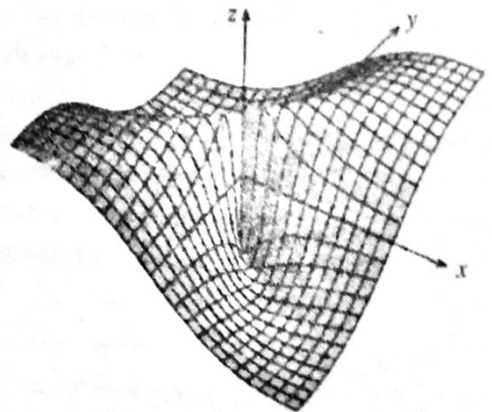


FIGURE 6

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

EXAMPLE 3 If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by approaching $(0, 0)$ along any line through the origin. If the line is not the y -axis, $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2}{1 + m^4x^2}$$

So

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along } y = mx$$

We get the same result as $(x, y) \rightarrow (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show

limit is 0, for if we now let $(x, y) \rightarrow (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist. ■

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\boxed{2} \quad \lim_{(x, y) \rightarrow (a, b)} x = a \quad \lim_{(x, y) \rightarrow (a, b)} y = b \quad \lim_{(x, y) \rightarrow (a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{that is, if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $x^2/(x^2 + y^2) \leq 1$ and therefore

$$\boxed{3} \quad \frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 \quad \blacksquare$$

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous

function is $\lim_{x \rightarrow a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D .

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a function of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3 y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = x^2 + y^2$ are continuous. Since any polynomial can be built up out of the simple functions f , g , and h by multiplication and addition, it follows that *all polynomials are continuous*. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

EXAMPLE 5 Evaluate $\lim_{(x, y) \rightarrow (1, 2)} (x^2 y^3 - x^3 y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2 y^3 - x^3 y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y) \rightarrow (1, 2)} (x^2 y^3 - x^3 y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 8 - 4 + 3 + 4 = 11$$

EXAMPLE 6 Where is the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$.

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y) \rightarrow (0, 0)} g(x, y)$ does not exist (see Example 1). ■

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on \mathbb{R}^2 . ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f , then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y -axis. ■

■ Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f . Because the distance between two points (x, y, z) and (a, b, c) in \mathbb{R}^3 is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, we can write the precise definition as follows: for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then } |f(x, y, z) - L| < \epsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = f(a, b, c)$$

Figure 8 shows the graph of the continuous function in Example 8.

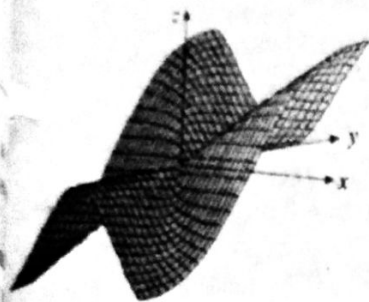


FIGURE 8

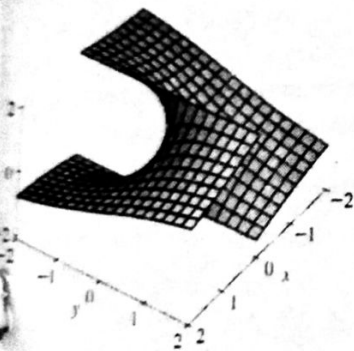


FIGURE 9

The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then the definitions of a limit for functions of two or three variables in a single case as follows.

5 If f is defined on a subset D of \mathbb{R}^n , then $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ then } |f(\mathbf{x}) - L| < \epsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$ and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (5) becomes Definition 1. If $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 EXERCISES

- Suppose that $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?
- Explain why each function is continuous or discontinuous.
 - The outdoor temperature as a function of longitude, latitude, and time
 - Elevation (height above sea level) as a function of longitude, latitude, and time
 - The cost of a taxi ride as a function of distance traveled and time

Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$. Then explain why your guess is correct.

3. $f(x, y) = \frac{x^2y^2 + x^2y^2 - 5}{2 - xy}$ 4. $f(x, y) = \frac{2xy}{x^2 + 2y^2}$

5-7 Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x,y) \rightarrow (0,0)} (x^2y^2 - 4y^2)$ 6. $\lim_{(x,y) \rightarrow (0,-1)} \frac{x^2y + xy^2}{x^2 - y^2}$

7. $\lim_{(x,y) \rightarrow (\pi, \pi/2)} y \sin(x - y)$

8. $\lim_{(x,y) \rightarrow (3,2)} e^{xy}$

9. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

10. $\lim_{(x,y) \rightarrow (0,0)} \frac{5}{x^2}$

11. $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$

12. $\lim_{(x,y) \rightarrow (1,0)} \frac{1}{x}$

13. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

14. $\lim_{(x,y) \rightarrow (0,0)} \frac{1}{x^2}$

15. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2 \cos y}{x^2 + y^4}$

16. $\lim_{(x,y) \rightarrow (0,0)} \frac{1}{x^2}$

17. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2 + y^8}$

19. $\lim_{(x,y,z) \rightarrow (0,0,1/3)} e^{yz} \tan(xz)$

20. $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{1}{x^2}$

$$21. \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2}$$

$$22. \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 y^2 z^2}{x^2 + y^2 + z^2}$$

23–24 Use a computer graph of the function to explain why the limit does not exist.

$$23. \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2} \quad 24. \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6}$$

25–26 Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

$$25. g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$$

$$26. g(t) = t + \ln t, \quad f(x, y) = \frac{1 - xy}{1 + x^2 y^2}$$

27 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

$$27. f(x, y) = e^{1/(x-y)} \quad 28. f(x, y) = \frac{1}{1 - x^2 - y^2}$$

29 Determine the set of points at which the function is continuous.

$$29. F(x, y) = \frac{xy}{1 + e^{x-y}} \quad 30. F(x, y) = \cos \sqrt{1 + x - y}$$

$$31. G(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2} \quad 32. H(x, y) = \frac{e^x + e^y}{e^{xy} - 1}$$

$$33. G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$$

$$34. G(x, y) = \ln(1 + x - y)$$

$$35. f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$$

$$36. f(x, y, z) = \sqrt{y - x^2} \ln z$$

$$37. f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$38. f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

39–41 Use polar coordinates to find the limit. (If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \rightarrow 0^+$ as $(x, y) \rightarrow (0, 0)$.)

$$39. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 + y^2}$$

$$40. \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2)$$

$$41. \lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$$

42. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed on the basis of numerical evidence that $f(x, y) \rightarrow 1$ as $(x, y) \rightarrow (0, 0)$. Use polar coordinates to confirm the value of the limit. Then graph the function.

43. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \frac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^2 \\ 1 & \text{if } 0 < y < x^2 \end{cases}$$

- (a) Show that $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^2$ with $0 < m < 4$.
 (b) Despite part (a), show that f is discontinuous at $(0, 0)$.
 (c) Show that f is discontinuous on two entire curves.

45. Show that the function f given by $f(\mathbf{x}) = \|\mathbf{x}\|$ is continuous on \mathbb{R}^n . [Hint: Consider $\|\mathbf{x} - \mathbf{a}\|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

46. If $c \in V_n$, show that the function f given by $f(\mathbf{x}) = c \cdot \mathbf{x}$ is continuous on \mathbb{R}^n .

3 Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the *temperature-humidity index*, or *humidex*, in some countries) to describe the combined

By taking $h = 5$ and -5 , we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70% , the heat index rises about 0.9°F for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y , suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x , namely, $g(x) = f(x, b)$. If g has a derivative at a , then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

2

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)** , denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96^\circ\text{F}$ and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \quad f_H(96, 70) \approx 0.9$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 If f is a function of two variables, its partial derivatives are the functions f_x and f_y , defined by

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_x or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$. But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_x = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_y = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 14.1 that the partial derivative with respect to x is just the *ordinary* derivative of the function of a single variable that we get by keeping y fixed. Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x , we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y , we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

■ Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S . By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S . (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2 . Both of the curves C_1 and C_2 pass through the point P . (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z / \partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z / \partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

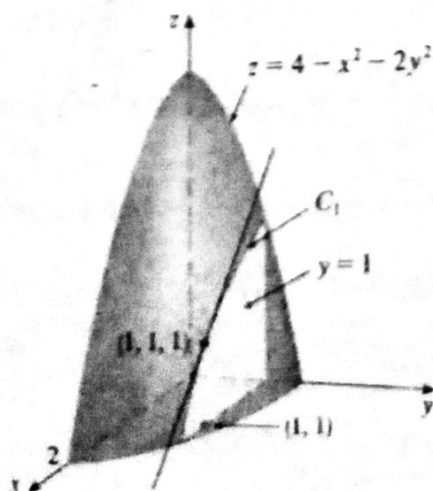


FIGURE 2

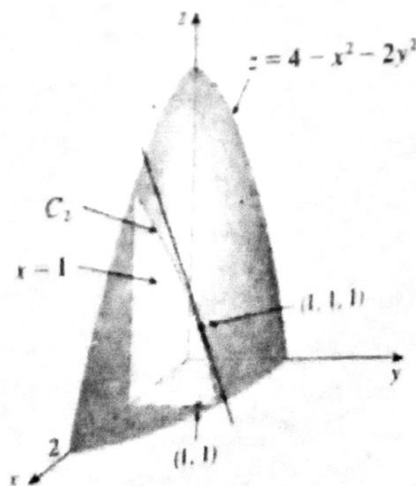


FIGURE 3

Now we regard m as a constant. The partial derivative with respect to h is

$$\frac{\partial B}{\partial h}(m, h) = \frac{\partial}{\partial h} \left(\frac{m}{h^2} \right) = m \left(-\frac{2}{h^3} \right) = -\frac{2m}{h^3}$$

so

$$\frac{\partial B}{\partial h}(64, 1.68) = -\frac{2 \cdot 64}{(1.68)^3} \approx -27 \text{ (kg/m}^2\text{)/m}$$

This is the rate at which the man's BMI increases with respect to his height when he weighs 64 kg and his height is 1.68 m. So if the man is still growing and his weight stays unchanged while his height increases by a small amount, say 1 cm, then his BMI will decrease by about $27(0.01) = 0.27$. ■

EXAMPLE 4 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x} \left(\frac{x}{1+y} \right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y} \left(\frac{x}{1+y} \right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

EXAMPLE 5 Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x , being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

■ Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x , y , and z , then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x . If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with

computer software can plot
defined by implicit equations
variables. Figure 6 shows such
the surface defined by the
Example 5.

$$x \cdot 6yz = 6yz + 6xz$$

respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \dots, x_n)$, its partial derivative with respect to the i th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 6 Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x ,

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f . If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f / \partial y \partial x$) means that we first differentiate with respect to x and then with respect to y , whereas in computing f_{yx} the order is reversed.

EXAMPLE 7 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \quad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x} (3x^2 + 2xy^3) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y} (3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x} (3x^2y^2 - 4y) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y} (3x^2y^2 - 4y) = 6x^2y - 4$$