12.1 Three-Dimensional Coordinate Systems

3D Space

To locate a point in a plane, we need two numbers. We know that any point in the can be represented as an ordered pair (a, b) of real numbers, where a is the x-coo and b is the y-coordinate. For this reason, a plane is called two-dimensional. To l point in space, three numbers are required. We represent any point in space by an o triple (a, b, c) of real numbers.

In order to represent points in space, we first choose a fixed point O (the origi three directed lines through O that are perpendicular to each other, called the **coord axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-a being horizontal and the z-axis as being vertical, and we draw the orientation of the as in Figure 1. The direction of the z-axis is determined by the **right-hand rule** trated in Figure 2: If you curl the fingers of your right hand around the z-axis in the tion of a 90° counterclockwise rotation from the positive x-axis to the positive then your thumb points in the positive direction of the z-axis.

FIGURE 1
Coordinate axes

FIGURE 2
Right-hand rule

The three coordinate axes determine the three **coordinate planes** illustrated i ure 3(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane co the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate divide space into eight parts, called **octants**. The **first octant**, in the foreground, is mined by the positive axes.

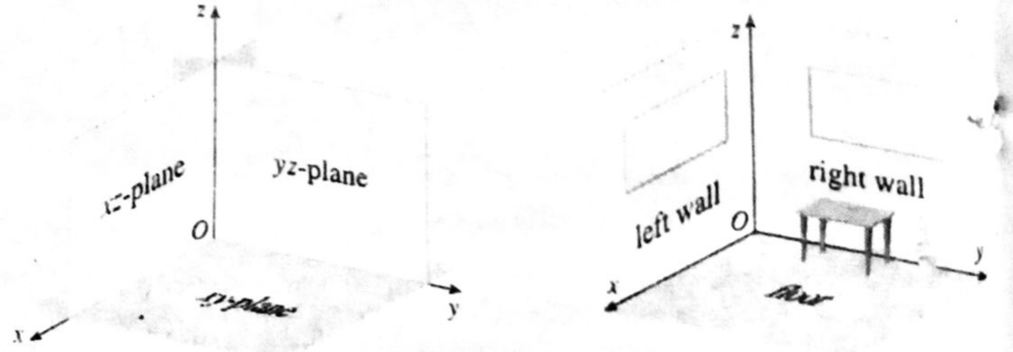


FIGURE 3 (a) Coordinate planes (b)

Because many people have some difficulty visualizing diagrams of three-dime figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any corner of a room and call the corner the origin. The wall on your left is in the x the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-a along the intersection of the floor and the left wall. The y-axis runs along the inte of the floor and the right wall. The z-axis runs up from the floor toward the ceili the intersection of the two walls. You are situated in the first octant, and you imagine seven other rooms situated in the other seven octants (three on the sa and four on the floor below), all connected by the common corner point O.

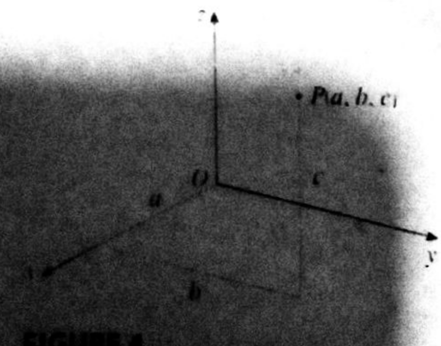


FIGURE 4

Now if P is any point in space, let a be the (directed) distance from the yz-pl let b be the distance from the xz-plane to P, and let c be the distance from the xy- P. We represent the point P by the ordered triple (a, b, c) of real numbers and a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and z-coordinate. Thus, to locate the point (a, b, c), we can start at the origin O an a units along the x-axis, then b units parallel to the y-axis, and then c units parall z-axis as in Figure 4.

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.

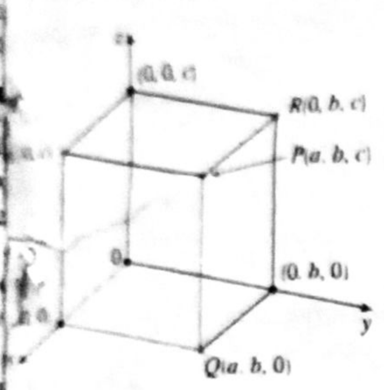


FIGURE 5

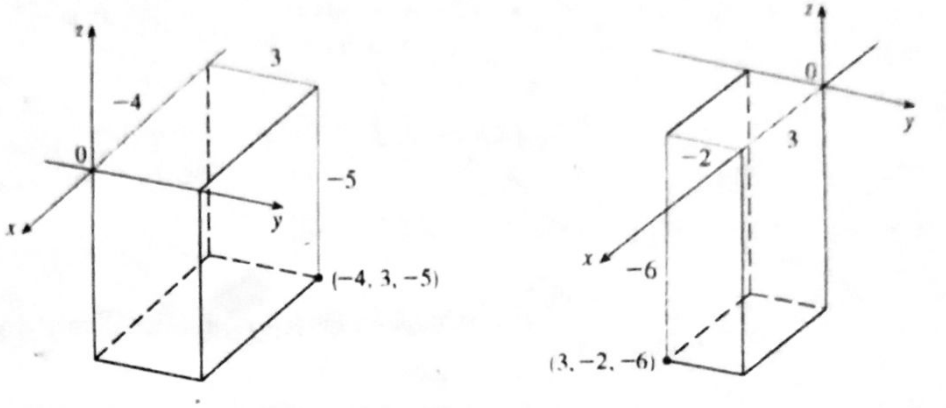


FIGURE 6

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Surfaces

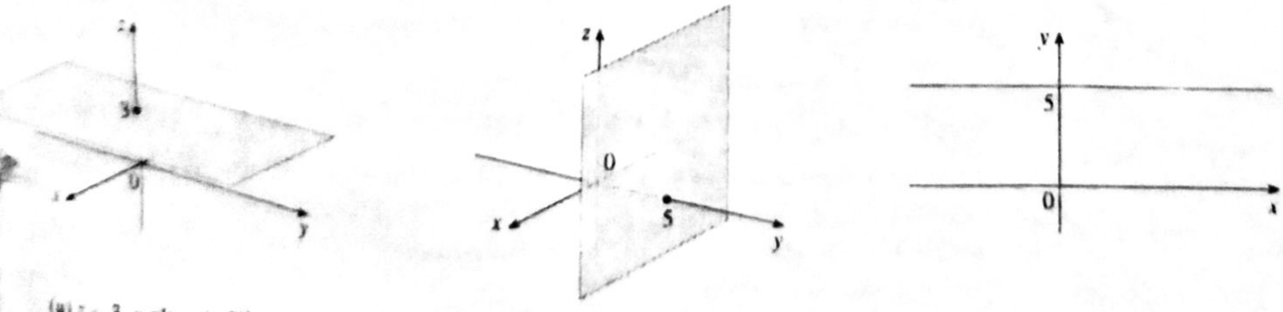
In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

EXAMPLE 1 What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ (b) $y = 5$

SOLUTION

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3 (x and y can each be any value). This is the horizontal plane that is parallel to the xy-plane and three units above it as in Figure 7(a).



(a) $z = 3$, a plane in $\mathbb{R}^3$ (b) $y = 5$, a plane in $\mathbb{R}^3$ (c) $y = 5$, a line in $\mathbb{R}^2$

FIGURE 7

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b). ■