DETERMINANTS

Instructor Memoona Nawaz

Contents

Introduction to Determinants

Determinant

Every square matrix has associated with it a scalar called its determinant.

Given a matrix **A**, we use **det(A)** or **|A|** to designate its determinant.

We can also designate the determinant of matrix **A** by replacing the brackets by vertical straight lines. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

Definition 1: The determinant of a 1×1 matrix [a] is the scalar a.

Definition 2: The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the scalar ad-bc.

For higher order matrices, we will use a recursive procedure to compute determinants.

Example

Evaluate the determinant : $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$

Solution : $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = 4 \times 5 - 2 \times (-3) = 20 + 6 = 26$

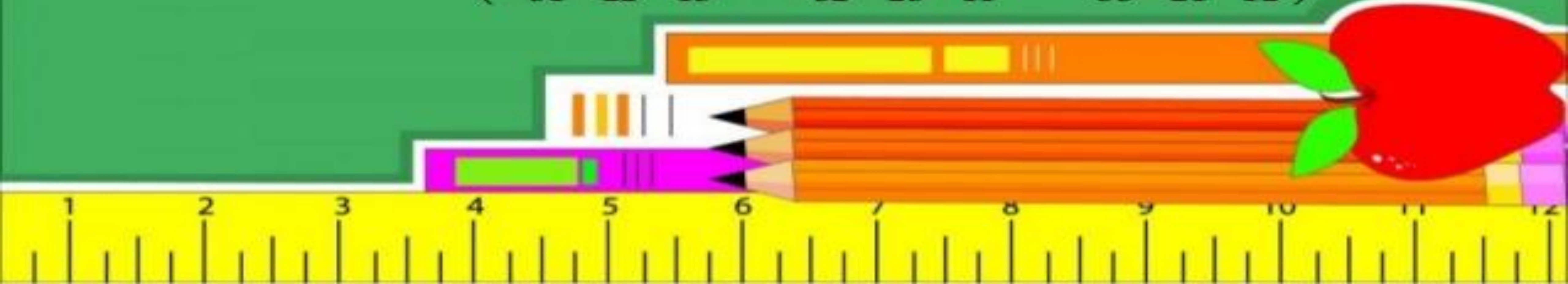
Solution

The **determinant of a 3 × 3 matrix *A*,** where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is a real number defined as

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}).$$



Solution

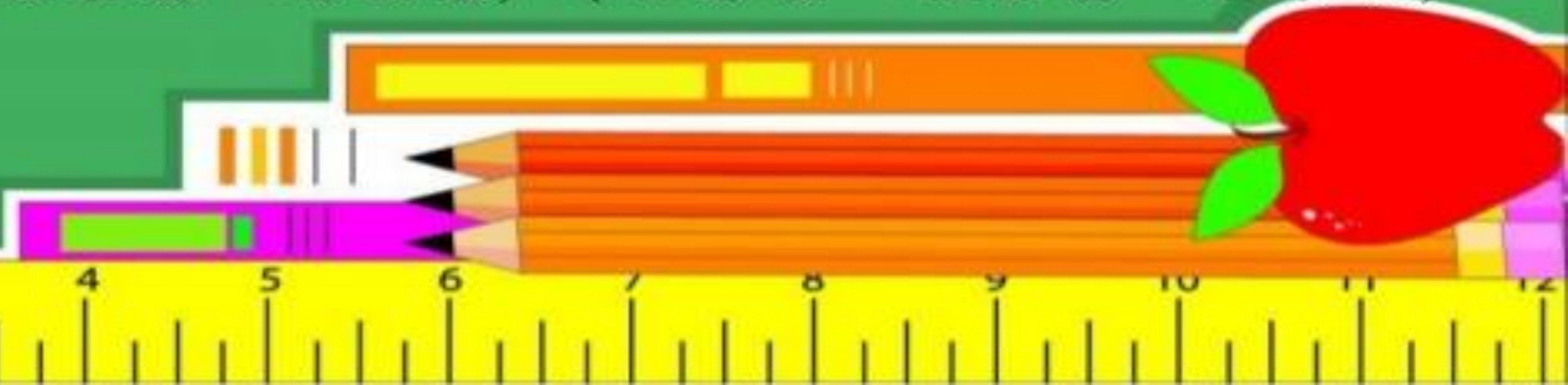
If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

[Expanding along first row]

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32}) - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22})$$



Example

Evaluate the determinant : $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$

Solution :

$$\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

[Expanding along first row]

$$= 2(1+8) - 3(7-6) - 5(28+3)$$
$$= 18 - 3 - 155$$
$$= -140$$

Properties of Determinants

Properties of Determinants

1. The value of a determinant remains unchanged, if its rows and columns are interchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{i.e. } |A| = |A'|$$

2. If any two rows (or columns) of a determinant are interchanged, then the value of the determinant is changed by minus sign.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\text{Applying } R_2 \leftrightarrow R_1]$$

Properties

3. If all the elements of a row (or column) is multiplied by a non-zero number k, then the value of the new determinant is k times the value of the original determinant.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which also implies

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{m} \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Properties

4. If each element of any row (or column) consists of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

$$\begin{vmatrix} a_1+x & b_1 & c_1 \\ a_2+y & b_2 & c_2 \\ a_3+z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$$

5. The value of a determinant is unchanged, if any row (or column) is multiplied by a number and then added to any other row (or column).

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1-nc_1 & b_1 & c_1 \\ a_2+mb_2-nc_2 & b_2 & c_2 \\ a_3+mb_3-nc_3 & b_3 & c_3 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + mC_2 - nC_3]$$

Properties

6. If any two rows (or columns) of a determinant are identical, then its value is zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

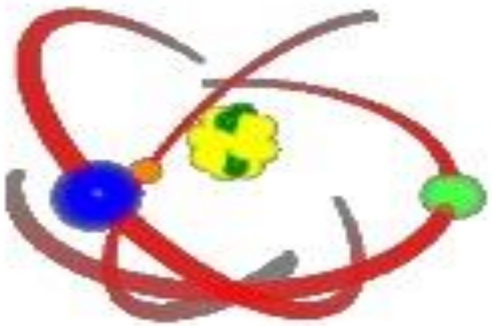
7. If each element of a row (or column) of a determinant is zero, then its value is zero.

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Properties

(8) Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ be a diagonal matrix, then

$$|A| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$



Minors and Cofactors

The Minor of an Element

- The determinant of each 3 × 3 matrix is called a **minor** of the associated element.
- The symbol M_{ij} represents the minor when the *i*th row and *j*th column are eliminated.

Element	Minor	Element	Minor
a_{11}	$M_{11} = \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$	a_{22}	$M_{22} = \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$
a_{21}	$M_{21} = \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$	a_{23}	$M_{23} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$
a_{31}	$M_{31} = \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$	a_{33}	$M_{33} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The Cofactor of an Element

Let M_{ij} be the minor for element a_{ij} in an $n \times n$ matrix. The **cofactor** of a_{ij}, written A_{ij}, is

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}.$$

- To find the determinant of a 3 × 3 or larger square matrix:
  1. Choose any row or column,
  2. Multiply the minor of each element in that row or column by a +1 or −1, depending on whether the sum of $i + j$ is even or odd,
  3. Then, multiply each cofactor by its corresponding element in the matrix and find the sum of these products. This sum is the determinant of the matrix.

Finding the Determinant

Example Evaluate $\det\begin{bmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{bmatrix}$, expanding by the second column.

Solution First find the minors of each element in the second column.

$$M_{12} = \det\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} = -1(2) - (-1)(-3) = -5$$

$$M_{22} = \det\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} = 2(2) - (-1)(-2) = 2$$

$$M_{32} = \det\begin{bmatrix} 2 & -2 \\ -1 & -3 \end{bmatrix} = 2(-3) - (-1)(-2) = -8$$

Finding the Determinant

Now, find the cofactor.

$$A_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot (-5) = 5$$
$$A_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \cdot (2) = 2$$
$$A_{32} = (-1)^{3+2} \cdot M_{32} = (-1)^5 \cdot (-8) = 8$$

The determinant is found by multiplying each cofactor by its corresponding element in the matrix and finding the sum of these products.

$$\det\begin{bmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{bmatrix} = a_{12} \cdot A_{12} + a_{22} \cdot A_{22} + a_{32} \cdot A_{32}$$
$$= -3(5) + (-4)(2) + (0)(8)$$
$$= -23$$

VALUE OF DETERMINANT IN TERMS OF MINORS AND COFACTORS

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A| = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{3} a_{ij} C_{ij}$$

$$= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}, \text{ for } i = 1 \text{ or } i = 2 \text{ or } i = 3$$

ROW (COLUMN) OPERATIONS

Following are the notations to evaluate a determinant:

(i) R_i to denote ith row

(ii) $R_i \leftrightarrow R_j$ to denote the interchange of ith and jth rows.

(iii) $R_i \leftrightarrow R_i + \lambda R_j$ to denote the addition of λ times the elements of jth row to the corresponding elements of ith row.

(iv) λR_i to denote the multiplication of all elements of ith row by λ.

Similar notations can be used to denote column operations by replacing R with C.

EVALUATION OF DETERMINANTS

If a determinant becomes zero on putting $x = \alpha$, then $(x - \alpha)$ is the factor of the determinant.

For example, if $\Delta = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$, then at $x = 2$

$\Delta = 0$ because C_1 and C_2 are identical at $x = 2$

Hence, $(x - 2)$ is a factor of determinant Δ.

SIGN SYSTEM FOR EXPANSION OF DETERMINANT

Sign System for order 2 and order 3 are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

EXAMPLE - 1

Find the value of the following determinants

(i) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 6\times 7 & 1 & 6 \\ 4\times 7 & 7 & 4 \\ 2\times 7 & 3 & 2 \end{vmatrix}$

$= 7\begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}$ [Taking out 7 common from C_1]

$= 7 \times 0$ $[\because C_1$ and C_3 are identical$]$

$= 0$

EXAMPLE – 1 (II)

(ii) $\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

$$= \begin{vmatrix} -3\times(-2) & -3 & 2 \\ -1\times(-2) & -1 & 2 \\ 5\times(-2) & 5 & 2 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$$ [Taking out -2 common from C_1]

$= (-2)\times 0$ [$\because$ C_1 and C_2 are identical]

$= 0$

EXAMPLE - 2

Evaluate the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Solution :

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \qquad [\text{Applying } c_3 \to c_2 + c_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \qquad [\text{Taking } (a+b+c) \text{ common from } C_3]$$

$$= (a+b+c) \times 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= 0$$

EXAMPLE - 3

Evaluate the determinant: $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution:

We have $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

$$= \begin{vmatrix} (a-b) & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$$

$$= (a-b)(b-c) \begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix} \quad \left[\begin{matrix}\text{Taking } (a-b) \text{ and } (b-c) \text{ common} \\ \text{from } C_1 \text{ and } C_2 \text{ respectively}\end{matrix}\right]$$

SOLUTION CONT.

$$=(a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ -(c-a) & b+c & c^2 \\ -(c-a) & -a & ab \end{vmatrix} \quad [\text{Applying } c_1 \to c_1 - c_2]$$

$$=-(a-b)(b-c)(c-a)\begin{vmatrix} 0 & 1 & c \\ 1 & b+c & c^2 \\ 1 & -a & ab \end{vmatrix}$$

$$=-(a-b)(b-c)(c-a)\begin{vmatrix} 0 & 1 & c \\ 0 & a+b+c & c^2-ab \\ 1 & -a & ab \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_3]$$

Now expanding along C_1 , we get

(a-b) (b-c) (c-a) [- (c^2 – ab – ac – bc – c^2)]

= (a-b) (b-c) (c-a) (ab + bc + ac)

EXAMPLE - 4

Without expanding the determinant,

prove that $\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3$

Solution :

$$\text{L.H.S} = \begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = \begin{vmatrix} 3x & 2x & x \\ 4x & 3x & 3x \\ 5x & 4x & 6x \end{vmatrix} + \begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y \begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ 6 & 4 & 6 \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y \times 0 \quad [\because C_1 \text{ and } C_2 \text{ are identical in II determinant}]$$

SOLUTION CONT.

$$= x^3 \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2]$$

$$= x^3 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2]$$

$$= x^3 \times (3-2) \qquad [\text{Expanding along } C_1]$$

$$= x^3 = \text{R.H.S.}$$

EXAMPLE - 5

Prove that : $\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^5 & 1 \end{vmatrix} = 0$, where ω is cube root of unity.

Solution :

$$\text{L.H.S} = \begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega^3 & \omega^3.\omega^2 \\ \omega^3 & 1 & \omega^3.\omega \\ \omega^3.\omega^2 & \omega^3.\omega^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{vmatrix} \qquad [\because \omega^3 = 1]$$

$$= 0 = \text{R.H.S.} \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

EXAMPLE - 6

Prove that : $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+C \end{vmatrix} = x^2(x+a+b+c)$

Solution :

$$L.H.S = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+C \end{vmatrix} = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

[Taking $(x+a+b+c)$ common from C_1]

SOLUTION CONT.

$$=(x+a+b+c)\begin{vmatrix}1 & b & c\\ 0 & x & 0\\ 0 & 0 & x\end{vmatrix}$$

$$\left[\text{Applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1\right]$$

Expanding along C_1, we get

$(x + a + b + c)\ [1(x^2)] = x^2(x + a + b + c)$

= R.H.S

EXAMPLE - 7

Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).$$

Solution :

$$L.H.S = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

SOLUTION CONT.

$$= 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ (c-b) & (a-c) & b+c \\ (a-c) & (b-a) & c+a \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$$

Now expanding along R_1, we get

$$2(a+b+c)\left[(c-b)(b-a)-(a-c)^2\right]$$

$$= 2(a+b+c)\left[bc - b^2 - ac + ab - (a^2 + c^2 - 2ac)\right]$$

$$= 2(a+b+c)\left[ab+bc+ac-a^2-b^2-c^2\right]$$

$$= R.H.S$$

EXAMPLE - 8

Using properties of determinants prove that

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

Solution :

$$\text{L.H.S} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

SOLUTION CONT.

$$=(5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -(x-4) & 0 \\ 0 & x-4 & -(x-4) \end{vmatrix} \quad \left[\text{Applying}\, R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2\right]$$

Now expanding along C_1, we get

$$(5x+4)\left[1(x-4)^2-0\right]$$

$$=(5x+4)(4-x)^2$$

$$=\text{R.H.S}$$

EXAMPLE - 9

Using properties of determinants, prove that

$$\begin{vmatrix} x+9 & x & x \\ x & x+9 & x \\ x & x & x+9 \end{vmatrix} = 243(x+3)$$

Solution :

$$L.H.S = \begin{vmatrix} x+9 & x & x \\ x & x+9 & x \\ x & x & x+9 \end{vmatrix}$$

$$= \begin{vmatrix} 3x+9 & x & x \\ 3x+9 & x+9 & x \\ 3x+9 & x & x+9 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

SOLUTION CONT.

$$=(3x+9)\begin{vmatrix}1 & x & x\\1 & x+9 & x\\1 & x & x+9\end{vmatrix}$$

$$=3(x+3)\begin{vmatrix}1 & x & x\\0 & 9 & 0\\0 & -9 & 9\end{vmatrix} \quad \left[\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2\right]$$

$$=3(x+3)\times 81 \qquad [\text{Expanding along } C_1]$$

$$=243(x+3)$$

$$=\text{R.H.S.}$$

SOLUTION CONT.

$$=(a^2+b^2+c^2)\begin{vmatrix}1 & a^2 & bc\\ 0 & (b-a)(b+a) & c(a-b)\\ 0 & (c-b)(c+b) & a(b-c)\end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2]$$

$$=(a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix}1 & a^2 & bc\\ 0 & -(b+a) & c\\ 0 & -(b+c) & a\end{vmatrix}$$

$$=(a^2+b^2+c^2)(a-b)(b-c)(-ab-a^2+bc+c^2) \quad [\text{Expanding along } C_1]$$

$$=(a^2+b^2+c^2)(a-b)(b-c)\left[b(c-a)+(c-a)(c+a)\right]$$

$$=(a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)=\text{R.H.S.}$$

EXAMPLE - 10

Show that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix} \left[\text{Applying } C_1 \to C_1 - 2C_3\right]$$

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2]$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Applications of Determinants

Applications of Determinants (Area of a Triangle)

The area of a triangle whose vertices are

(x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the expression

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$$

Example

Find the area of a triangle whose vertices are (-1, 8), (-2, -3) and (3, 2).

Solution :

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -1 & 8 & 1 \\ -2 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-1(-3-2)-8(-2-3)+1(-4+9)]$$

$$= \frac{1}{2}[5+40+5] = 25 \text{ sq.units}$$

Condition of Collinearity of Three Points

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three points,

then A, B, C are collinear

$\Leftrightarrow$ Area of triangle $ABC = 0$

$$\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example

If the points (x, -2) , (5, 2), (8, 8) are collinear, find x , using determinants.

Solution :

Since the given points are collinear.

$$\therefore \begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x(2-8)-(-2)(5-8)+1(40-16)=0$$

$$\Rightarrow -6x-6+24=0$$

$$\Rightarrow 6x=18 \Rightarrow x=3$$

Solution of System of 2 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1x + b_1y = c_1 \qquad \ldots(i)$$

$$a_2x + b_2y = c_2 \qquad \ldots(ii)$$

Then $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$ provided $D \neq 0$,

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Cramer's Rule

Note :

(1) If $D \neq 0$,

then the system is consistent and has unique solution.

(2) If $D = 0$ and $D_1 = D_2 = 0$,

then the system is consistent and has infinitely many solutions.

(3) If $D = 0$ and one of $D_1, D_2 \neq 0$,

then the system is inconsistent and has no solution.

Example

Using Cramer's rule , solve the following system of equations 2x-3y=7, 3x+y=5

Solution :

$$D=\begin{vmatrix}2 & -3\\3 & 1\end{vmatrix}=2+9=11\neq 0$$

$$D_1=\begin{vmatrix}7 & -3\\5 & 1\end{vmatrix}=7+15=22$$

$$D_2=\begin{vmatrix}2 & 7\\3 & 5\end{vmatrix}=10-21=-11$$

$$\because D\neq 0$$

$$\therefore \text{By Cramer's Rule } x=\frac{D_1}{D}=\frac{22}{11}=2 \text{ and } y=\frac{D_2}{D}=\frac{-11}{11}=-1$$

Solution of System of 3 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \qquad \ldots(i)$$

$$a_2x + b_2y + c_2z = d_2 \qquad \ldots(ii)$$

$$a_3x + b_3y + c_3z = d_3 \qquad \ldots(iii)$$

Then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$ provided $D \neq 0$,

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Cramer's Rule

Note:

(1) If $D \neq 0$, then the system is consistent and has a unique solution.

(2) If $D=0$ and $D_1 = D_2 = D_3 = 0$, then the system has infinite solutions or no solution.

(3) If $D = 0$ and one of D_1, D_2, $D_3 \neq 0$, then the system is inconsistent and has no solution.

(4) If $d_1 = d_2 = d_3 = 0$, then the system is called the system of homogeneous linear equations.

(i) If $D \neq 0$, then the system has only trivial solution $x = y = z = 0$.

(ii) If $D = 0$, then the system has infinite solutions.

Example

Using Cramer's rule , solve the following system of equations

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Solution :

$$D = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$$= 5(18+10) + 1(12-25)+4(-4 -15)$$
$$= 140 - 13 - 76 = 140 - 89$$
$$= 51 \neq 0$$

$$D_1 = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ -1 & -2 & 6 \end{vmatrix}$$

$$= 5(18+10)+1(12+5)+4(-4 +3)$$
$$= 140 + 17 - 4$$
$$= 153$$

Solution

$$D_2 = \begin{vmatrix} 5 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & -1 & 6 \end{vmatrix}$$

$= 5(12 + 5) + 5(12 - 25) + 4(-2 - 10)$
$= 85 + 65 - 48 = 150 - 48$
$= 102$

$$D_3 = \begin{vmatrix} 5 & -1 & 5 \\ 2 & 3 & 2 \\ 5 & -2 & -1 \end{vmatrix}$$

$= 5(-3 + 4) + 1(-2 - 10) + 5(-4 - 15)$
$= 5 - 12 - 95 = 5 - 107$
$= -102$

$\because D \neq 0$

$\therefore$ By Cramer's Rule $x = \frac{D_1}{D} = \frac{153}{51} = 3$, $y = \frac{D_2}{D} = \frac{102}{51} = 2$

and $z = \frac{D_3}{D} = \frac{-102}{51} = -2$

Example

Solve the following system of homogeneous linear equations:

$x + y - z = 0, x - 2y + z = 0, \ 3x + 6y + -5z = 0$

Solution:

$$\text{We have } D = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 6 & -5 \end{bmatrix} = 1(10-6) - 1(-5-3) - 1(6+6)$$

$$= 4 + 8 - 12 = 0$$

$\therefore$ The system has infinitely many solutions.

Putting $z = k$, in first two equations, we get

$x + y = k, x - 2y = -k$

$$\therefore \text{By Cramer's rule} \quad x = \frac{D_1}{D} = \frac{\begin{vmatrix} k & 1 \\ -k & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-2k+k}{-2-1} = \frac{k}{3}$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 1 & k \\ 1 & -k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-k-k}{-2-1} = \frac{2k}{3}$$

These values of x, y and z = k satisfy (iii) equation.

$$\therefore x = \frac{k}{3},\ y = \frac{2k}{3},\ z = k,\ \text{where } k \in R$$

Self Assessment

Find the determinant of each matrix.

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix} \qquad \begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix} \qquad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix} \qquad \begin{pmatrix} 2 & 1 & 4 & 8 \\ 0 & 2 & 5 & 19 \\ 0 & 0 & 3 & -1 \\ 2 & 1 & 4 & 0 \end{pmatrix}.$$

THE END……!