Chapter 4

Eigenvalue Problems

4.1 INTRODUCTION

Computation of eigenvalues and the corresponding eigenvectors of a matrix is of practical importance. For example, in solid mechanics, where we consider an element in a continuum, subjected to normal and shear stresses, usually one will be interested in finding the principal stresses, which are the maximum and minimum stresses in an element.

Consider the wedge of unit thickness, subjected to normal and shear stresses. If it has to be in equilibrium, a system of equations written in matrix notation as

$$\begin{bmatrix} \sigma_x - \sigma_\theta & \sigma_{xy} \\ \sigma_{xy} & \sigma_y - \sigma_\theta \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1}$$

has to be satisfied. For non-trivial solution to exist, the determinant of the matrix must be zero. Its characteristic equation is

$$\sigma_\theta^2 - \sigma_\theta(\sigma_x + \sigma_y) + (\sigma_x\sigma_y - \sigma_{xy}^2) = 0 \tag{4.2}$$

This is a quadratic equation whose roots give two eigenvalues corresponding to principal stresses.

In general, let $[A]$ be an $n \times n$ square matrix. Suppose, there exists a scalar λ and a vector $X = (x_1 \; x_2 \; \ldots \; x_n)^T$ such that

$$[A]\,(X) = \lambda\,(X) \tag{4.3}$$

then λ is the eigenvalue and X is the corresponding eigenvector of the matrix $[A]$. Equation (4.3) can also be written as

$$[A - \lambda I]\,(X) = (O) \tag{4.4}$$

This represents a set of n homogeneous equations possessing non-trivial solution, provided

$$|A - \lambda I| = 0 \tag{4.5}$$

This determinant, on expansion, gives an nth degree polynomial in λ, which is called *characteristic* polynomial of $[A]$, which has n roots. Corresponding to each root, we can solve Eq. (4.4) in principle, and determine a vector called *eigenvector*. However, finding the roots of the characteristic equation is laborious. Hence, we look for better methods suitable from the point of view of computation. Depending upon the type of matrix $[A]$ and on what one is looking for, various numerical methods are available. Power method, Jacobi's method, Given's method, Householder, Lanczos method, Ruthishauser and Francis method are well known in the literature.

In this chapter, we shall consider only real and real-symmetric matrices and discuss power method and Jacobi's method in detail. For further study, one can consult Wilkinson (1965).

4.2 POWER METHOD

To compute the largest eigenvalue and the corresponding eigenvector of the system

$$[A](X) = \lambda (X)$$

where $[A]$ is a real, symmetric or unsymmetric matrix, the power method is widely used in practice. It is an iterative technique. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of an $(n \times n)$ matrix $[A]$, such that

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| \tag{4.6}$$

and suppose $v_1, v_2, \ldots, v_n$ are the corresponding eigenvectors. Power method is applicable if the above eigenvalues are real and distinct, and hence, the corresponding eigenvectors are linearly independent. Then, any eigenvector v in the space spanned by the eigenvectors $v_1, v_2, \ldots, v_n$ can be written as their linear combination. Therefore,

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \tag{4.7}$$

Pre-multiplying Eq. (4.7) by A and substituting

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2, \ldots, \quad Av_n = \lambda_n v_n$$

We get

$$Av = \lambda_1 \left(c_1 v_1 + c_2 \frac{\lambda_2}{\lambda_1} v_2 + \cdots + c_n \frac{\lambda_n}{\lambda_1} v_n \right) \tag{4.8}$$

Again, pre-multiplying by A and simplifying, we obtain

$$A^2 v = \lambda_1^2 \left[c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^2 v_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^2 v_n \right]$$

Similarly, we have

$$A^r v = \lambda_1^r \left[c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^r v_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^r v_n \right] \tag{4.9}$$

and

$$A^{r+1} v = (\lambda_1)^{r+1} \left[c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^{r+1} v_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^{r+1} v_n \right] \tag{4.10}$$

Since $\lambda_i/\lambda_1 < 1$ for $i = 2, 3, \ldots, n$ and as $r \to \infty$, the right-hand sides of Eqs. (4.9) and (4.10) tend to $\lambda_1^r c_1 v_1$ and $\lambda_1^{r+1} c_1 v_1$ respectively. Now, the eigenvalue λ_1 can be computed as the limit of the ratio of the corresponding components of $A^r v$ and $A^{r+1} v$. That is,

$$\lambda_1 = \frac{\lambda_1^{r+1}}{\lambda_1^r} = \underset{r\to\infty}{\mathrm{Lt}} \frac{(A^{r+1}v)_p}{(A^r v)_p}, \qquad p = 1, 2, \ldots, n \tag{4.11}$$

Here, the index p stands for the pth component in the corresponding vector.

Sometimes, we may be interested in finding the least eigenvalue and the corresponding eigenvector. In that case, we proceed as follows. We note that $[A]\,(X) = \lambda\,(X)$. Pre-multiplying by $[A^{-1}]$, we get

$$[A^{-1}]\,[A]\,(X) = [A^{-1}]\,\lambda\,(X) = \lambda\,[A^{-1}]\,(X)$$

or

$$(X) = \lambda\,[A^{-1}]\,(X)$$

which can be rewritten as

$$[A^{-1}]\,(X) = \frac{1}{\lambda}(X) \tag{4.12}$$

which shows that the inverse matrix has a set of eigenvalues which are the reciprocals of the eigenvalues of $[A]$. Thus, for finding the eigenvalue of the least magnitude of the matrix $[A]$, we have to apply power method to the inverse of $[A]$. In order to see that the power method converges fast, the following numerical algorithm is adopted, particularly when working with numerical examples.

Step 1: Choose the initial vector such that the largest element is unity.

Step 2: This normalized vector $v^{(0)}$ is pre-multiplied by the given matrix $[A]$.

Step 3: The resultant vector is again normalized.

Step 4: This process of iteration is continued and the new normalized vector is repeatedly pre-multiplied by the matrix $[A]$ until the required accuracy is obtained. At this point, the result looks like

$$u^{(k)} = [A]v^{(k-1)} = q_k v^{(k)}$$

Here, q_k is the desired largest eigenvalue and $v^{(k)}$ is the corresponding eigenvector.

Example 4.1 Find the eigenvalue of largest modulus, and the associated eigenvector of the matrix

$$[A] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

by power method.

Solution We choose an initial vector $v^{(0)}$ as $(1, 1, 1)^T$. Then, compute the first iteration

$$u^{(1)} = [A]v^{(0)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$$

and normalize the resultant vector to get

$$u^{(1)} = 14 \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = q_1 v^{(1)}$$

The second iteration gives,

$$u^{(2)} = [A]v^{(1)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{39}{7} \\ \frac{67}{7} \\ \frac{171}{14} \end{pmatrix} = 12.2143 \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix} = q_2 v^{(2)}$$

Similarly, continuing this procedure, the third and subsequent iterations are given as

$$u^{(3)} = [A]v^{(2)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 5.263158 \\ 9.175438 \\ 11.935672 \end{pmatrix}$$

$$= 11.935672 \begin{pmatrix} 0.44096 \\ 0.76874 \\ 1.0 \end{pmatrix} = q_3 v^{(3)}$$

$$u^{(4)} = [A]v^{(3)} = \begin{pmatrix} 5.18814 \\ 9.07006 \\ 11.86036 \end{pmatrix} = 11.86036 \begin{pmatrix} 0.437435 \\ 0.764737 \\ 1.0 \end{pmatrix} = q_4 v^{(4)}$$

$$u^{(5)} = [A]v^{(4)} = \begin{pmatrix} 5.16908 \\ 9.04395 \\ 11.84178 \end{pmatrix} = 11.84178 \begin{pmatrix} 0.436512 \\ 0.763732 \\ 1.0 \end{pmatrix} = q_5 v^{(5)}$$

After rounding-off, we take the largest eigenvalue as $\lambda = 11.84$ and the corresponding eigenvector as

$$(X) = \begin{pmatrix} 0.44 \\ 0.76 \\ 1.00 \end{pmatrix}$$

accurate to two decimals.

4.3 JACOBI'S METHOD

Definition 4.1 An $(n \times n)$ matrix $[A]$ is said to be *orthogonal* if

$$[A]^{\mathrm{T}}\,[A] = [I], \qquad \text{i.e. } [A]^{\mathrm{T}} = [A]^{-1}$$

In order to compute all the eigenvalues and the corresponding eigenvectors of a real symmetric matrix, Jacobi's method is highly recommended. It is based on an important property from matrix theory, which states that, if $[A]$ is an $(n \times n)$ real symmetric matrix, its eigenvalues are real, and there exists an orthogonal matrix $[S]$ such that $[S^{-1}]\,[A]\,[S]$ is a diagonal matrix $[D]$. This diagonalization can be carried out by applying a series of orthogonal transformations $S_1, S_2, \ldots, S_n$, as explained below.

Let A be an $(n \times n)$ real symmetric matrix. Suppose $|a_{ij}|$ be numerically the largest element amongst the off-diagonal elements of A. We construct an orthogonal matrix S_1 defined as

$$s_{ij} = -\sin\theta, \quad s_{ji} = \sin\theta, \quad s_{ii} = \cos\theta, \quad s_{jj} = \cos\theta \tag{4.13}$$

while each of the remaining off-diagonal elements are zero, the remaining diagonal elements are assumed to be unity. Thus, we construct S_1 as under

$$S_1 = \begin{array}{ccccccccc}
 & & & i\text{th column} & & j\text{th column} & & & \\
 & & & \downarrow & & \downarrow & & & \\
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \\
0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & \cos\theta & \cdots & -\sin\theta & \cdots & 0 & \leftarrow i\text{th row} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & \sin\theta & \cdots & \cos\theta & \cdots & 0 & \leftarrow j\text{th row} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 &
\end{array} \tag{4.14}$$

where $\cos\theta$, $-\sin\theta$, $\sin\theta$ and $\cos\theta$ are inserted in (i, i), (i, j), (j, i), (j, j)th positions respectively, and elsewhere it is identical with a unit matrix. Now, we compute

$$D_1 = S_1^{-1} A S_1 = S_1^{\mathrm{T}} A S_1$$

since S_1 is an orthogonal matrix, such that $S_1^{-1} = S_1^{\mathrm{T}}$. After the transformation, the elements at the positions (i, j), (j, i) get annihilated, that is, d_{ij} and d_{ji} reduce to zero, which is seen as follows:

$$\begin{bmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} a_{ii}\cos^2\theta + 2a_{ij}\sin\theta\cos\theta + a_{jj}\sin^2\theta & (a_{jj} - a_{ii})\sin\theta\cos\theta + a_{ij}\cos 2\theta \\ (a_{jj} - a_{ii})\sin\theta\cos\theta + a_{ij}\cos 2\theta & a_{ii}\sin^2\theta + a_{jj}\cos^2\theta - 2a_{ij}\sin\theta\cos\theta \end{bmatrix}$$

Therefore, $d_{ij} = 0$, only if,

$$a_{ij}\cos 2\theta + \frac{a_{jj} - a_{ii}}{2}\sin 2\theta = 0$$

That is, if

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} \tag{4.15}$$

Thus, we choose θ such that, Eq. (4.15) is satisfied, thereby, the pair of off-diagonal elements d_{ij} and d_{ji} reduces to zero.

However, though it creates a new pair of zeros, it also introduces non-zero contributions at formerly zero positions. Also, Eq. (4.15) gives four values of θ, but to get the least possible rotation, we choose $-\pi/4 \leq \theta \leq \pi/4$.

As a next step, the numerically largest off-diagonal element in the newly obtained rotated matrix D_1 is identified and the above procedure is repeated using another orthogonal matrix S_2 to get D_2. That is, we obtain

$$D_2 = S_2^{-1}D_1S_2 = S_2^{\mathrm{T}}(S_1^{\mathrm{T}}AS_1)S_2$$

Similarly, we perform a series of such two-dimensional rotations or orthogonal transformations. After making r transformations, we obtain

$$\begin{aligned} D_r &= S_r^{-1}S_{r-1}^{-1} \;\ldots\; S_2^{-1}S_1^{-1}AS_1S_2 \;\ldots\; S_{r-1}S_r \\ &= (S_1S_2 \;\ldots\; S_{r-1}S_r)^{-1}A(S_1S_2 \;\ldots\; S_{r-1}S_r) \\ &= S^{-1}AS \end{aligned} \tag{4.16}$$

where $S = S_1S_2 \ldots S_{r-1}\, S_r$. Now, as $r \to \infty$, D_r approaches to a diagonal matrix, with the eigenvalues on the main diagonal. The corresponding eigenvectors are the columns of S.

It is estimated that the minimum number of rotations required to transform the given $(n \times n)$ real symmetric matrix $[A]$ into a diagonal form is $n(n-1)/2$.

Example 4.2 Find all the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

by Jacobi's method.

Solution The given matrix is real and symmetric. The largest off-diagonal element is found to be $a_{13} = a_{31} = 2$. Now, we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0} = \infty$$

which gives, $\theta = \pi/4$. Thus, we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The first rotation gives,

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we may observe that the elements d_{13} and d_{31} got annihilated. To make sure that our calculations are correct up to this step, we may also observe that the sum of the diagonal elements of D_1 is same as the sum of the diagonal elements of the original matrix A.

As a second step, we choose the largest off-diagonal element of D_1 and is found to be $d_{12} = d_{21} = 2$, and compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{4}{0} = \infty$$

which again gives $\theta = \pi/4$. Thus, we construct the second rotation matrix as

$$S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At the end of second rotation, we get

$$D_2 = S_2^{-1} D_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1}$$

which turned out to be a diagonal matrix, and therefore, we stop the computation. From (1) we notice that the eigenvalues of the given matrix are 5, 1 and –1. The eigenvectors are the column vectors of $S = S_1 S_2$. Therefore,

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example 4.3 Find all the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

by Jacobi's method.

Solution In this example, we find that all the off-diagonal elements are of the same order of magnitude. Therefore, we can choose any one of them. Suppose, we choose a_{12} as the largest element and compute

$$\tan 2\theta = \frac{-1}{0} = \infty$$

which gives, $\theta = \pi/4$. Then $\cos\theta = \sin\theta = 1/\sqrt{2}$ and we construct an orthogonal matrix S_1 such that

$$S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$$

Now, we choose $d_{13} = -1/\sqrt{2}$ as the largest element of D_1 and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{-\sqrt{2}}{1 - 2}$$

which gives, $\theta = 27° \, 22' \, 41''$.

Now we construct another orthogonal matrix S_2, such that

$$S_2 = \begin{bmatrix} 0.888 & 0 & -0.459 \\ 0 & 1 & 0 \\ 0.459 & 0 & 0.888 \end{bmatrix}$$

At the end of second rotation, we obtain

$$D_2 = S_2^{-1} D_1 S_2 = \begin{bmatrix} 0.634 & -0.325 & 0 \\ 0.325 & 3 & -0.628 \\ 0 & -0.628 & 2.365 \end{bmatrix}$$

Now, the numerically largest off-diagonal element of D_2 is found to be $d_{23} = -0.628$ and compute

$$\tan 2\theta = \frac{-2 \times 0.628}{3 - 2.365}$$

we get, $\theta = -31° \, 35' \, 24''$. Thus, the orthogonal matrix S_3 is seen to be

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.852 & 0.524 \\ 0 & -0.524 & 0.852 \end{bmatrix}$$

At the end of third rotation, we get

$$D_3 = S_3^{-1} D_2 S_3 = \begin{bmatrix} 0.634 & -0.277 & 0 \\ 0.277 & 3.386 & 0 \\ 0 & 0 & 1.979 \end{bmatrix}$$

To reduce D_3 to a diagonal form, some more rotations are required. However, we may take 0.634, 3.386 and 1.979 as eigenvalues of the given matrix.

Example 4.4 Find all the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

by Jacobi's method.

Solution The given matrix is

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

In this example, the largest off-diagonal element is found to be $a_{13} = a_{31} = 1$. Now, we compute

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2}{5-5} = \frac{2}{0} = \infty$$

which gives $\theta = \pi/4$. Following Jacobi's method, we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos(\pi/4) & 0 & -\sin(\pi/4) \\ 0 & 1 & 0 \\ \sin(\pi/4) & 0 & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

The first rotation gives

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad (1)$$

Which is a diagonal matrix and hence we stop further computation. From (1), we observe that 6, –2 and 4 are the eigenvalues of the given matrix and the corresponding eigenvectors are respectively the column vectors of

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

4.4 GERSCHGORIN'S THEOREM

This is one of the useful theorems on the bounds for eigenvalues of a square matrix.

Let λ_i be an eigenvalue of the $n \times n$ matrix [A] and let x_i be the corresponding eigenvector. Suppose R_s be the sum of the moduli of the terms along sth row, excluding the diagonal element a_{ss}. Then, every eigenvalue of $[A]$ lies inside or on the boundary of atleast one of the circles $|\lambda - a_{ss}| = R_s$

Proof: Given that

$$[A]x_i = \lambda_i x_i \tag{4.17}$$

Let $v_1, v_2, ..., v_n$ are the components of x_i, then the above equation can be expanded as

$$\left.\begin{aligned} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n &= \lambda_i v_1 \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n &= \lambda_i v_2 \\ \vdots \qquad\qquad & \qquad \vdots \\ a_{s1}v_1 + a_{s2}v_2 + \cdots + a_{sn}v_n &= \lambda_i v_s \\ \vdots \qquad\qquad & \qquad \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n &= \lambda_i v_n \end{aligned}\right\} \tag{4.18}$$

Suppose v_s be the largest in modulus of $v_1, v_2, ..., v_n$. Now, let us divide the sth equation by v_s and get

$$\lambda_i = a_{s1}\left(\frac{v_1}{v_s}\right) + a_{s2}\left(\frac{v_2}{v_s}\right) + \cdots + a_{ss} + \cdots + a_{sn}\left(\frac{v_n}{v_s}\right) \tag{4.19}$$

Since $\left|\frac{v_i}{v_s}\right| \le 1,\ i = 1, 2, ..., n$, it follows that

$$|\lambda_i| \le |a_{s1}| + |a_{s2}| + \cdots + |a_{ss}| + \cdots + |a_{sn}| \tag{4.20}$$

Eq. (4.19) can also be written as

$$|\lambda_i - a_{ss}| \le |a_{s1}| + |a_{s2}| + \cdots + |a_{sn}|$$

or

$$|\lambda_i - a_{ss}| \le \sum_{\substack{j=1 \\ j \ne s}}^{n} |a_{sj}| = R_s \tag{4.21}$$

Hence the proof. This theorem also holds for any column.

An immediate consequence of Gerschgorin's theorem, when applied to identity matrix or permutation matrix is that, its eigenvalues lie within a circle having center at 1 and radius 0. Here follows an example.

Example 4.4 Apply Gerschgorin's theorem to the matrix

$$[A] = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

Solution This matrix has a dominant diagonal.

In this example $a_{ss} = 4$, Max. $R_s = 3$. Thus, Gerschgorin's theorem states that all the eigenvalues of the given matrix lie inside the circle with center at 4 and radius 3. In view of its symmetry, the eigenvalues are also real.

Definition 4.2 (Spectral Norm). Let λ_i be the largest eigenvalue of AA^* or A^*A, where A^* is the conjugate transpose of A, then the spectral norm of the matrix A, denoted by $\sigma(A)$ is defined as

$$\sigma(A) = (\lambda_i)^{1/2} \tag{4.22}$$

Definition 4.3 (Determinant). The determinant of an $n \times n$ matrix A is the product of its eigenvalues.

Definition 4.4 (Trace of a Matrix). The sum of the diagonal elements of an $n \times n$ matrix A is called the trace of the matrix A. Trace of the matrix A is also defined as the sum of its eignvalues.

EXERCISES

4.1 Find the largest eigenvalue of the matrix

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

and the corresponding eigenvector, by power method after sixth iteration.

4.2 Find the largest eigenvalue of the matrix

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

and the corresponding eigenvector, by power method after fourth iteration starting with the initial vector $v^{(0)} = (0, 0, 1)^T$.

4.3 Find the dominant eigenvalue and the corresponding eigevector of the matrix

$$\begin{bmatrix} 8 & 1 & 2 \\ 0 & 10 & -1 \\ 6 & 2 & 15 \end{bmatrix}$$

by power method with unit vector as the initial vector.

4.4 Find the largest eigenvalue and the corresponding eigenvector of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

by power method at the end of sixth iteration, taking unit vector as the initial vector.

4.5 Using Jacobi's method, find all the eigenvalues and eigenvectors of the Hilbert matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Give result after two rotations.

4.6 Use Jacobi's method to find all the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

4.7 Find all the eigenvalues and the corresponding eigenvectors of the matrix

(i) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

by Jacobi's method. Give results at the end of third rotation.

4.8 Find the dominant eigenvalue of

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

and the corresponding eigenvector.