Least Square Approximation

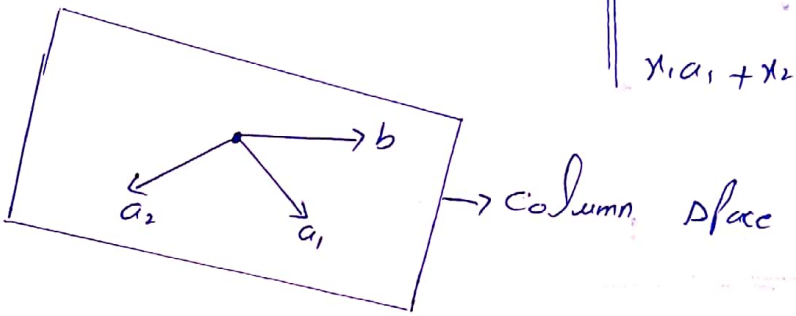
Case-I

$Ax = b$ solution will exist iff b is in column space of A.

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

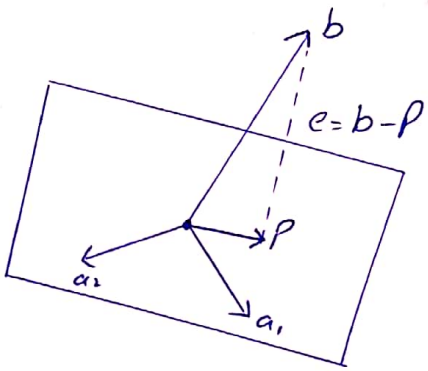
basis of col space $\{a_1, a_2\}$

$x_1 a_1 + x_2 a_2 = b$



Case-II

no solution



$\|e\| = \|b - p\|$

$= \sqrt{(b_1 - p_1)^2 + (b_2 - p_2)^2}$

↓

Least Square

$Ax = b$ has no solution

$A^T A x^* = A^T b$

Derivation of above equation:

$p = x_1^* a_1 + x_2^* a_2 = A x^*$

$e = b - p$

$\langle a_1 \cdot e \rangle = 0$

$\langle a_2 \cdot e \rangle = 0$

$$\left.\begin{aligned} a_1^T \cdot e &= 0 \\ a_2^T \cdot e &= 0 \end{aligned}\right\} A^T e = 0$$

$$A^T(b-p) = 0$$

$$A^T(b - Ax^*) = 0$$

$$A^Tb - A^TA\,x^* = 0$$

$$A^TA\,x^* = A^Tb$$

Projection matrix

$$x^* = (A^TA)^{-1}A^Tb$$

$$p = Ax^* = \underbrace{A(A^TA)^{-1}A^T}_{\text{Projection}}b$$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find $Ax = b$ by least approximation method.

Sol:

$$A^TAx^* = A^Tb$$

$$A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} x^* = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}^{-1} = \frac{1}{18-16} \begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}$$

$$x^* = \frac{1}{2} \begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -2 & 3/2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 18-16 \\ -12 + \frac{24}{2} \end{bmatrix}$$

$$x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Projection Matrix:

$$P = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 \\ -1 & 1 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$