Gram-Schmidt Orthoganalization Process

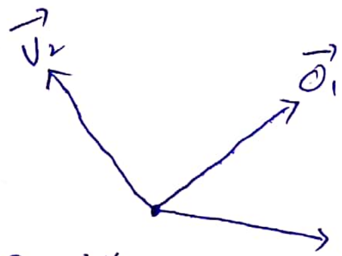
This is method for orthogonalizing set of vectors in an inner product space.

$$V = a_1V_1 + a_2V_2 + \cdots + a_nV_n$$

"How to find a_is"

- Provide us a short method to find values of $a_1, a_2, \ldots, a_n$.
- Help us to convert normal basis to orthogonal basis.

<u>idea:</u>



$$\{O_1,\ V_2 - Proj_{O_1}^{V_2},\ V_3 - Proj_{O_1}^{V_3} - Proj_{O_2}^{V_3}\}$$

Let the vectors $V_1, V_2, V_3, \ldots, V_n$

$$\vec{O_1} = \vec{V_1}$$

$$O_2 = V_2 - Proj_{O_1}^{V_2}$$

$$O_3 = V_3 - Proj_{O_1}^{V_3} - Proj_{O_2}^{V_3}$$

$\vdots$

$$O_n = V_n - Proj_{O_1}^{V_n} - Proj_{O_2}^{V_n} - \cdots - Proj_{O_{n-1}}^{V_n}$$

Example: Given the following basis of a vector space V, find the orthogonal basis of V.

Basis of V.

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

Sol:

$$O_1 = V_1$$

$$O_2 = V_2 - Proj_{O_1}(V_2)$$

$$= V_2 - \frac{\langle V_2 \cdot O_1 \rangle}{\langle O_1 \cdot O_1 \rangle} \times O_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{2+0+0}{1+1+0} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$O_3 = V_3 - Proj_{O_1}(V_3) - Proj_{O_2}(V_3)$$

$$= V_3 - \frac{\langle V_3 \cdot O_1 \rangle}{\langle O_1 \cdot O_1 \rangle} \times O_1 - \frac{\langle V_3 \cdot O_2 \rangle}{\langle O_2 \cdot O_2 \rangle} \times O_2$$

$$= \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \frac{3+3+0}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3-3-6}{1+1+4} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3+1 \\ -3+3+1 \\ 3-0-2 \end{bmatrix}$$

$$O_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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$\langle O_1 . O_2 \rangle = 0$

$\langle O_1 . O_3 \rangle = 0$

$\langle O_2 . O_3 \rangle = 0$

b Creat orthogonal matrix.

$$O_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad O_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad O_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$n_1 = \frac{O_1}{\|O_1\|} \Rightarrow \|O_1\| = \sqrt{\langle O_1 . O_1 \rangle} = \sqrt{2}$$

$$= \frac{O_1}{\|O_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$n_2 = \frac{O_2}{\|O_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad n_3 = \frac{O_3}{\|O_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

orthogonal matrix

$$m = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$m^{-1} = m^t$

$\langle n_1 . n_1 \rangle = 1$

$\langle n_1 . n_2 \rangle = 0$