Orthagonal Projection

By theorem point $V = W \oplus W^{\perp}$

$V = w + u$

The vector w is called as orthogonal projection of v in w written as "$Proj_{W} V$"

$$Proj_{W} V = W = (V, w_1)w_1 + (V, w_2)w_2 + \cdots + (V, w_n)w_n.$$

$\Rightarrow \{w_1, w_2, w_3, \ldots, w_n\}$ is orthonormal basis of W.

If basis is orthogonal then,

$$Proj_{W} V = \frac{(V, w_1)}{(W_1, W_1)} w_1 + \frac{(V, w_2)}{(W_2, W_2)} w_2 + \cdots + \frac{(V, w_n)}{(W_n, W_n)} w_n$$

Example: Let W is 2-D subspace of R^3 with orthonormal basis w_1, w_2.

$$w_1 = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Formula: $Proj_{W} V = (V, w_1)w_1 + (V, w_2)w_2$

$$(V, w_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} = 4/3 + (-1/3) + (-\tfrac{6}{3}) \Rightarrow \frac{4 - 1 - 6}{3} = -\frac{7+4}{3} = -1$$

$$(V, w_1)w_1 = -1 \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2/3 \\ +1/3 \\ +2/3 \end{bmatrix}$$

$$(V, W_2) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow 2/\sqrt{2} + 0 + \frac{3}{\sqrt{2}}$$

$$\frac{2 + 0 + 3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$$(V \cdot W_2) W_2 = \frac{5}{\sqrt{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 5/2 \\ 0 \\ 5/2 \end{bmatrix}$$

Now,

$$(V \cdot W_1) W_1 + (V \cdot W_2) W_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 5/2 \\ 0 \\ 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 11/6 \\ 1/3 \\ 19/6 \end{bmatrix}$$