# Inner Product Space

**Def:** Let $V(F)$ be a vector space, where $F$ is either the field of real numbers or the field of complex numbers.

## Properties:

an inner product space on $V$ is a function from $V \times V$ into $F$, which assigns to each ordered pair of vectors $\alpha, \beta$ in $V$, a scaler $\langle \alpha, \beta \rangle$ in a such way that:

### [1] Conjugate - Symmetry:

$$\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$$

here $\overline{\langle \beta, \alpha \rangle}$ denotes the conjugate complex of the number $\langle \beta, \alpha \rangle$, $\forall \alpha, \beta \in V$

### [2] Linearity:

$$\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$$

$$\forall \alpha, \beta, \gamma \in V, \quad a, b \in F.$$

### [3] Non-Negativity:

$$\langle \alpha, \alpha \rangle \geq 0$$

and

$$\langle \alpha, \alpha \rangle = 0$$

$$\Rightarrow \alpha = 0, \quad \forall \alpha \in V$$

Also, the vector space $V$ is then said to be an inner product space, with respect to the specified inner product defined on it.

# Orthogonality

**Def:** Let $\alpha$ and $\beta$ be vectors in an inner Product space $V$, then $\alpha$ is said to be orthogonal to $\beta$, If

$$\langle \alpha, \beta \rangle = 0$$

The relation of orthogonality in an inner product space is symmetric.

We have

$\alpha$ is orthogonal to $\beta$.

$$\Rightarrow \langle \alpha, \beta \rangle = 0$$

$$\Rightarrow \overline{\langle \alpha, \beta \rangle} = 0$$

$$\Rightarrow \langle \beta, \alpha \rangle = 0$$

$\beta$ is orthogonal to $\alpha$.

## Note:

1): If $\alpha$ is orthogonal to $\beta$, then every scaler multiple of $\alpha$ is orthogonal to $\beta$.

Let $k$ be any scaler

$$\langle k\alpha, \beta \rangle = k\langle \alpha, \beta \rangle$$

$$\Rightarrow \quad k \cdot 0 = 0 \qquad \because \langle \alpha, \beta \rangle = 0$$

$\therefore$ $k\alpha$ is orthogonal to $\beta$.

2): The zero vector is orthogonal to every vector.

$$\Rightarrow \langle 0 \cdot \alpha \rangle = 0$$

3): The zero-vector is the only vector which is orthogonal to itself.

we have,

$\alpha$ is orthogonal to $\alpha$.

$$\langle \alpha, \alpha \rangle = 0$$

$$\Rightarrow \alpha = 0 \quad (\text{By def. of IPS})$$

# Example:

Prove that given vector's are orthogonal or not?

$$u = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\langle u.v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$$\langle u.v \rangle = 0(1) + 2(0) + 3(0) + 0(1)$$

$$= 0 + 0 + 0 + 0$$

$$\langle u.v \rangle = 0$$

# Orthogonal Set's:

Let $V$ be an IPS. A subset of $V$ is said to be orthogonal set if every 2 distinct vectors of $S$ are orthogonal.

# Example:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\{x_1, x_2, x_3\}$ is an orthogonal set.

$$\langle x_1 . x_2 \rangle = 1(-2) + 0(0) + 2(1) = 0$$

$$\langle x_2 . x_3 \rangle = -2(0) + 0(1) + 1(0) = 0$$

$$\langle x_3 . x_1 \rangle = 0(1) + 1(0) + 0(2) = 0$$

# Orthogonal Complement: (Also known as $\perp$ar complement:)

Let $V$ = vector space (IPS)

$W$ = Subspace of $V$.

If set of all the vector's in $V$ are orthogonal to every vector in $W$, then $V$ is called orthogonal complement of $W$, and is denoted by $W^{\perp}$ (read as

"W PreP".

## Important result's:

1) $W^\perp$ is subspace of V.

2) $W \cap W^\perp = \{0\}$

3) $V = W \oplus W^\perp$, W finite dimensional.

4) $(W^\perp)^\perp = W$

Let A be an nxn matrix.

1) The null-space of A is orthogonal complement of Row-space of A.

2) The null-space of $A^T$ is orthogonal complement of column-space of A.

**Example:** $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

Row(A) = $R_1 = (1, 3, 0, -1)$, $R_2 = (0\ 0\ 1, 0)$

col (A) = $C_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}$

Null(A) = $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Null($A^T$) = $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

So, Proof-1:

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} = -3+3+0-0 = 0$$

Proof-2:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \Rightarrow = 5-4-1 = 0$$

# Orthonormal - Set

**Def:** Let $V$ be an IPS. A subset $S$ of $V$ is said to be orthonormal set if

i) $S$ is orthogonal
ii) Length of every vector is one.

**Example:** Set 'S' containing following vectors $V_1, V_2, V_3$

$$S = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} +1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

i)

$$\langle V_1 \cdot V_2 \rangle = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0(0) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) = 0$$

$$\langle V_2 . V_3 \rangle = \frac{1}{\sqrt{2}}(0) + 0(1) + \left(-\frac{1}{\sqrt{2}}\right)(0) = 0$$

$$\langle V_3 \cdot V_1 \rangle = 0\left(-\frac{1}{\sqrt{2}}\right) + 1(0) + 0(1/\sqrt{2}) = 0$$

ii)

$$|V_1| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$|V_2| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(-\frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$|V_3| = \sqrt{(0)^2 + (1)^2 + (0)^2} = 1$$

Proved