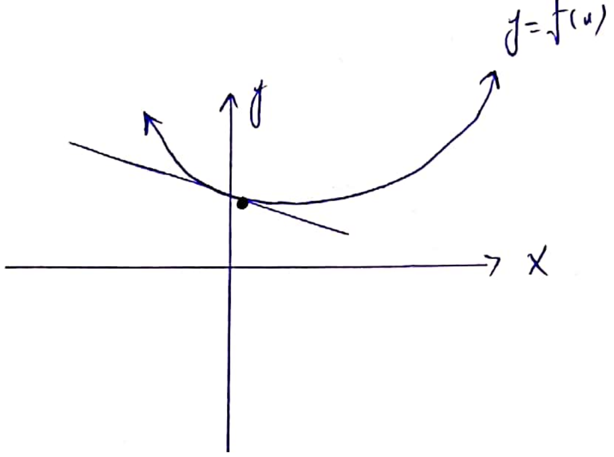
Concavity & Point of Inflection

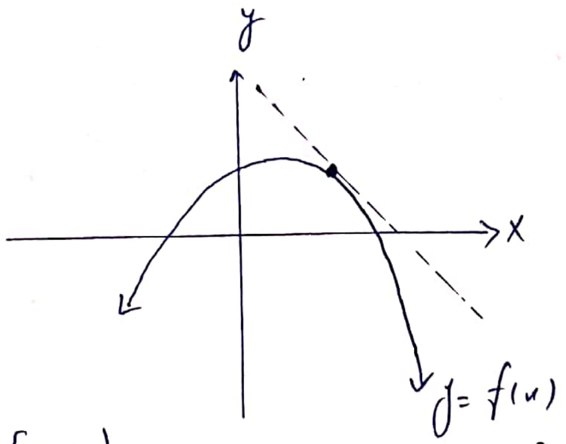
The graph of a function $f(x)$ is concave UP on an open interval I if the graph of f lies above all its tangent line on I.

★ Concave upward on I if $f''(x) > 0$ on I



The graph of a function $f(x)$ is concave down on an open interval I if the graph of f lies below all its tangent line on I.

★ Concave downward on I if $f''(x) < 0$ on I



A point $(c, f(c))$ is a Point of inflection of f if the concavity of f changes at $x = c$.

Mathematically: $f''(x) = 0$

Example: Find the point of inflection of $f(x) = x^3 - 6x^2 - 3x + 1$

where is $f(x)$ is concave up?

where is $f(x)$ is concave down?

$$f(x) = x^3 - 6x^2 - 3x + 1$$

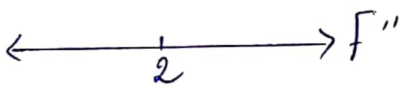
$$f'(x) = 3x^2 - 12x - 3$$

$$f''(x) = 6x - 12$$

$$6x - 12 = 0$$

$$6x = 12$$

$$x = 2$$



$f''(0) = -12 < 0$ $\quad$ $f'' < 0$ on $(-\infty, 2)$

$f''(3) = 6 > 0$ $\quad$ $f'' > 0$ on $(2, \infty)$

$$f(2) = (2)^3 - 6(2)^2 - 3(2) + 1$$

$$= -21$$

So the point of inflection $(2, -21)$

Second Derivative Test for Extrema

Suppose that f is differentiable on an open interval containing 'C' and that $f'(c) = 0$.

i) If $f''(c) < 0$ then f has a local maximum at "c".

ii) If $f''(c) > 0$ then f has a local minimum at "c".

iii) If $f''(c) = 0$, then second derivative test is not applicable.

Example:

a) Use the second derivative test to find the local extrema.

b) Find the intervals on which the graph of f is concave upward or downward.

c) Find the x-coordinate of the point of inflection.

d) Sketch the graph of f.

$$f(x) = -4x^3 + 9x^2 + 12x \longrightarrow (1)$$

$$f'(x) = -12x^2 + 18x + 12 \longrightarrow (2)$$

$$f''(x) = -24x + 18 \longrightarrow (3)$$

So, for critical number $f'(x) = 0$, put in eq. (2).

$$-12x^2 + 18x + 12 = 0$$

$$x = 2 \; \& \; x = -1/2$$

a) Extrema: For extrema put critical number in $f''(x)$.

At $c = 2$: $f''(2) = -24(2) + 18 = -30 < 0$

by 2nd derivative test, f has local max at at $x = 2$, so $f(2) = -4(2)^3 + 9(2)^2 + 12(2)$

$F(2) = 28$ local max.

At $x = -\frac{1}{2}$: $F''(-\frac{1}{2}) = -24(-1/2) + 18 = 30$

so f has local minimum at $x = -1/2$

$F(-1/2) = -13/4$

c) For x-coordinate of Point of inflection

$f''(x) = 0$, Put in (3)

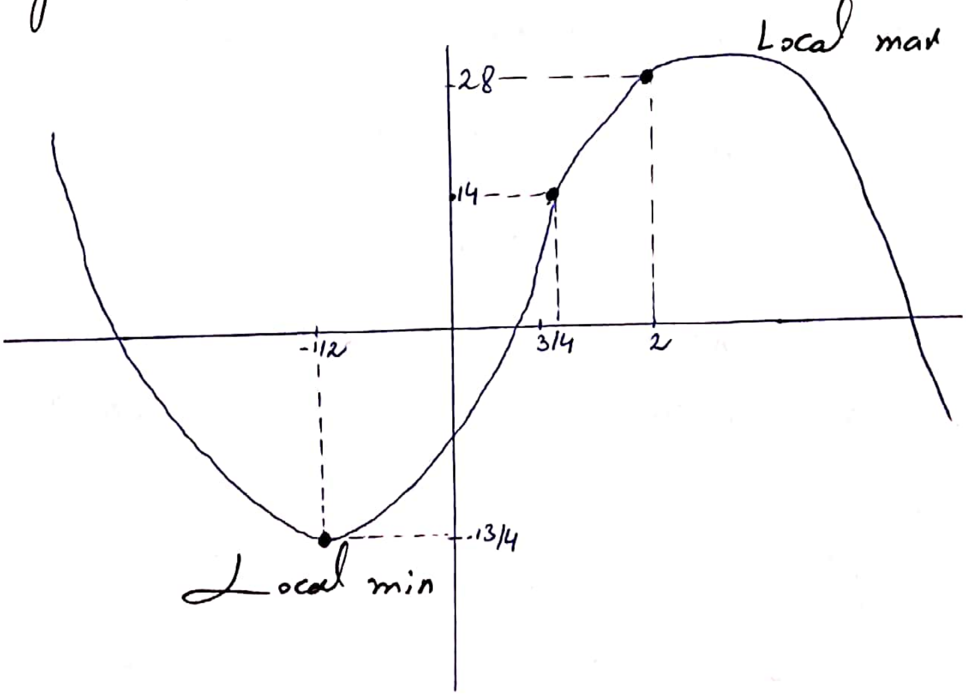
$$-24x + 18 = 0$$

$$x = 3/4$$

So $f(3/4) = 14$

b) For Concavity:

Interval	$(-\infty, 3/4)$	$(3/4, \infty)$
Testing value k	0	1
Sign of $f''(k)$	$f''(0) = 18 > 0$ (+ve)	$f''(1) = -6 < 0$ (-ve)
Concavity	Upward	Down



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