Sequences & Series

chapter name.

is a fn whose domain is the set of Natural numbers. It is denoted as

$\{S_n\}_{n=1}^{\infty}$ or $\{S_n ; n \in N\}$ or

$\{S_1, S_2, \ldots\}$ or $\{S_n\}$

i) $\{n\} = \{1, 2, 3, \ldots\}$

ii) $\{\frac{1}{n}\} = \{1, 1/2, 1/3, \ldots\}$

iii) $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$

Sub-Sequence:-

It is a seq whose terms are contained in given sequence. A sub-seq of $\{S_n\}$ is written as

$\{S_{n_k}\}$

Increasing

$\{S_n\}$ is $\uparrow$ seq if

$S_{n+1} \geq S_n \ \forall\, n \geq 1$

e.g $\left(1+\frac{1}{n}\right)^n$

$\downarrow$

Increasing

Decreasing seq

$\{S_n\}$ is $\downarrow$ seq if

$S_{n+1} \leq S_n \ \forall\, n \geq 1$

e.g $S_n = \left(1+\frac{1}{n}\right)^{n+1}$

$\downarrow$

Decreasing.

Monotonic seq:-

A seq is said to be monotonic seq if its either ↑ or ↓.

→ $\{S_n\}$ is monotonically ↑ if

$$S_{n+1} \geq S_n \Rightarrow \frac{S_{n+1}}{S_n} \geq \frac{S_n}{S_n}$$

$$\Rightarrow \frac{S_{n+1}}{S_n} \geq 1 \quad \forall n \geq 1$$

→ $\{S_n\}$ is monotonically ↓ if

$$S_n \geq S_{n+1} \Rightarrow \frac{S_n}{S_{n+1}} \geq 1 \quad \forall n \geq 1$$

Strictly ↑ or ↓:-

if $S_{n+1} > S_n$ (↑) or $S_{n+1} < S_n$ (↓)

Bernoulli's Inequality:-

let $p \in \mathbb{R}$, $p \geq -1$ & $p \neq 0$ then for $n \geq 2$,

$$(1+p)^n > 1+np$$

proof:-

we will prove it by mathematical induction

if $n = 2$

$$L.H.S = (1+p)^2 = 1+p^2+2p \quad \text{---} (1)$$

$$R.H.S = 1+np = 1+2p \quad \text{---} (2)$$

from (1) & (2)

$$L.H.S > R.H.S$$

$$\Rightarrow (1+p)^2 > 1+2p$$

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Suppose it is true for $n=k$

$\Rightarrow (1+p)^k > 1+kp$ ——(3) , $k \geq 2$

Now let

$(1+p)^{k+1} = (1+p)^k (1+p)$

$> (1+kp)(1+p)$ (using (3))

$= 1+p+kp+kp^2$

$= 1+p(k+1)+kp^2$

$\geq 1+(k+1)p$

$\Rightarrow (1+p)^{k+1} > 1+(k+1)p$

we conclude that

$(1+p)^n > 1+np.$

Example:-

$S_n = \{(1+\frac{1}{n})^n\}, n \geq 1$

prove that S_n is $\uparrow$ seq.

proof:-

To prove we will use Bernoulli's inequality with $p = -1/n^2, n \geq 2$

$\Rightarrow (1+p)^n > (1+np)$

$(1-\frac{1}{n^2})^n > (1-\frac{n}{n^2})$

$\Rightarrow [(1-\frac{1}{n})(1+\frac{1}{n})]^n > 1-\frac{1}{n}$

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$\Rightarrow \left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n}$

$= \left(\frac{n-1}{n}\right)^{1-n}$

$= \left(\frac{n}{n-1}\right)^{n-1}$

$= \left(1+\frac{1}{n-1}\right)^{n-1}$

$1-\frac{1}{n^2}$

$\frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n^2}$

$\left(\frac{(n-1)}{n^2}\right)\left(\frac{n+1}{n^2}\right)$

$\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)$

$\Rightarrow \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1}$

$S_n > S_{n-1}$

$\Rightarrow S_n$ is $\uparrow$ seq.

Example: $C_n = \left(1+\frac{1}{n}\right)^{n+1}$, $n \geq 1$

Show that it's $\downarrow$ seq

proof:-

Use $p = \frac{1}{n^2-1}$ in Bernoulli's inequality

$\Rightarrow (1+p)^n > (1+np)$

$\left(1+\frac{1}{n^2-1}\right)^n > \left(1+\frac{n}{n^2-1}\right)$ — (1)

$1+\frac{1}{n^2-1} = \frac{n^2-1+1}{n^2-1} = \frac{n^2}{n^2-1} = \left(\frac{n}{(n-1)}\right)\left(\frac{n}{n+1}\right)$

$\Rightarrow \left(1+\frac{1}{n^2-1}\right)\left(\frac{n}{n+1}\right)^{-1} = \frac{n}{(n-1)}$

$\left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right) = \frac{n}{n-1}$ — (2)

Now let

$$t_{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1+1} = \left(\frac{n-\cancel{1}+\cancel{1}}{n-1}\right)^n = \left(\frac{n}{n-1}\right)^n$$

as ② is from ②; -

$$= \left(\left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right)\right)^n \qquad \because \frac{n}{n-1} = \left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right)$$

$$= \left(1+\frac{1}{n^2-1}\right)^n\left(\frac{n+1}{n}\right)^n$$

$$> \left(1+\frac{n}{n^2-1}\right)\left(\frac{n+1}{n}\right)^n \quad \text{(from ①)}$$

$$> \left(1+\frac{1}{n}\right)\left(\frac{n+1}{n}\right)^n$$

$$= \left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)^n = \left(\frac{n+1}{n}\right)^{n+1} = t_n$$

$$n^2-1 < n^2$$

$$\cancel{n^2}/1$$

$$\frac{1}{n^2-1} > \frac{1}{n^2}$$

$$\frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$$

$$\frac{n}{n^2-1} > \frac{1}{n}$$

$t_{n-1} > t_n$

$\Rightarrow \downarrow$ seq.

Bounded seq:-

A seq $\{S_n\}$ is said to be bounded iff there exist a positive real NO d s.t

$$|S_n| \le d \quad \forall \; n \in N. \text{ i.e}$$

$$-d \le S_n \le d \quad \forall \; n \in N$$

$-d$ = lower bound, d = upper bound

Convergent seq:- in $\mathbb{R}$

A seq $X = (x_n)$ is said to cgs to $x \in \mathbb{R}$ or 'x' is said to be a Limit pt of (x_n) iif for every $\varepsilon > 0$, There exist a natural number $K(\varepsilon)$ s.t $\forall$ $n \geq K(\varepsilon)$ The Term (x_n) satisfy $|x_n - x| < \varepsilon$

e.g $\lim\limits_{n \to \infty} x_n = x$ $\rightarrow$ notation for seq which has limit.

Uniqueness of Limits:-

A cgt seq has one & only one lim pt.

proof:-

Suppose (x_n) cgs to Two limits x' & x''.

for each $\varepsilon > 0$, There exist K' s.t

$|x_n - x'| < \varepsilon/2$ $\forall$ $n \geq K'$; $K' = K(\varepsilon)$

for each $\varepsilon > 0$, $\exists$ K'' s.t

$|x_n - x''| < \varepsilon/2$ $\forall$ $n \geq K''$;

let's Take $K = \max(K', K'')$

Then for $n \geq K$ we apply Triangle inequality

$|x' - x''| = |x' - x_n + x_n - x''|$

$\leq |x' - x_n| + |x_n - x''|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\because \varepsilon > 0$ arbitrary +ve No

$\Rightarrow x' - x'' = 0$

$\Rightarrow x'' = x'$

$\Rightarrow$ Lim is unique.