LECTURE-04

The Extended Real No system:-

The extended real No system consists of real field $\mathbb{R}$ & two symbols $+\infty$ & $-\infty$ i.e (in this system, these symbols treated as actual numbers while in real line $-\infty < x < \infty$ ($-\infty$ & ∞ are just symbols & not considered).

The system is $\Rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Now, we make the following conventions.

(i) If 'x' is real then:-

$x + \infty = \infty$, $x - \infty = -\infty$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0$

(ii) if $x > 0 \Rightarrow x(\infty) = \infty$ & $x(-\infty) = -\infty$

(iii) if $x < 0 \Rightarrow x(-\infty) = \infty$, $x(\infty) = -\infty$

Euclidean Space:-

For each +ve integer 'k', let $\mathbb{R}^k$ be the set of all ordered k-tuple.

i.e

$\mathbb{R}^k = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ (k Times)

if $\underline{x} = (x_1, x_2, \dots, x_k)$

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$\Rightarrow (x_1, x_2, \dots, x_k)$ are real numbers. It's called co-ordinates of $\underline{x}$. The elements of $\mathbb{R}^k$ are called points or vectors specially $k > 1$. Now if $y = (y_1, y_2, \dots, y_k)$ & α is real No.

$$x + y = (x_1, x_2, \dots, x_k) + (y_1, y_2, \dots, y_k)$$
$$= (x_1 + y_1, x_2 + y_2, \dots, x_k + y_k)$$

$$\left.\begin{array}{l} \alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_k), \text{ so} \\ x + y \in \mathbb{R}^k \text{ \& } \alpha x \in \mathbb{R}^k \end{array}\right\} \text{--- (1)}$$

where eq (1) is also def of vector space, so these operations make this $\mathbb{R}^k$ into a vector space over field $\mathbb{R}$.

The inner/scalar/dot prod of x & y is defined as

$$x \cdot y = (x_1 y_1 + x_2 y_2 + \dots + x_k y_k)$$
$$= \sum_{i=1}^{k} x_i y_i \text{ \&}$$

Norm of x is defined as

$$\|x\| = (x \cdot x)^{1/2} = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}$$

$$\{\underbrace{\text{vector space + inner prod + Norm}}_{\text{Euclidean } k\text{-space.}}\}$$

Theorem

(i) $\|x\|^2 = x \cdot x$

$\because \|x\| = (x \cdot x)^{1/2}$

$\Rightarrow \|x\|^2 = \left((x \cdot x)^{1/2}\right)^2 = x \cdot x$

(ii)

$\|x \cdot y\| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz Inequality)

proof:-

for $\lambda \in \mathbb{R}$, choose

$0 \leq \|x - \lambda y\|^2 = (x - \lambda y) \cdot (x - \lambda y)$

$= x(x - \lambda y) + (-\lambda y)(x - \lambda y)$

$= x \cdot x + x \cdot (-\lambda y) + (-\lambda y)x + (-\lambda y) \cdot (-\lambda y)$

$\|x - \lambda y\| \geq 0$

$\|x - \lambda y\|^2 \geq 0$

$= (x+y) \cdot (x+y)$

$x(x+y) + y(x+y$

$= \|x\|^2 - 2\lambda (x \cdot y) + \lambda^2 \|y\|^2$ — (1)

Now put $\lambda = \dfrac{x \cdot y}{\|y\|^2}$ in (1); as

(1) $\Rightarrow 0 \leq \|x\|^2 - 2\dfrac{(x \cdot y)}{\|y\|^2} \cdot (x \cdot y) + \dfrac{(x \cdot y)^2}{\|y\|^4} \cdot \cancel{\|y\|^2}$

$0 \leq \|x\|^2 - \dfrac{2(x \cdot y)^2}{\|y\|^2} + \dfrac{(x \cdot y)^2}{\|y\|^2}$

$0 \leq \|x\|^2 - \dfrac{(x \cdot y)^2}{\|y\|^2}$

$\Rightarrow 0 \leq \|x\|^2 \cdot \|y\|^2 - \|x \cdot y\|^2$

$\Rightarrow \|x\|^2 \cdot \|y\|^2 - \|x \cdot y\|^2 \geq 0$

$-\|x \cdot y\|^2 \geq -\|x\|^2 \cdot \|y\|^2$

$\Rightarrow \|x \cdot y\|^2 \leq \|x\|^2 \cdot \|y\|^2$

$\Rightarrow \|x \cdot y\| \leq \|x\| \cdot \|y\|$

Assignment (H.w)

(a) $\|x+y\| \leq \|x\| + \|y\|$

(b) $\|x-z\| \leq \|x-y\| + \|y-z\|$

Hint for (b) $\rightarrow$ replace x by $x-y$ & y by $y-z$ in (a).

Question

If r is rational & x is irrational Then prove that

(a)

$r+x$ is irrational

proof:-

let $r+x$ be rational

$\Rightarrow r+x = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, $(a, b) = 1$

$\Rightarrow x = \frac{a}{b} - r$ —— (1)

$\because r$ is rational, so $r = \frac{c}{d}$, $c, d \in \mathbb{Z}$, $d \neq 0$, $(c, d) = 1$

(1) $\Rightarrow x = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

$\Rightarrow x$ is rational, which is contradiction to the fact that x is irrational.

(ii)

$+x$ is irrational (Do it by your self).

<u>Question</u>

If 'n' is +ve integer which is not perfect square then prove that $\sqrt{n}$ is irrational.

<u>Proof</u>:-

Case-I $\Rightarrow$ when 'n' contain no square factor greater than '1'. let's suppose $\sqrt{n}$ is rational

$\Rightarrow \sqrt{n} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $(p, q) = 1$

$\Rightarrow n = \frac{p^2}{q^2}$ $\Rightarrow p^2 = nq^2$ — (1)

$q^2 = p^2/n$

$\Rightarrow n \mid p^2 \Rightarrow n \mid p$ — (2)

Now suppose that $\frac{p}{n} = c$, $c \in \mathbb{Z}$

$\Rightarrow p = nc \Rightarrow p^2 = n^2c^2$ put in (1)

(1) $\Rightarrow n^2c^2 = nq^2$

$c^2 = q^2/n \Rightarrow n \mid q^2$

$\Rightarrow n \mid q$ — (3)

from (2) & (3), we get
$(p,q)=n$, which is contradiction to the fact that $(p,q)=1$, so $\sqrt{n}$ is irrational.

Case-II: when 'n' contain square factor greater than '1'.
let's suppose that
$n=k^2 m, \quad m>1$
$\Rightarrow \sqrt{n}=k\sqrt{m}$
k is rational, $\sqrt{m}$ is irrational
prod is irrational
$\Rightarrow \sqrt{n}$ is irrational.

Assignment:-
prove that: $\sqrt{12}$ & $\sqrt{32}$ are irrational.

Question

let E be non-empty set & suppose 'α' is lower bound of E & β is upper bound of E. Prove that
$\alpha \le \beta$

Proof:-

$\because$ E is non-empty so $x \in E$

$\because$ α is LB of E

$\Rightarrow \alpha \leq x$ — 1

$\because$ β is UB of E

$x \leq \beta$ — 2

from 1 & 2

$\Rightarrow \alpha \leq x \leq \beta$

$\Rightarrow \alpha \leq \beta$

{ if $x < y$ & $y < z$ $\Rightarrow x < z$