# Existence of Real field:-

There exists an ordered field $\mathbb{R}$ which has LUB property & it contains $\mathbb{Q}$ (as sub-field).

## Theorem:-

(a) If $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x > 0$, $\exists$ a +ve integer 'n' s.t. $nx > y$ that is called Archmedean property.

Lecture-03

proof:-

let $A = \{nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R}\}$

suppose that given statement is false i.e

$nx \leq y$

$\Rightarrow y$ is an upper bound of $A$ $\left(\begin{matrix} x \leq \beta \\ x \in E \\ \beta \in S \\ \beta \text{ is } UB \end{matrix}\right)$

Since we are dealing with the set of reals, therefore it has the LUB property.

let $\alpha = \sup A$

$\because x > 0 \Rightarrow -x < 0$

$\Rightarrow \alpha - x < \alpha + 0$

$\alpha - x < \alpha$ (if $\alpha = \sup A$. Then any element less then $\alpha$ is not upper bound). so

$\Rightarrow \alpha - x$ is not upper bound of $A$.

$\Rightarrow \alpha - x < mx$ where $mx \in A$ for some +ve integer $m$.

$\Rightarrow \alpha - \cancel{x} + x < mx + x$

$\Rightarrow \alpha < (m+1)x$ where $(m+1)$ is integer so $(m+1)x \in A$

so which is impossible b/c $\alpha$ is LUB of A i.e $\alpha = \sup A$.
so our contradiction is wrong & given statement is right.

(b) Given any number $x \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ ~~s.t~~ satisfying $n > x$

(c)

Given any $y \in \mathbb{R}$, $y > 0$, $\exists\ n \in \mathbb{N}$, s.t $0 < \frac{1}{n} < y$

(d)

Given any real No $x \in \mathbb{R}$, $\exists\ m \in \mathbb{Z}$, s.t $m-1 \leq x < m$

Density of $\mathbb{Q}$ in $\mathbb{R}$

If $x, y \in \mathbb{R}$, $x < y$ Then $\exists\ p \in \mathbb{Q}$ s.t $x < p < y$, i.e b/w any two real, there is a rational or $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof..

$\because x < y$ then we can write

$\Rightarrow x - y < 0 \Rightarrow y - x > 0$

Now by using (c) ; as

$y - x > 0$ so $\exists\, n \in \mathbb{N}$, s.t

$\frac{1}{n} < (y - x)$ or

$n > \frac{1}{y - x}$

$\Rightarrow n(y - x) > 1$

$\Rightarrow ny - nx > 1$

$\Rightarrow ny > 1 + nx$ — 1

Now by (d), ; as $x \in \mathbb{R}$, $x > 0$ $\exists\, m \in \mathbb{Z}$

s.t $m - 1 \leq x < m$

Here as $n > 0$, $x > 0$, $nx > 0$, so

$\Rightarrow m - 1 \leq \underbrace{nx < m}$

$\Rightarrow nx < m$ — 2 & from

$\underbrace{m - 1 \leq nx} < m$

$\Rightarrow m \leq 1 + nx$ — 3

from 1 & 3

$\Rightarrow m \leq 1 + nx < ny$

$\Rightarrow m < ny$ — 4 so from 2 & 4

$\Rightarrow nx < m < ny$

$\Rightarrow x < \frac{m}{n} < y \Rightarrow x < p < y.$

proved.

H.W ⇒ b/w any two reals, there exist irrational i.e

$x < u < y$

Hint:

(i) if we divide any $x \in \mathbb{R}$ by any irrational, result is still real.

(ii) prod of rational & irrational is irrational

Theorem :-

For every real $x$, there is a set $E$ of rational numbers s.t

$x = \sup E$.

Proof:-

let $E = \{q \in \mathbb{Q} : q < x\}$

Then $E$ is bounded above.

$\because E \subset \mathbb{R}$, therefore $\sup E$ exists in $\mathbb{R}$. Now suppose $\sup E = d$

$\Rightarrow d \le x$

if $d = x$ then nothing to prove

$E \subset S$
$\forall x \in E$
$\beta \in S$
$x \le \beta$
$E$ is sup set
$\beta$ is sup

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if $d < x$, $\exists$, a $q \in Q$ s.t $d < q < x$, which is not possible b/c as def:-

if $x$ is upper bound of $E$ & if there is any upper bound $d$ Then $x \le d$, so

$$x = \sup E$$

Theorem:-

For every real $x > 0$, every integer $n > 0$, there is one & only one real '$y$' s.t $y^n = x$

proof:-

let $y_1, y_2 \in \mathbb{R}$ s.t $0 < y_1 < y_2$

$\Rightarrow (y_1)^n < (y_2)^n$ i.e there is almost one $y \in \mathbb{R}$ s.t

$y^n = x$ (uniqueness of '$y$').

let's suppose that $E$ be the set of all the reals s.t $t^n < x$, i.e

$E = \{t : t \in \mathbb{R},\ t^n < x\}$

$\because x > 0 \Rightarrow 1 + x > x > 0$

$\Rightarrow 0 < x < 1+x$

$\Rightarrow 0 < \frac{x}{1+x} < \frac{1+x}{1+x}$ ($\div$ by $1+x$).

$\Rightarrow 0 < \frac{x}{1+x} < 1$

Then let $t = \frac{x}{1+x}$

$\Rightarrow 0 < t < 1$.

$\Rightarrow t^n < t$ & $t^n < x$

$\Rightarrow t^n < t < x$

$t = 0.5$
$t^3 = 0.125$
$t^3 < t$ ($n=3$, $t^n < t$)

$\Rightarrow t \in E$ & $E$ is non-empty

Now, consider;

if $t > 1+x$ Then $t^n > t > x$, so $t \notin E$.

$\Rightarrow 1+x$ is an upper bound of $E$

($\because x$ is upper bound. Then $1+x$ is upper bound of $x \leq 1+x$).

so $E$ is non-empty & bounded above. Therefore $\sup E$ exist.

~~Th~~ Take $y = \sup E$

To show that $y^n = x$, ~~we~~ we will show that each of The inequality $y^n < x$ & $y^n > x$ leads to the contradiction.

consider

$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \dots + a^{n-1})$, $n \in \mathbb{Z}$.

which yields inequality (each $a$ is replaced by $b$ in R.H.S of above)

$\cancel{b^n - a^n < (b-a)(nb^{n-1})}$ — ① , $0<a<b$

$a>0$
$b>0$
$a<b$
$b-a>0$

$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}b + b^{n-3}b^2 + \dots + b^{n-1})$

$b^n - a^n = (b-a)(b^{n-1} + b^{n-1} + b^{n-1} + \dots + b^{n-1})$

$b^n - a^n < (b-a)(nb^{n-1})$ — ① $0<a<b$.

Now suppose $y^n < x$

choose $h$ so that $0<h<1$

& $h < \dfrac{x-y^n}{n(y+1)^{n-1}}$

$\because y^n < x$, $x>0$

$x - y^n > 0$

put $a=y$ & $b=y+h$ in ①; as

$\cancel{(y+h)^n - y^n < (y+h)}$

① $\Rightarrow nb^{n-1} > 0$

$\Rightarrow n(y+1)^{n-1} > 0$

$\dfrac{x-y^n}{n(y+1)^{n-1}} > 0$

$(y+h)^n - y^n < (y+h-y)(n)(y+h)^{n-1}$

$< nh(y+1)^{n-1}$

$(\because h<1)$

as $h>0$

$\dfrac{x-y^n}{n(y+1)^{n-1}} > h$

$< x - y^n$

$\Rightarrow (y+h)^n < x - \cancel{y^n} + \cancel{y^n}$

$\Rightarrow (y+h)^n < x$

$\Rightarrow y+h \in E$

$\because y+h > y$ so it contradicts the fact that $y$ is sup $E$.

$\Rightarrow y^n < x$ is impossible.

Now suppose $y^n > x$

put $k = \dfrac{y^n - x}{ny^{n-1}}$, $0 < k < y$

$h < \dfrac{x - y^n}{n(y+1)^{n-1}}$

$nh(y+1)^{n-1} < x - y^n$

if $t \geq y-k \Rightarrow y-k \leq t$

① $\Rightarrow b^n - a^n < (b-a)(nb^{n-1})$

Now if $b = y$, $a = t$

$\Rightarrow y^n - t^n < (y-t)(ny^{n-1})$

$\leq kny^{n-1}$

$= y^n - x$

$-t^n < \cancel{y^n} - x - \cancel{y^n}$

$\Rightarrow -t^n < -x \Rightarrow t^n > x$ 4

$t \notin E$

$\Rightarrow y-k$ is upper bound of $E$ but $y-k < y$ which contradict that $y = \sup E$, so

$y^n = x$

$y-k \leq t$

$y-t \leq k$

$\therefore h < \cancel{\frac{x-y^n}{}}$

$k = \frac{y^n - x}{ny^{n-1}}$

$nky^{n-1} = y^n - x$