Lecture: 02

~~Sum~~ Supremum (Least upper Bound):-

let 'S' be a non-empty subset of $\mathbb{R}$.
if S is bounded above, Then a number α is said to be a supremum of S if it satisfies:-

i) α is an upper bound of S.

ii) if γ is any upper bound of S, Then $\alpha \leq \gamma$

$S = \{1, 2, 3\}$
$\alpha = 3$ if $\gamma = 4$
$\alpha \leq \gamma$

Infimum:-

i) α' is lower bound of S

ii) If γ' is any lower bound of S, Then $\gamma' \leq \alpha'$

$\mathbb{R}$

1 2 3

$\alpha' = 1$
$\gamma' = 0$ $\gamma' < \alpha'$

Example:

$E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$, find sup & inf
$E_2 = \{r : r \in \mathbb{Q} \wedge r \geq 0\}$ of E_1 & E_2

Sol:-

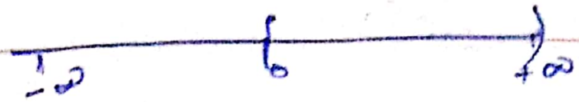
for E_1:

E_1

$-\infty$ 0 $+\infty$

upper bounds of E_1 = ~~(-∞, 0]~~ $[0, \infty)$
→ E_1 is bounded above, & upper bounds of E_1 are numbers of E_2, al

for E_2:



so

we can observe that '0' is the smallest number of E_2 or '0' is least upper bound for which according to definition:-

$x \leq \beta \ \forall x \in E_1$, $\beta \in S$, $\beta = '0'$.

$\Rightarrow$ '0' is sup of E_1.

As E_1 has no greatest lower bound so inf $E_1 = \infty$ doesn't exist. (Infact E_1 is not bouned below).

$E_2 = [0, \infty) \Rightarrow$ lower bounds $= (-\infty, 0]$

$\Rightarrow E_2$ has no smallest number, except '0' so

inf $E_2 = 0$ & sup E_2 = not exist as it has no upper bound.

Example:-

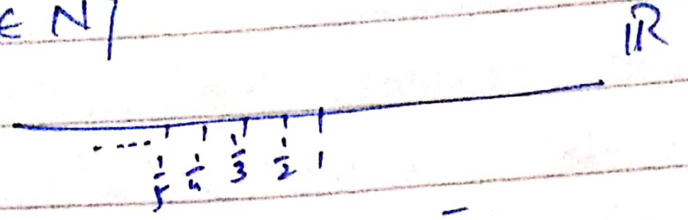
let E be the set of all numbers of the form $\frac{1}{n}$, where $n \in N$

i.e $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

find sup, inf of E.

Sol:-

$E = \{ \frac{1}{n} : n \in N \}$



Here E is ordered set ; as

$E : 1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots$

~~g.l.b~~ = $\sup E = 1$ as all other numbers are less than '1'.

g.l.b = inf = 0 as '0' is the only number from which all numbers are greater while '0' $\notin E$, '0' is so b/c as $n \to \infty$ so that $\frac{1}{n} \to 0$

Theorem:

Suppose S is an ordered set with LUB property. $B \subset S$, B is non-empty & is bounded below. let L be the set of all lower bounds of B Then $a = \sup L$ exists in S & also $a = \inf B$

OR

A set which has LUB also has GLB property.

Proof:-

$\because$ B is bounded below, therefore L is non-empty

$\Rightarrow$ L consists of those $y \in S$ which satisfy the condition

$y \le x \ \forall x \in B$ (b/c L is set of all lower bounds of B)

$\Rightarrow$ every $x \in B$ is an upper bound of L. (as $y \in L$)

$\Rightarrow$ L is bounded above.

Now as S is ordered & non-empty so L has sup in S, that is

$a = \sup L$

If $g < a$, then g is not upper bound of L. (as per def)

$\Rightarrow g \notin B$

$\Rightarrow a \le x \ \forall x \in B$

$\Rightarrow a \in L$

Now if $a < b$

Then $b \notin L$, $a = \sup L$

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$B = \{4, 5, 6\}$

$L = \{4, 3, 2, 1, \dots\}$

upper bounds of $L = (-\infty, 4)$

$\sup L = 4$, $g = 3$, $a = 4$

if $b = 5$, $b \notin L$

$g \notin B \Rightarrow 3 \notin B$

$\inf B = 4 = a$.

$\sup B = 6 \in S$,

$4 \in S$

We have show that $a \in L$

but $b \notin L$, so

$a = \inf B$

proved.

Assignment

1

$A = \{p : p \in Q, p^2 < 2\}$

$B = \{p : p \in Q, p^2 > 2\}$

find inf & sup of A & B

2

$S = \{x \in \mathbb{R}, x \geq 0\}$

show that

$\inf S = 0$ & find sup.

Field :-

A set F is field if it is abelian group w.r.t '+' & '×', and also holds distributive property.

Theorem:

(a) If $x+y = x+z$ then $y = z$

proof

suppose $x+y = x+z$ —— ①

$\because \quad y = 0+y$

$= (-x+x)+y \qquad \because -x+x=0$

$= -x+(x+y) \quad$ (asso prop)

$= -x+(x+z) \quad$ (from ①)

$= (-x+x)+z \quad$ (asso prop

$= 0+z \qquad \because -x+x=0$

$= z$

so $y = z$

(b)

if $x+y = x$, Then $y=0$

take $z=0$ in (a)

(a) $\Rightarrow \quad x+y = x+0$

$\Rightarrow x+y = x$

$\Rightarrow x-x+y = x-x = 0$

(c) If $x+y=0$ Then $y=-x$

take $z=-x$ in (a)

(d)

$-(-x)=x$ (do it by your self)

Theorem:-

axioms of multiplication implies

(i) If $x \neq 0$ & $xy=xz$ Then $y=z$

proof

suppose $xy=xz$ —— (1)

$\because y = 1 \cdot y = \left(\frac{1}{x} \cdot x\right) y$ $\quad \because \frac{1}{x} \cdot x = 1$

$= \frac{1}{x}(xy)$ (asso)

$= \frac{1}{x}(xz)$ (from (1))

$= \left(\frac{1}{x} \cdot x\right) z$ (asso)

$= 1z = z$

$y = z$

(ii)

If $x \neq 0$, $xy = x$, Then $y = 1$

take $z = 1$ in (i)

(1) $\Rightarrow xy = x(1) = x$

$\frac{\cancel{x}y}{\cancel{x}} = \frac{x}{\cancel{x}}$

$y = 1$

(iii) If $x \neq 0$, Then $\frac{1}{1/x} = x$

(iv) If $x \neq 0$, $xy = 1$ Then $y = \frac{1}{x}$

(Do it by your self).

Theorem:

axioms of field imply —

(i) $0 \cdot x = 0$

$0 \cdot x + 0 \cdot x = (0+0)x$

$\Rightarrow 0x + 0x = 0x$

$\Rightarrow 0x = 0$

(ii)

If ~~x+y~~ $x \neq 0, y \neq 0$, Then $x \cdot y \neq 0$

(by your self)

(iii)

$(-x)y = -(xy) = x(-y)$

proof

$\because (-x)y + xy = (-x+x)y = 0 \cdot y = 0$ — (1)

also $x(-y) + xy = x(-y+y) = x \cdot 0 = 0$ — (2)

Also $-(xy) + xy = 0$ — (3)

Combining (1) & (2); as

$(-x)y + xy = x(-y) + xy$

$\Rightarrow (-x)y = x(-y)$ — (4)

Scanned by CamScanner

Combining 4 ② & ③ : as

$x(-y) + xy = -(xy) + xy$

$x(-y) = -(xy)$ — ⑤

from ④ & ⑤

$(-x)y = x(-y) = -xy$

Ordered field:-

An ordered field F is field which is also an ordered set. s.t the following properties hold.

$x+y < x+z$ if $x, y, z \in F$, $y < z$

$xy > 0$ if $x, y \in F$, $x > 0, y > 0$

e.g $\mathbb{Q}$ is an ordered field.

Theorem:-

(i) if $x > 0$, Then $-x < 0$ & if $x < 0$, $-x > 0$

(ii) If $x > 0$, $y < z$ Then $xy < xz$

(iii) If $x < 0$, $y < z$, Then $xy > xz$

(iv) If $x \neq 0$, Then $x^2 > 0$ in particular $1 > 0$

(do it by yourself)

(v)

If $0 < x < y$ Then $0 < \frac{1}{y} < \frac{1}{x}$

Scanned by CamScanner

proof

If $y>0$ & $v\le 0$

$\Rightarrow yv \le 0$

but $y\left(\frac{1}{y}\right)=1>0$

$\Rightarrow \frac{1}{y}>0$

likewise $\frac{1}{x}>0$ as $x>0$

$y>0, v\le 0$

$y=1, v=-2$

$yv=(1)(-2)=-2$

$-2\le 0$

If we multiply b.s of inequality by the +ve quantity $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$

$\Rightarrow \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$

$\frac{1}{y} < \frac{1}{x}$

finally

$0<\frac{1}{y}<\frac{1}{x}$

Existence of Real field:-

There exists an ordered field $\mathbb{R}$ which has LUB property & it contains $\mathbb{Q}$ (as sub-field).

Theorem:-

(a) If $x\in\mathbb{R}$, $y\in\mathbb{R}$, $x>0$, $\exists$ a +ve integer 'n' s.t $nx>y$ that is called Archmedean property.

proof:-

let $A = \{nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R}\}$
suppose that given statement is
false i.e

$nx \leq y$

$\Rightarrow y$ is an upper bound of A $\begin{pmatrix} x \leq \beta \\ x \in E \\ \beta \in S \\ \beta \text{ is } UB \end{pmatrix}$
Since we are dealing with
the set of reals, therefore it
has the LUB property.

let $\alpha = \sup A$

$\because x > 0 \Rightarrow -x < 0$

$\Rightarrow \alpha - x < \alpha + 0$

$\alpha - x < \alpha$ (if $\alpha = \sup A$, Then
any element less then α is not
upper bound). so

$\Rightarrow \alpha - x$ is not upper bound, of A.

$\Rightarrow \alpha - x < mx$ where $mx \in A$ for
some +ve integer m.

$\Rightarrow \alpha - \cancel{x} + x < mx + \cancel{x}$

$\Rightarrow \alpha < (m+1)x$ where $(m+1)$
is integer so $(m+1)x \in A$

so which is impossible b/c α is LUB of A i.e $\alpha = \sup A$.
so our contradiction is wrong & given statement is right.