Real Analysis

Chapter # 01
Real Number System

Theorem:-
There is no rational p s.t $p^2 = 2$

proof:-
we will prove it by contradiction.
let us suppose that there exist a rational p s.t $p^2 = 2$,
we can write ; as

$p = \frac{m}{n}$ —① , $m, n \in \mathbb{Z}$, where m, n has no common factor. Then

$p^2 = 2$

$\Rightarrow \left(\frac{m}{n}\right)^2 = 2$ (from ①)

$\frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even

$\Rightarrow m$ is divisible by 2, so m^2 is divisible by 4.

$\Rightarrow 2n^2$ is divisible by 4 so n^2 is divisible by 2.

$\Rightarrow n^2$ is even

$\Rightarrow$ n is even
$\Rightarrow$ as m & n are both even &
has common factor 2 which is
contradiction, that (m, n) has no
common factor, so our hypothesis
is wrong & there is no 'p'
s.t. $p^2 = 2$ or $p = \sqrt{2}$
$\therefore$ Hence, 'p' is irrational number.

Theorem:-

let A be the set of all +ve rationals p s.t $p^2<2$ & let B contains of rationals p s.t $p^2>2$, then A contain no largest number & B contains no smallest.

Proof:-

Here, according to given condition:-

$A=\{p: p\in Q \wedge p^2<2\}$

$B=\{p: p\in Q \wedge p^2>2\}$

we have to show that for every 'p' in A, there exist a rational $q\in A$ s.t $p<q$ & for all

$p \in B$, we can find rational $q \in B$ s.t $q < p$.

associate with each rational $p > 0$, the number

$$q = p - \frac{p^2 - 2}{p+2} = \frac{\cancel{p^2} + 2p - \cancel{p^2} + 2}{p+2}$$

$$\boxed{q = \frac{2p+2}{p+2}} \quad \text{--- (1)}$$

Now $q^2 - 2 = \left(\frac{2p+2}{p+2}\right)^2 - 2$

$$= \frac{4p^2 + 4 + 8p}{p^2 + 4 + 4p} - 2$$

$$= \frac{4p^2 + 4 + 8p - 2p^2 - 8p - 8}{p^2 + 4 + 4p}$$

$$= \frac{2p^2 - 4}{(p+2)^2} = \frac{2(p^2 - 2)}{(p+2)^2}$$

$$\boxed{q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2}} \quad \text{--- (2)}$$

Now if $p \in A$ Then $p^2 < 2$

$\Rightarrow p^2 - 2 < 0$

from ① $\Rightarrow q = p - \frac{p^2-2}{p+2}$, $q > p$
&

$$\frac{2(p^2-2)}{(p+2)^2} < 0$$

$\Rightarrow q^2 - 2 < 0$ (from ②)

$\Rightarrow q^2 < 2$

$\Rightarrow q \in A$

Now if $p \in B$ Then $p^2 > 2$

$\Rightarrow p^2 - 2 > 0$

from ①;-

$$q = p - \frac{p^2-2}{p+2} \Rightarrow q < p$$

$$\Rightarrow \frac{2(p^2-2)}{(p+2)^2} > 0 \Rightarrow q^2 - 2 > 0 \text{ (from ②)}$$

$\Rightarrow q^2 > 2 \Rightarrow q \in B$

The purpose is to show that The rational Number system has certain gap. These are filled by irrationals

so

There is always irrational b/w Two rational.

Order of a set:-

let 'S' be a non empty set. An order on 'S' is a relation denoted by "$\leq$" with the following two properties.

i) If $x \in S$ & $y \in S$, then one & only one statement

ii) $x<y$, $x=y$, $y<x$ is true.

If $x, y, z \in S$ & if $x<y$, $y<z$

$\Rightarrow x<z$

Ordered set:-

A set 'S' is said to be ordered if an order is defined on S.

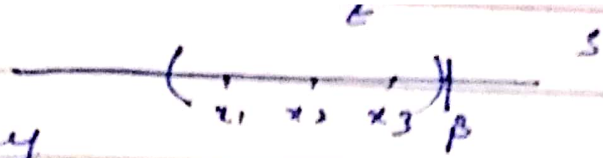
e.g: set of Naturals.

Bound:-

let 'S' be an ordered & E is subset of S. ($E \subset S$). If there exist $\beta \in S$ s.t

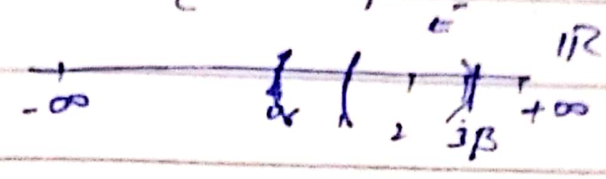
$x \leq \beta \;\forall\; x \in E$, Then

E is bounded above & β is upper bound of E



let's say

$S = \mathbb{R}$, $E = \{1, 2, 3\}$

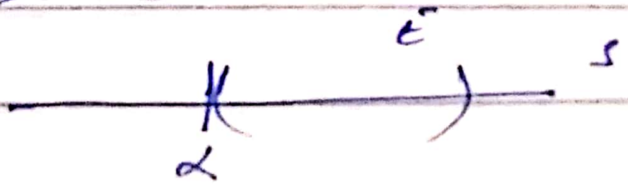


$\because x \le \beta \ \forall x \in E$
$\beta \in S$

upper bound $= [3, \infty)$

Lower Bound:-

if there exist $\alpha \in S$ s.t
$\alpha \le x \ \forall x \in E$, then
E is bounded below & α is
lower bound



lower bound $= (-\infty, 1]$ $\because \alpha \le x \ \forall x \in E$
$\alpha \in S$