Lecture

Commutator of Given Dynamical variables 4 momentum operator can be written as $[q_j, P_k] = i\hbar\,\delta_{jk}$

For it, first we define kronecker delta

$$\delta_{ij} = (\psi_i, \psi_j) = 1; \quad (i = j)$$
$$= \quad " \quad " = 0; \quad (i \neq j)$$

This is orthonormal condition

Already, we have proved that

$$[P_x, x] = -i\hbar \quad \text{or} \quad [x, P_x] = i\hbar$$

we generalize it

$$[x, P_x] = [y, P_y] = [z, P_z] \quad \text{—} (1)$$

Now we check $[y, P_z]$

By definition

$$[y, P_z]\psi = (yP_z - P_z y)\psi$$
$$= \left(y \frac{\hbar}{i}\frac{\partial}{\partial z} - \frac{\hbar}{i}\frac{\partial y}{\partial z}\right)\psi$$
$$= y\frac{\hbar}{i}\frac{\partial \psi}{\partial z} - \frac{\hbar}{i}\frac{\partial}{\partial z}(y\psi)$$
$$= y\frac{\hbar}{i}\frac{\partial \psi}{\partial z} - \frac{\hbar}{i}(\psi\cdot 0) - \frac{\hbar}{i}y\frac{\partial \psi}{\partial z}$$
$$= y\frac{\hbar}{i}\frac{\partial \psi}{\partial z} - \frac{\hbar}{i}y\frac{\partial \psi}{\partial z}$$
$$= y\frac{\hbar}{i}\frac{\partial \psi}{\partial z} - y\frac{\hbar}{i}\frac{\partial \psi}{\partial z} = 0$$

$$[y, P_z] = 0$$

we generalize it; as (2)

$$[x, P_y] = [x, P_z] = [y, P_x] = [y, P_z] = [z, P_x] = [z, P_y] = 0$$

combing (1) & (2); we get; as

$$[q_j, P_k] = i\hbar \quad , (j = k)$$
$$= 0 \quad ; (j \neq k)$$

$$\Rightarrow [q_j, P_k] = i\hbar\delta_{jk}$$

To check functions are linear or non-L, operator :-

Operator :-

An operator is a rule which when applied on a fn, changes its value of fn. As T is an op,

$$T[f(x)] = g(x)$$

As square of a fn is an operator

$$T[f(x)] = [f(x)]^2$$

Prove That

$$T\psi = \frac{d\psi}{dx} + 2\psi \text{ is Lin-op}$$

proof:-

For linearity, we write

$$T(c_1\psi_1 + c_2\psi_2) = c_1T\psi_1 + c_2T\psi_2$$

we use above given fn

L.H.S:-

$T(c_1\psi_1+c_2\psi_2) = \frac{d}{dx}(c_1\psi_1+c_2\psi_2)+2(c_1\psi_1+c_2\psi_2)$

$= c_1\frac{d}{dx}\psi_1 + c_2\frac{d\psi_2}{dx} + 2c_1\psi_1 + 2c_2\psi_2$

$= c_1\left(\frac{d\psi_1}{dx}+2\psi_1\right) + c_2\left(\frac{d\psi_2}{dx}+2\psi_2\right)$

$= c_1 T\psi_1 + c_2 T\psi_2$

$=$ R.H.S

so given op is Lin-op.

Prove that:

square of fn is not Lineal.

Proof:-

$T\psi = \psi^2$

L.H.S:-

$T\psi = T(c_1\psi_1+c_2\psi_2) = (c_1\psi_1+c_2\psi_2)^2$

$= c_1^2\psi_1^2 + c_2^2\psi_2^2 + 2c_1c_2\psi_1\psi_2$

R.H.S:-

$c_1T\psi_1 + c_2T\psi_2$

$= c_1\psi_1^2 + c_2\psi_2^2$

as L.H.S $\neq$ R.H.S, so

$T\psi = \psi^2$ is not Lin-op.

Assigment:-

$T\psi : \psi + x$ is not a Lin-op

Poisson Brackets:-

The observable physical quantity is under rule of Hamilton & Matrix.

Let P & Q be two operators, they can be written; as

$$[P, Q] = i\hbar \{p, q\}$$

Here p & q be dynamical variables. P & q are written in brackets called poisson Brackets.

Similarly in Q.M, there are many operators.

Normalization Constant & Equation:-

Schrodinger equation is:-

$H\psi = E\psi$, ψ is its sol which may be $N\psi$, where N is const.

$$\int (N\psi)^* (N\psi)\, d^3r = 1$$

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$$= |N|^2 \int |\psi|^2 d^3r = 1$$

This equation is called Normalization equation & N is called Normalization const.

Example:

Normalization constant of $\psi(x) = \exp(-ax^2)$ is $N = \left(\frac{2a}{\pi}\right)^{1/4}$, where a is real const.

Sol:-

To find Normalization constant, we can write; as

$$N\psi = N\exp(-ax^2)$$

Normalization equation is:-

$$|N|^2 \int |\psi|^2 dx = 1$$

$$|N|^2 \int \{\exp(-ax^2)\}^2 dx = 1$$

$\exp(x) = e^x$

$\{\exp(x)\}^2 = e^{2x}$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\frac{N^2}{\sqrt{2a}}(\sqrt{\pi}) = 1 \quad \text{or}$$

$$N = \left(\frac{2a}{\pi}\right)^{1/4}$$

proved

Example: A wave fn is given ; as $\psi(x) = \sin\left(\frac{n\pi x}{L}\right)$ of a particle moving in range of $0 \le x \le L$. Prove that Normalization constant $N = \sqrt{\frac{2}{L}}$ also prove that $\langle P_x \rangle = 0$

Proof:-

First we write Normalization equation

$$N^2 \int |\psi|^2 dx = 1$$

$$N^2 \int_0^L \sin^2 \frac{n\pi x}{L} \, dx = 1$$

$$N^2 \int_0^L \left(\frac{1 - \cos \frac{2n\pi x}{L}}{2} \right) dx = 1$$

$$\frac{N^2}{2} \left(\int_0^L 1 \, dx - \int_0^L \left(1 \cos \frac{2n\pi x}{L} \right) dx \right) = 1$$

$$\frac{N^2}{2} \left(L - \frac{L}{2n\pi} \sin \frac{2n\pi L}{L} \right) = 1$$

$$\frac{N^2}{2} (L - 0) = 1$$

$$\frac{N^2}{2} (L) = 1 \Rightarrow N^2 = \frac{2}{L}$$

$$N = \sqrt{\frac{2}{L}}$$

proved

Now

(v)

We find average value of momentum.

$$\langle p \rangle = \int_0^L (N\psi)^* p (N\psi)\, dx$$

$$= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot \frac{\hbar}{i}\frac{\partial}{\partial x}\left(\sin\left(\frac{n\pi x}{L}\right)\right) dx$$

$$= \frac{\hbar}{i}\frac{2}{L}\cdot\frac{n\pi}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\hbar}{i}\frac{n\pi}{L^2}\int_0^L \sin 2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\hbar}{i}\frac{n\pi}{L^2}\left\{\frac{-\cos 2\left(\frac{n\pi}{L}\right)x}{2(n\pi/L)}\right\}\Bigg|_0^L$$

$$= \frac{\hbar}{2iL}\left(-1-(-1)\right)$$

$$= \frac{\hbar}{2iL}(-1+1) = \frac{\hbar}{2iL}(0) = 0$$

$$\langle p \rangle = 0$$

proved.