Lecture-09

1

Definition of Heisenburg Uncertainty :-

Heisenburg uncertainty states :-

"If Accuracy in one quantity higher than accuracy in other quantity is lower."

This is interesting & important result of Q.M. But in C.M, position & momentum are measured with an abitrary accuracy, This is not the case in Q.M.

Heisenberg Uncertainty Principle :-

Statement :-

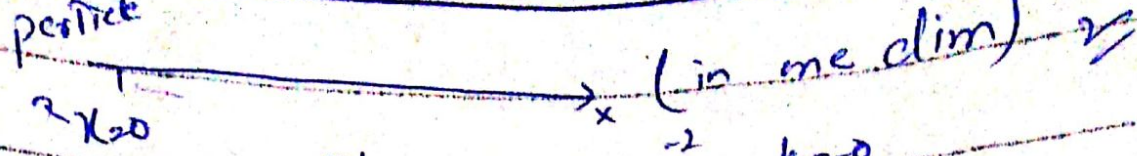
Uncertainty in position & momentum can never be equal to zero.

$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$ (positive Number).

proof :-

Let ψ be a wave fn in one dim.

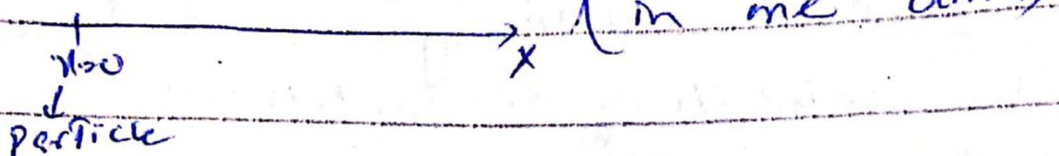
To prove it, we take some assumptions.



m = 5kg , a = 9ms^{-2} , t=0

after 5 seconds, where will be the particle,

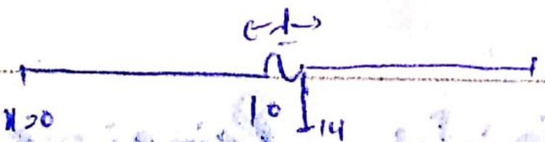
$x = ut + \frac{1}{2}at^2$

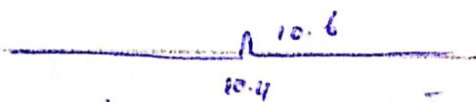


m=5kg , v=50m/s

p=mv (but in C.M)

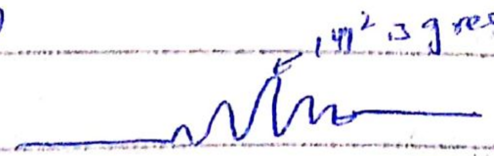
Now





(If we know exact wavelength, then we can't determine position (there will be some error) & vice versa

Now



(possibility of existance of particle)

let $\langle x \rangle = 0$ & also $\langle p \rangle = 0$ }

let's consider $\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x \psi \, dx$

$$\int_{-\infty}^{\infty} -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx = -\frac{\hbar}{i}\psi^* x \psi \Big|_{-\infty}^{\infty} - \left(-\frac{\hbar}{i}\right)\int_{-\infty}^{\infty} \psi^*\left(\psi + x\frac{\partial \psi}{\partial x}\right) dx$$

(we have applied formula of integration by parts)

$$\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx = \frac{\hbar}{i}\int |\psi|^2 dx - \int -\frac{\hbar}{i}\psi^* x \frac{\partial \psi}{\partial x} dx$$

$$\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx = \frac{\hbar}{i}\int |\psi|^2 dx - \left(\int \frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx\right)^*$$

$$\frac{\hbar}{i}\int |\psi|^2 dx = \int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx - \left[\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx\right]^*$$

$$\left\{ \because z - \bar{z} = 2i\, \mathrm{Im}\, z \;, \; \int |\psi|^2 dx = 1 \quad \mathrm{Im}\, z = b \Rightarrow |\mathrm{Im}\, z| = \sqrt{b^2} \right\}$$

$$\frac{\hbar}{i}(1) = 2i\,\mathrm{Im}\left[\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx\right]$$

where Im is an imaginary part

Taking moduli & square, as

$$\frac{\hbar^2}{4} = \left|\mathrm{Im}\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} \cdot x\psi \, dx\right|^2$$

we formed right hand side as
1 state, as we know that

$|Im\, z| \le |z|$; so

$$\frac{\hbar^2}{4} \le \left|\int -\frac{\hbar}{i}\frac{\partial \psi^*}{\partial x} x\psi \, dx\right|^2$$

$$\frac{\hbar^2}{4} \le \left|\int \left(\frac{\hbar}{i}\frac{\partial \psi}{\partial x}\right)^* x\psi \, dx\right|^2$$

applying Schwarz Inequality, we get

$$\frac{\hbar^2}{4} \le \left(\int \left|\frac{\hbar}{i}\frac{\partial \psi}{\partial x}\right|^2 dx\right)\left(\int |x\psi|^2 dx\right)$$

$$\because (\Delta x)^2 = \int |(x-\langle x\rangle)|^2 |\psi|^2 \, dx$$

$$\frac{\hbar^2}{4} \le \left(\int |(p-\langle p\rangle)\psi|^2 dx\right)\left(\int |(x-\langle x\rangle)\psi|^2 dx\right)$$

$$\frac{\hbar^2}{4} \le (\Delta p)^2 (\Delta x)^2$$

or

$$(\Delta x)^2 (\Delta p)^2 \ge \frac{\hbar^2}{4}$$

$$(\Delta x)(\Delta p) \ge \frac{\hbar}{2}$$

proved.

Energy-Time Uncertainty Principle:

as we know that frequency =

$$\nu = \frac{n}{t} = \frac{\text{No of waves}}{\text{unit time}}$$

It can be written ; as

$$\Delta \nu = \frac{\Delta n}{\Delta t} \quad \text{—— (1)}$$

let us consider an ideal machine which counts the number of waves. Number of waves cannot be fraction so $\Delta n = 1$

$① \Rightarrow \Delta \nu = \frac{1}{\Delta t} \Rightarrow (\Delta \nu)(\Delta t) = 1$

By Plank's formula of Energy :-

$E = h\nu \Rightarrow \Delta E = h \Delta \nu$

$\Delta E = h \cdot \frac{1}{\Delta t} \Rightarrow (\Delta E)(\Delta t) = h$

$(\Delta E)(\Delta t) \geq \frac{h}{2(2\pi)} = \frac{\hbar}{2}$, so

$(\Delta E)(\Delta t) \geq \frac{\hbar}{2}$

Commulator Operator :-

let T_1 & T_2 be two operators, their commutator is defined; as

$[T_1, T_2] = T_1 T_2 - T_2 T_1$

Q: Prove That :-

$[P, x] = -i\hbar$

proof:-

let ψ be a wave fn., then

$[P, x]\psi = (px - xp)\psi$

$= \frac{\hbar}{i}\frac{\partial}{\partial x}(x\psi) - x\frac{\hbar}{i}\frac{\partial \psi}{\partial x}$

$$= \frac{\hbar}{i}\left(\psi + x\frac{\partial \psi}{\partial x}\right) - x\frac{\hbar}{i}\frac{\partial \psi}{\partial x}$$

$$= \frac{\hbar}{i}\psi + \cancel{x\frac{\hbar}{i}\frac{\partial \psi}{\partial x}} - \cancel{x\frac{\hbar}{i}\frac{\partial \psi}{\partial x}}$$

$$\Rightarrow [P, x] = \frac{\hbar}{i} = -i\hbar$$

$\Rightarrow$ This is origin of uncertainty.

Scanned with CamScanner