Schwarz-Inequality for finite sum :- 1

Integral form of Schwarz inequality :-

$$\left|\int \phi^* \psi \, d^3\underline{r}\right|^2 \leq \left(\int |\phi|^2 d^3\underline{r}\right)\left(\int |\psi|^2 d^3\underline{r}\right)$$

$\int$ = sums

$\sum_n$ = sums

Schwarz Inequality for finite sums takes the form:-

$$\left|\sum a_n b_n\right|^2 \leq \left(\sum_n |a_n|^2\right)\left(\sum_n |b_n|^2\right)$$

Equality is obtained when ϕ is parallel to ψ. i.e $\phi = c\psi$

L.H.S:-

$$\left|\int (c\psi)^* \psi \, d^3\underline{r}\right|^2 = |c|^2 \left|\int |\psi|^2 d^3\underline{r}\right|^2$$

R.H.S:-

$$\left[\int |c\psi|^2 d^3\underline{r}\right]\left[\int |\psi|^2 d^3\underline{r}\right]$$

$$|c|^2 \left|\int |\psi|^2 d^3\underline{r}\right|^2$$

$\Rightarrow$ L.H.S = R.H.S

$\Rightarrow$ Equality holds

Hermition Operator:-

An operator T is called Hermition if

$$\int \phi^* T \psi \, d^3r = \int (T\phi)^* \psi \, d^3r = \left(\int \psi^* T \phi \, d^3r\right)^*$$

⇒ Such operator T is called Hermition op.

Prove that:-

Momentum operator is Hermition operator.

proof:-

Momentum operator $= P = \frac{\hbar}{i}\frac{d}{dx} = T$

We take ϕ & ψ (two wave fn.)

$$L.H.S: \int \phi^* T \psi \, d^3r = \int \phi^* \frac{\hbar}{i}\frac{d}{dx}\psi \, dx$$

$$= \frac{\hbar}{i}\int_{-\infty}^{\infty} \phi^* \frac{d\psi}{dx} \, dx \quad \text{(in one dim)}$$

While integrating by parts; as

$$= -i\hbar\left\{ \cancel{\phi^*\psi\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} \frac{d\phi^*}{dx}\psi \, dx \right\}$$

required $\int \phi^* \psi \, d^3r < \infty$

so $\phi, \psi \to 0$ as $x \to \pm\infty$, so

ϕ, ψ are well define

$$= i\hbar \int_{-\infty}^{\infty} \frac{d\phi^*}{dx}\psi \, dx = \int_{-\infty}^{\infty} \left(i\hbar\frac{d}{dx}\right)\phi^* \psi \, dx$$

$$= \int_{-\infty}^{\infty} \left(-i\hbar\frac{d\phi}{dx}\right)^* \psi \, dx = \int \left(\frac{\hbar}{i}\frac{d\phi}{dx}\right)^* \psi \, dx$$

⇒ P is Hermition op.

Probability or average / Expectation Value:-

let $f(\underline{r}, p)$ be a fn. Its expectation value is denoted by $\langle f \rangle$. It's defined by:-

$$\langle f \rangle = \frac{\int \phi^*(\underline{r}) f(\underline{r}, \frac{\hbar}{i}\nabla) \phi(\underline{r})\, d^3\underline{r}}{\int \phi^*(r)\phi(r)\, d^3 r}$$

if $\int |\phi(\underline{r})|^2 d^3\underline{r} = 1$, Then it's normalized condition is:-

$$\langle f \rangle = \int \phi^*(\underline{r})\, f(r, \frac{\hbar}{i}\nabla)\, \phi(\underline{r})\, d^3\underline{r}$$

If fn is written $f(\underline{r})$, Then

$$\langle f \rangle = \frac{\int \phi^*(\underline{r}) f(\underline{r}) \phi(r)}{\int |\phi(\underline{r})|^2 d^3\underline{r}} \quad \text{or}$$

$$\langle f \rangle = \frac{\int |\phi(\underline{r})|^2 f(\underline{r})\, d^3 r}{\int |\phi(r)|^2 d^3 r}$$

or

$$\langle f \rangle = \int |\phi(r)|^2 \, f(\underline{r})\, d^3 r$$

Example:

If T_1 & T_2 are two Hermitian op Then prove That

(ii) $T_1T_2 + T_2T_1$ is Hermitian

proof:-

We are given T_1 & T_2 be two Hermitian operators. Then

$$\int \phi^* T_1 \psi \, dx = \int (T_1\phi)^* \psi \, dx \quad \&$$

$$\int \phi^* T_2 \psi \, dx = \int (T_2\phi)^* \psi \, dx$$

$\because \int \phi^* T \psi \, dx = \int (T\phi)^* \psi \, dx$

where ϕ & ψ be two wave fns.

Now we take

$$\int \phi^* (T_1T_2 + T_2T_1) \psi \, dx$$

$$= \int \phi^* T_1 T_2 \psi \, dx + \int \phi^* T_2 T_1 \psi \, dx$$

here $T_1T_2 = T_2T_1$

$$= \int (T_2T_1\phi)^* \psi \, dx + \int (T_1T_2\phi)^* \psi \, dx$$

$$= \int T_2^* T_1^* \phi^* \psi \, dx + \int T_1^* T_2^* \phi^* \psi \, dx$$

$$= \int (T_2^* T_1^* + T_1^* T_2^*) \phi^* \psi \, dx$$

$$= \int \{(T_2T_1 + T_1T_2)\phi\}^* \psi \, dx$$

$$= \int \{(T_1T_2 + T_2T_1)\phi\}^* \psi \, dx$$

so

$T_1T_2 + T_2T_1$ is Hermitian operator.

ii) $T_1T_2 - T_2T_1$ is Not Hermitian.

proof

$$\int \phi^*(T_1T_2 - T_2T_1)\psi\, dx$$
$$= \int \phi^* T_1T_2\psi - \int \phi^* T_2T_1\psi\, dx$$

$$= \int (T_2T_1\phi)^*\psi\, dx - \int (T_1T_2\phi)^*\psi\, dx$$

$$= \int T_2^* T_1^* \phi^* \psi\, dx - \int T_1^* T_2^* \phi^* \psi\, dx$$

$$= \int (T_2^*T_1^* - T_1^*T_2^*)\phi^*\psi\, dx$$
$$= \int \{(T_2T_1 - T_1T_2)\phi\}^*\psi\, dx$$
$$= -\int \{(T_1T_2 - T_2T_1)\phi\}^*\psi\, dx$$

so

$T_1T_2 - T_2T_1$ is not Hermitian.

Variances or Uncertainty:-

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$\because \langle x\rangle = \frac{\sum x}{n}$

$\langle x^2\rangle = \frac{\sum x^2}{n}$

$\langle x\rangle^2 = \frac{(\sum x)^2}{n}$

Example:-

Prove That :-

$$(\Delta x)^2 = \int (x - \langle x\rangle)^2 |\psi|^2 dx$$

proof:-

R.H.S:-

$$\int (x-\langle x\rangle)^2 |\psi|^2 dx$$

$$= \int (x^2 + \langle x\rangle^2 - 2x\langle x\rangle)|\psi|^2 dx$$

$$= \int x^2 |\psi|^2 dx + \langle x\rangle^2 \int |\psi|^2 dx - 2\langle x\rangle \int x |\psi|^2 dx$$

$$\left\{ \because \langle x\rangle = \frac{\int x|\psi|^2 dx}{\int |\psi|^2 dx} \right.$$

$$\left\{ \begin{array}{l} \int |\psi|^2 dx = \text{Normalized condition} = 1 \\ \langle x\rangle = \int x|\psi|^2 dx \;,\; \langle x^2\rangle = \int x^2 |\psi|^2 dx \end{array} \right.$$

$$= \langle x^2\rangle + \langle x\rangle^2(1) - 2\langle x\rangle\langle x\rangle$$

$$= \langle x^2\rangle + \langle x\rangle^2 - 2\langle x\rangle^2$$

$$= \langle x^2\rangle - \langle x\rangle^2 = (\Delta x)^2$$

Similarly

$$\int (p-\langle p\rangle)^2 |\psi|^2 dx = (\Delta p)^2$$

Inner product space:-

Characteristics features of Hilbert space is that it is a vector space having I.P.S, is defined as:-

$$\int \phi^* \psi \, d^3 r = (\phi, \psi)$$

It satisfies following properties:

i) Positivity:-

$(\varphi, \varphi) = \int |\varphi|^2 d^3r \geq 0$

$\Rightarrow (\varphi, \varphi) = 0$ iff $\varphi = 0$

ii) Linearity:-

a) $(\alpha\varphi, \psi) = \alpha^*(\varphi, \psi)$

b) $(\varphi, (\alpha\psi + \beta\chi)) = \alpha(\varphi, \psi) + \beta(\varphi, \chi)$

take L.H.S:-

$(\varphi, (\alpha\psi + \beta\chi)) = \int \varphi^*(\alpha\psi + \beta\chi) d^3r$

$= \alpha \int \varphi^* \psi d^3r + \beta \int \varphi^* \chi d^3r$

$= \alpha(\varphi, \psi) + \beta(\varphi, \chi)$

iii) Reality:-

$(\varphi, \psi) = (\psi, \varphi)^*$