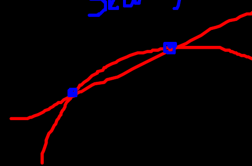
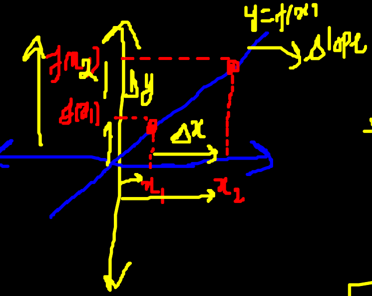
Chapter 03: THE DERIVATIVE

1. TANGENT LINES AND RATES OF CHANGE

Basic Concept of secant line:

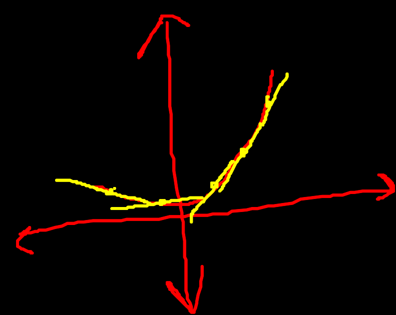
secant line



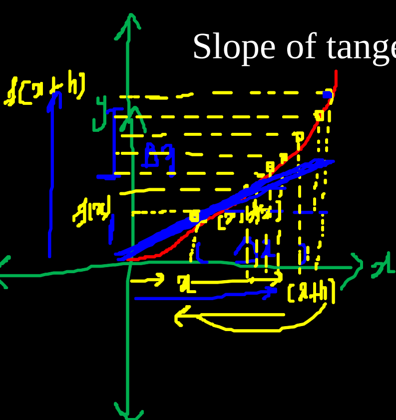


$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\boxed{m_{secant} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$



Slope of tangent:



$P(x, x+h)$

$P(x_1, f(x_1))$ $\quad Q(x_2, f(x_2))$

$$\text{slope of line} = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

$$\text{slope of line} = \frac{f(x+h) - f(x)}{h}$$

$$\text{slope of line} = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$$

Slope of Tangent

$$\boxed{m_{tang} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f'(x) = \frac{dy}{dx}$$

Tangent

Tangent line: m ; $P(x_0, y_0)$

$$\boxed{y = y_0 + m(x - x_o)}$$

Eq of T.L

Calculate the slope of tangent?

Example:01

Find the slope of the tangent $y = x^2$ at the point (2,4) , if limit exists find the tangent line?

$P(2,4) \Rightarrow x_0 = 2 \; ; \; y_0 = 4$

Sol:

$$\boxed{m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$f(x) = x^2$

$f(2) = (2)^2 = 4$

$f(x+h) = (x+h)^2 = x^2 + h^2 + 2xh$

$f(2+h) = 4 + h^2 + 4h$

$$m\Big|_{(2,4)} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + h^2 + 4h - \cancel{4}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+4)}{\cancel{h}} = \lim_{h \to 0} (h+4)$$

$$= 0 + 4 = 4$$

Ans

$\boxed{m = 4}$; $P(\underset{x_0}{2}, \underset{y_0}{4})$

Eq of Tangent line: $y = y_0 + m(x - x_0)$

$$\boxed{y = 4 + 4(x-2)}$$

or

$y - 4 = 4(x-2)$

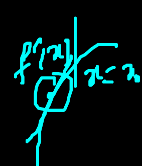
or

$y - 4 = 4x - 8$

$y - 4x = 4 - 8 \Rightarrow \boxed{y - 4x = -4}$

All of these to refer to the same things

1) The Slope of $y = f(x)$ at $x = x_0$
2) The slope of tangent $y = f(x)$ at $x = x_0$
3) the rate of change of $f(x)$ w.r.t "x" at $x = x_0$
4) The derivative of $f(x)$ at $x = x_0$



$$\frac{dy}{dx} = f'(x) = \boxed{m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}}$$

Slope of tangent at a point

$$\frac{dy}{dx}\Big|_{x = x_0}$$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Questions:01

Find the equation of the tangent to the curve $y = x^2 + 1$; at $(2,5)$

Sol: $P(2,5) \Rightarrow x_0 = 2$; $y_0 = 5$; $m =$

Eq. Tangent:-

$$y = y_0 + m(x - x_0)$$

$$\boxed{y = 5 + 4(x-2)}$$

$$y - y_0 = m(x - x_0)$$

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \qquad f(x+h)$$

$$f(x) = x^2 + 1$$

$$\boxed{f(2) = 4 + 1 = 5}$$

$$f(2+h) = (2+h)^2 + 1$$

$$f(2+h) = (2+h)^2 + 1 = 5 + h^2 + 2\cdot 2h$$

$$m = \lim_{h\to 0}\left(\frac{\cancel{5} + h^2 + 2\cdot 2h - \cancel{5}}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+4)}{\cancel{h}}$$

$$\boxed{m = 4}$$

Question:02

Find the slope of the equation, if limit exists then find the tangent line

Where $g(x) = \frac{x}{x-2}$; $(3,3)$ $\qquad x_0 = 3$; $y_0 = 3$

Sol:

$$g(x) = \frac{x}{x-2}$$

$$g(3) = \frac{3}{3-2} = 3$$

$$\boxed{g(3) = 3}$$

$$g(x+h) = \frac{x+h}{x+h-2}$$

$$\boxed{g(3+h) = \frac{3+h}{3+h-2} = \frac{3+h}{h+1}}$$

$$m = \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h}$$

$$m = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$$

$$m = \lim_{h\to 0} \frac{\frac{3+h}{h+1} - 3}{h} = \lim_{h\to 0} \frac{3+h-3(h+1)}{(h+1)h}$$

$$m = \lim_{h\to 0} \frac{\cancel{3}+h-3h-\cancel{3}}{h(h+1)} = \lim_{h\to 0} \frac{\cancel{h}(1-3)}{\cancel{h}(h+1)}$$

$$m = \frac{1-3}{0+1} = -2$$

$$\boxed{m = -2}$$

Eq,s of Tangent line :-

$$y = y_0 + m(x - x_0)$$
$$y = 3 + (-2)(x - 3)$$
$$\boxed{y = 3 - 2(x - 3)}$$

Question:03

Find the slope of the curve at the point indicate

$$y = 5x^2 \; at \; x = -1$$

$P(x, y)$
$y = ?$
$(-1)^2 = 1$
$(-1)^3 = -1$

at $x = -1$
$$y = 5(-1)^2 = 5$$
$$\boxed{y = 5}$$

$P(-1, 5)$

$f(x) = 5x^2$
$f(1) = 5$
$f(x+h) = 5(x+h)^2$
$f(-1+h) = 5(-1+h)^2$
$= 5 + 5h^2 - 10h$

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\cancel{5} + 5h^2 - 10h - \cancel{5}}{h}$$

$$m = \lim_{h \to 0} \frac{\cancel{h}(5h - 10)}{\cancel{h}} = -10$$

$$\boxed{m = -10}$$

Question 04: $\frac{x-1}{x+1}$ $at\ x = 0$

$y = \frac{x-1}{x+1}$

$f(0, -1)$

at $x = 0$

$\boxed{y = -1}$

$\boxed{m = 2}$

There are two types of tangent slope

1) Vertical slope

The vertical slope of the tangent curve at point x= x_0 is

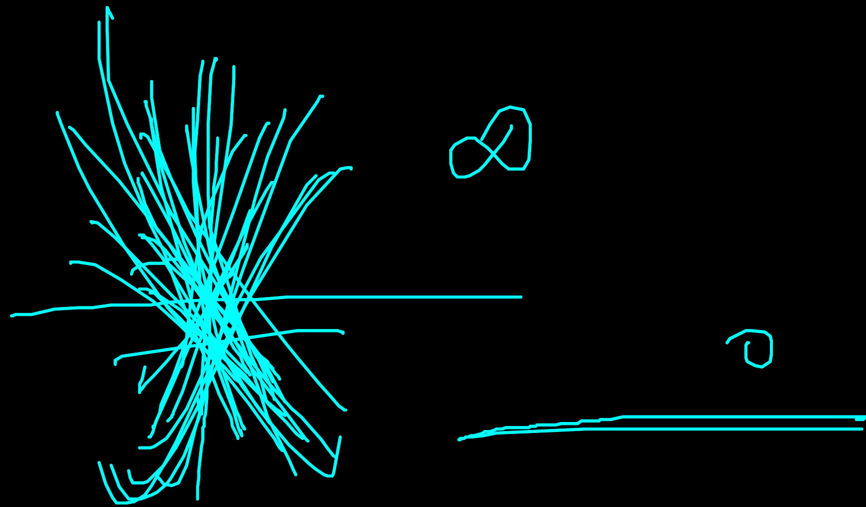
$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h} = \infty$$

$$-\infty$$

2) Horizontal slope:

The horizontal slope of the tangent curve at point x = x_0 is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h} = 0$$



$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = -\infty$

Horizontal Asys