11

Pricing Practices

We have thus far discussed output and pricing decisions under some very simplistic assumptions. We have assumed, for example, that a firm is a profit maximizer, that it produces and sells a single good or service, that all production takes place in a single location, that the firm operates within a well-defined market structure, and that management has precise knowledge about the firm's production, revenue, and cost functions. In addition, we assumed that the firm sells its output at the same price to all consumers in all markets. These conditions, however, are rarely observed in reality. These in the next two chapters we apply the tools of economic analysis developed earlier to more specific real-world situations, including multiplant and multiproduct operations, differential pricing, and non-profit-maximizing behavior.

PRICE DISCRIMINATION

For firms with market power, price discrimination refers to the practice of tailoring a firm's pricing practices to fit specific situations for the purpose of extracting maximum profit. Price discrimination may involve charging different buyers different prices for the same product or charging the same consumer different prices for different quantities of the same product. Price discrimination may involve pricing practices that limit the consumers' ability to exercise discretion in the amounts or types of goods and services purchased. In whatever guise price discrimination is practiced, it is often viewed by the consumer, when the consumer understands what is going on, as somehow nefarious, or at the very least "unfair."

Definition: Price discrimination occurs when profit-maximizing firms charge different individuals or groups different prices for the same good or service.

The literature generally discusses three degrees of price discrimination. First-degree price discrimination, which involves charging each individual a different price for each unit of a given product, is potentially the most profitable of the three types of price discrimination. First-degree price discrimination is the least often observed because of very difficult informational requirements. Second-degree price discrimination differs from first-degree price discrimination in that the firm attempts to maximize profits by "packaging" its products, rather than selling each good or service one unit at a time. Finally, third-degree price discrimination occurs when firms charge different groups different prices for the same good or service. While not as profitable as first-degree and second-degree price discrimination, third-degree price discrimination is the most commonly observed type of differential pricing. A recurring theme in most, but not all, price discriminatory behavior is the attempt by the firm to extract all or some consumer surplus.

FIRST-DEGREE PRICE DISCRIMINATION

We have noted that price discrimination occurs when different groups are charged different prices for the same product subject to certain conditions. Theoretically, price discrimination could take place at any level of group aggregation. Price discrimination at its most disaggregated level occurs when each "group" consists a single individual. First-degree price discrimination occurs when firms charge each individual a different price for each unit purchased. The price charged for each unit purchased is based on the seller's knowledge of each individual's demand curve. Because it is virtually impossible to satisfy this informational requirement, first-degree price discrimination is extremely rare. Nevertheless, an analysis of first-degree price discrimination is important because it underscores the rationale underlying differential pricing.

Definition: First-degree price discrimination occurs when a seller charges each individual a different price for each unit purchased.

The purpose of first-degree price discrimination is to extract the total amount of *consumer surplus* from each individual customer. The concept of consumer surplus was introduced in Chapter 8. Consumer surplus represents the dollar value of benefits received from purchasing an amount of a good or service in excess of benefits actually paid for. In Figure 11.1, which illustrates an individual's demand (marginal benefit) curve for a particular product, the market price of the product is $3. At that price, the consumer

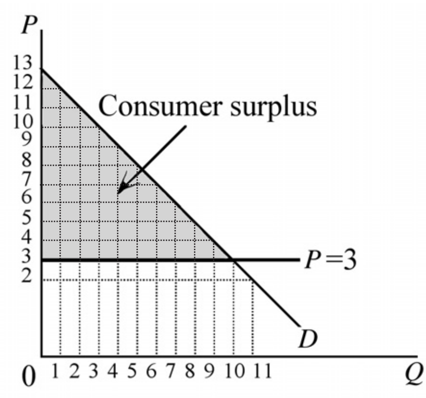

FIGURE 11.1 Consumer surplus.

purchases 10 units of the product. The total expenditure by the consumer, and therefore the total revenues to the firm, is $\$3 \times 10 = \$30$. It is clear from Figure 11.1, however, that the individual would have been willing to pay much more for the 10 units purchased at \$3. In fact, as we will see, only the tenth unit was worth \$3 to the consumer. Each preceding unit was worth more than \$3.

Suppose that we lived in a world of truth tellers. The consumer whose behavior is represented in Figure 11.1 enters a shop to purchase some amount of a particular product. The consumer is completely knowledgeable of his or her preferences and the value (to the consumer) of each additional unit. The process begins when the shopkeeper inquires how much the consumer is willing to pay for the first unit of the good. The consumer truthfully states a willingness to pay \$12. A deal is struck, the sale is made, and the consumer expends \$12, which becomes \$12 in revenue to the shopkeeper. The process continues. The shopkeeper then inquires how much the consumer is willing to pay for the second unit. By the law of diminishing marginal utility, the consumer truthfully acknowledges a willingness to pay \$11. Once again, a deal is struck, the sale is made, and the consumer expends an additional \$11, which becomes an additional \$11 in revenue to the shopkeeper.

This process continues until the tenth unit is purchased for \$3. The consumer will not purchase an eleventh unit, since the amount paid (\$3) will exceed the dollar value of the marginal benefits received (\$2). By proceeding in this manner, the consumer has paid for each item purchased an amount equivalent to the marginal benefit received, or a total expenditure of \$75. This amount is \$45 greater than would have been paid in a conventional market transaction. In other words, the shopkeeper was able extract \$45 in consumer surplus.

Definition: Consumer surplus is the value of benefits received per unit of output consumed minus the product's selling price.

Of course, this mind experiment is unrealistic in the extreme. Moreover, the amount of consumer surplus we calculated is only a rough approximation. With the price variations made arbitrarily small, the actual value of consumer surplus is the value of the shaded area in Figure 11.1. Our scenario, however, underscores the benefits to the firm being able to engage in first-degree price discrimination.

Alas, we do not live in a world of truth tellers. Even if we were completely cognizant of our individual utility functions, we would more than likely understate the true value of the next additional unit offered for sale. Moreover, even if the firm knew each consumer's demand equation, the realities of actual market transactions make it extremely unlikely that the firm would be able to extract the full amount of consumer surplus. Transactions are seldom, if ever, conducted in such a piecemeal fashion.

More formally, for discrete changes in sales (Q), consumer surplus may be approximated as

$$CS = \sum_{i=1\to n} (P_i \times \Delta Q) - P_n Q_n \tag{11.1}$$

where Q_n is the quantity demanded by individual i at the market price, P_n. If we assume that the individual's demand function is linear, that is,

$$P_i = b_0 + b_1 Q_i \tag{11.2}$$

then consumer surplus is approximated as

$$CS = \sum_{i=1\to n} (b_0 + b_1 Q_i)\Delta Q - P_n Q_n \tag{11.3}$$

Examination of Equation (11.3) suggests that the smaller ΔQ, the better the approximation of the shaded area in Figure 11.1. It can be easily demonstrated, and can be seen by inspection, that for a linear demand equation, as $\Delta Q \to 0$ the value of the shaded area in Figure 11.1 may be calculated as

$$CS = 0.5(b_0 - P_n)Q_n \tag{11.4}$$

In Chapter 2 we introduced the concept of the integral as accurately representing the area under a curve. The concept of the integral can be applied in this instance to calculate the value of consumer surplus. Defining the demand curve as $P = f(Q)$, consumer surplus may be defined as

$$CS = \int f(Q)dQ - P^*Q^*$$

where P_n and Q_n are the equilibrium price and quantity, respectively. Substituting Equation (11.2) into the integral equation yields

$$
\begin{aligned}
CS &= \int_0^n (b_0 + b_1 Q_i) dQ - P_n Q_n \\
&= [b_0 Q_i + 0.5 b_1 Q_i^2]_0^n - P_n Q_n \\
&= [b_0 Q_n + 0.5 b_1 Q_n^2] - \left[b_0(0) + 0.5 b_1 (0)^2\right] - P_n Q_n \\
&= [b_0 Q_n + 0.5 b_1 Q_n^2] - P_n Q_n
\end{aligned}
$$

If we assume that the demand equation is linear and that the firm is able to extract consumer surplus, how can we find the profit-maximizing price and output level? If the firm is able to extract consumer surplus, total revenue is

$$TR = PQ + 0.5(b_0 - P)Q \tag{11.5}$$

If we assume that total cost as an increasing function of output, then the total profit function is

$$\pi(Q) = TR(Q) - TC(Q) \tag{11.6}$$

Substituting Equations (11.4) and (11.5) into Equation (11.6) yields

$$
\begin{aligned}
\pi &= (b_0 - b_1 Q)Q + 0.5[b_0 - (b_0 + b_1 Q)Q] - TC \\
&= b_0 Q + 0.5 b_1 Q^2 - TC
\end{aligned} \tag{11.7}
$$

The first- and second-order conditions for profit maximization are

$$\frac{d\pi}{dQ} = b_0 + b_1 Q - MC = 0 \tag{11.8a}$$

$$\frac{d\pi^2}{dQ^2} = \frac{b_1 - dMC}{dQ} < 0 \tag{11.8b}$$

Solving Equation (11.8a) for output yields

$$Q^* = \frac{MC - b_0}{b_1} \tag{11.9}$$

Substituting Equation (11.9) into Equation (11.2) yields

$$P^* = b_0 + b_1\left(\frac{MC - b_0}{b_1}\right) = b_0 + (MC - b_0) = MC \tag{11.10}$$

Under the circumstances, the firm attempting to extract consumer surplus does not actually charge a price equal to marginal cost. Instead, the firm will calculate consumer surplus by substituting Equation (11.10) into Equation (11.4). It should be noted that Equation (11.10) looks similar to the one the profit-maximizing firm operating in a perfectly competitive industry. Of course, the crucial difference is that $P > MC$ for a profit-maximizing firm facing a downward-sloping demand curve for its product.

Problem 11.1. Assume that an individual's demand equation is

$$P_i = 20 - 2Q_i$$

Suppose that the market price of the product is $P_n = \$6$.
a. Approximate the value of this individual's consumer surplus for $\Delta Q = 1$.
b. What is value of consumer surplus as $\Delta Q \to 0$?

Solution

a. The equation for approximating the value of consumer surplus for discrete changes in Q when the demand function is linear is

$$CS = \sum_{i=1 \to n} (b_0 + b_1 Q_i)\Delta Q - P_n Q_n$$

For $P_n = \$6$ and $\Delta Q = 1$ this equation becomes

$$CS = \sum_{i=1 \to n} (20 - 2Q_i) - 42$$

For values of Q_i from 0 to 7 this becomes

$$\begin{aligned} CS &= [20-2(1)] + [20-2(2)] + [20-2(3)] + [20-2(4)] \\ &\quad + [20-2(5)] + [20-2(6)] + [20-2(7)] - 42 \\ &= 18 + 16 + 14 + 12 + 10 + 8 + 6 - 42 = \$42 \end{aligned}$$

The approximate value of consuming 7 units of this good is approximately \$84 dollars. If the consumer pays \$6 for 7 units of the good, then the individual's total expenditure is \$42. The approximate dollar value of benefits received, but not paid for, is \$42.

b. The value of the individual's consumer surplus as $\Delta Q \to 0$ is given by the expression

$$CS = 0.5(b_0 - P_n)Q_n$$

Substituting into this expression we obtain

$$CS = 0.5(20-6)7 = 0.5(14)7 = \$49$$

The actual value of consumer surplus is \$49, compared with the approximated value of \$42 calculated in part a.

SECOND-DEGREE PRICE DISCRIMINATION

Sometimes referred to as *volume discounting*, *second-degree price discrimination* differs from first-degree price discrimination in the manner in which the firm attempts to extract consumer surplus. In the case of second-

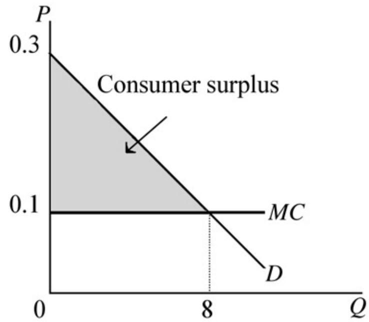

FIGURE 11.2 Block pricing.

degree price discrimination, sellers attempt to maximize profits by selling product in "blocks" or "bundles" rather than one unit at a time. There are two common types of second-degree price discrimination: *block pricing* and *commodity bundling*.

Definition: Second-degree price discrimination occurs when firms sell their product in "blocks" or "bundles" rather than one unit at a time.

Block Pricing

Block pricing, or selling a product in fixed quantities, is similar to first-degree price discrimination in that the seller is trying to maximize profits by extracting all or part of the buyer's consumer surplus. Eight frankfurter rolls in a package and a six-pack of beer are examples of block pricing.

The rationale behind block pricing is to charge a price for the package that approximates, but does not exceed, the total benefits obtained by the consumer. Suppose, for example, that the estimated demand equation of the average consumer for frankfurter rolls is given as $Q = 24 - 80P$. Solving this equation for P yields $P = 0.3 - 0.0125Q$. Suppose, further, that the marginal cost of producing a frankfurter roll is constant at \$0.10. This situation is illustrated in Figure 11.2.

With block pricing the firm will attempt to get the consumer to pay for the full value received for the eight frankfurter rolls by charging a single price for the package. If frankfurter rolls were sold for \$0.10 each, the total expenditure by the typical consumer would be \$0.80. The firm will add the value of consumer surplus to the package of eight frankfurter rolls, as follows:

$$\text{Block price} = TR = PQ + CS = PQ + 0.5(b_0 - P)Q$$
$$= 0.1(8) + 0.5(0.3 - 0.1)8 = \$1.60$$

The profit earned by the firm is

$$\pi = TR - TC = PQ + 0.5(b_0 - P)Q - (MC \times Q) = \$1.60 - \$0.80 = \$0.80$$

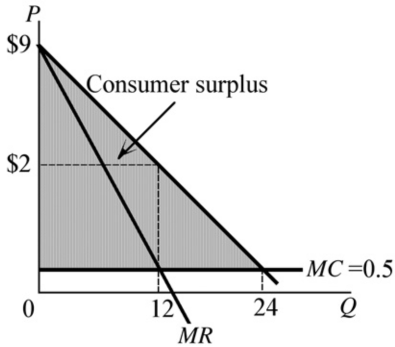

FIGURE 11.3 Amusement park pricing.

If this firm operated in a perfectly competitive industry and frankfurter rolls were sold individually, the selling price would be $0.10 per roll and the firm would break even. In other words, the firm would earn only normal profits, since $TR = TC$.

One interesting variation of block pricing is amusement park pricing. While it is not possible for the management of an amusement park to know the demand equation for each individual entering the park, and therefore first-degree price discrimination is out of the question, suppose that management had estimated the demand equation of the average park visitor. Figure 11.3 illustrates such a demand relationship.

In Figure 11.3 the marginal cost to the amusement park of providing a ride is assumed to be $0.50. If the amusement park is a profit maximizer, it will set the average price of a ride at $2 per ride (i.e., where $MR = MC$). At $2 per ride, the average park visitor will ride 12 times for an average total expenditure of $24 per park visitor. The total profit per visitor is

$$\pi = TR - TC = PQ - (MC \times Q) = 2(12) - 0.5(12) = \$18$$

At the profit-maximizing price, however, the average park visitor will enjoy a consumer surplus on the first 11 rides. The challenge confronting the managers of the amusement park is to extract this consumer surplus.

Rather than charging on a per-ride basis, many amusement parks charge a one-time admission fee, which allows park visitors to ride as often as they like. What admission fee should the amusement park charge? The park will calculate consumer surplus as if the price per ride is equal to the marginal cost to the amusement park of providing a single ride. Substituting Equation (11.22) into Equation (11.16), the amount of consumer surplus is

$$CS = 0.5(9 - 0.5)24 = \$102$$

The one-time admission fee charged by the amusement park should equal the marginal cost of providing a ride multiplied by the number of rides, plus the amount of consumer surplus. On average, the amusement park expects each guest to ride approximately 24 times. Thus, the amusement park should charge a one-time admission of \$114 [$(MC \times Q) + CS =$ \$0.5(24) + \$102].

The main difference between the block pricing of frankfurter rolls and admission to an amusement park is that while frankfurter rolls are very much a private good, amusement park rides take on the characteristics of a *public good*. The distinction between private and public goods will be discussed in greater detail in Chapter 15. For now, it is enough to say that the ownership rights of private goods are well defined. The owner of the private property rights to a good or service is able to exclude all other individuals from consuming that particular product. Moreover, once the product has been consumed, as in this case frankfurter rolls, there is no more of the good available for anyone else to consume. In other words, private goods have the properties of *excludability* and *depletability*.

The situation is quite different with public goods. For one thing, use by one person of a public good such as commercial radio programming or television broadcasts does not decrease its availability to others. Another important characteristic of a public good is unlimited access by individuals who have not paid for the good. This is the characteristic of nonexcludability. While cable television broadcasts possess the characteristic of nondepletability, they are not public goods because nonpayers can be excluded from their use.

In the case of public goods, private markets often fail because consumers are unwilling to reveal their true preferences for the good or service, which makes it difficult, if not impossible, to correctly price the good. This phenomenon is often referred to as the *free-rider problem*. In the case of pure public goods, the government is often obliged to step in to provide the good or service. The most commonly cited examples of public goods are national defense and police and fire protection. The provision of public goods is financed through tax levies.

Block pricing by amusement parks is similar to block pricing by cable television companies in that the success of this pricing policy depends crucially on management's ability to deny access to nonpayers. This is usually accomplished by controlling access to the park. It is not unusual for large amusement parks, such as the Six Flags, Busch Gardens, or Disney World theme parks, to be isolated from densely populated areas. Access to the park is typically limited to one or a few points, and the perimeter of the park is characterized by high walls, fences, or *a* natural obstacle, such as a lake, constantly guarded by security personnel. It is much more difficult for older amusement parks, which are usually located in densely populated metropolitan areas, to engage in a one-time admission fee pricing policy because of the difficulty associated with controlling access to park grounds. In such cases, an alternative pricing policy to extract consumer surplus is necessary. One such technique is to sell identifying bracelets that enable park visitors to ride as often as they like for a limited period of time, say, two hours. This approach is often advertised as a POP (pay-one-price) plan. Thus, access to rides is not controlled at the park entrance, but at the entrance to individual rides.

Ironically, whatever technique is used to extract consumer surplus by amusement parks, it is good public relations. Park visitors like the convenience of not having to pay per ride. What is more, most park visitors believe that this pricing practice is a by-product of the management's concern for the comfort and convenience of guests, which is probably true. Finally, and most important, many amusement park visitors believe that they are getting their money's worth by being able to ride as many times as they like, which is, of course, true. But do they get more than their money's worth? This may also be true, but it should not be forgotten that the purpose of this type of pricing is to maximize amusement park profits by extracting as much consumer surplus as possible.

Problem 11.2. Seven Banners High Adventure has estimated the following demand equation for the average summer visitor to its theme park

$$Q = 27 - 3P$$

where Q represents the number or rides by each guest and P the price per ride in U.S. dollars. The total cost of providing a ride is characterized by the equation

$$TC = 1 + Q$$

Seven Banners is a profit maximizer considering two different pricing schemes: charging on a per-ride basis or charging a one-time admission fee and allowing park visitors to ride as often as they like.

a. How much should the park charge on a per-ride basis, and what is the total profit to Seven Banners per customer?
b. Suppose that Seven Banners decides to charge a one-time admission fee to extract the consumer surplus of the average park guest. What is the estimated average profit per park guest? How much should Seven Banners charge as a one-time admission fee? What is the amount of consumer surplus of the average park guest?

Solution

a. Solving the demand equation for P yields

$$P = 9 - \frac{Q}{3}$$

The per-customer total revenue equation is

$$TR = PQ = \left(9 - \frac{Q}{3}\right)Q = 9Q - \frac{Q^2}{3}$$

The per-customer total profit equation is

$$\pi = TR - TC = 9Q - \frac{Q^2}{3} - (1 + Q) = -1 + 8Q - \frac{Q^2}{3}$$

The first- and second-order conditions for profit maximization are $d\pi/dQ = 0$ and $d^2\pi/dQ^2 < 0$, respectively. The profit-maximizing output level is

$$\frac{d\pi}{dQ} = 8 - \frac{2Q}{3} = 0$$

$$Q^* = 12$$

To verify that this is a local maximum, we write the second derivative of the profit function

$$\frac{d^2\pi}{dQ^2} = \frac{-2}{3} < 0$$

which satisfies the second-order condition for a local maximum. The profit-maximizing price per ride is, therefore,

$$P^* = 9 - \frac{12}{3} = 5$$

The estimated average profit per Seven Banners guest with per-ride pricing is

$$\pi = -1 + 8(12) - \frac{(12)^2}{3} = \$47$$

b. If Seven Banners charges a one-time admission fee, it will attempt to extract the total amount of consumer surplus. Since the demand equation is linear, the estimated consumer surplus per average rider is given by the equation

$$CS = 0.5(b_0 - P)Q$$ 1

From Equation (11.7) the profit equation for Seven Banners is

$$\pi = TR - TC = (b_0 + b_1Q)Q + 0.5[b_0 - (b_0 + b_1Q)]Q - TC$$
$$= \left(9 - \frac{Q}{3}\right)Q + 0.5\left[9 - \left(9 - \frac{Q}{3}\right)\right]Q - (1 + Q)$$
$$= 9Q - \frac{Q^2}{3} + \frac{0.5Q^2}{3} - 1 - Q = 8Q - \frac{Q^2}{6} - 1$$

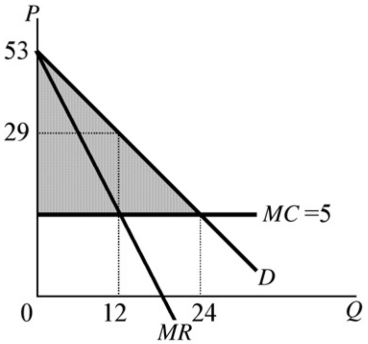

FIGURE 11.4 Two-part pricing.

The first-order condition for profit maximization is

$$\frac{d\pi}{dQ} = 8 - \frac{Q}{3} = 0$$

$$Q^* = 24$$

After substituting this value into the demand equation we get

$$P^* = 9 - \frac{24}{3} = 1 = MC$$

Total profit is, therefore,

$$\pi = 8Q - \frac{Q^2}{6} - 1 = 8(24) - \frac{(24)^2}{6} - 1 = 192 - 96 - 1 = \$95$$

The one-time admission fee should equal the total cost per guest of providing 24 rides plus the total amount of consumer surplus, that is,

$$\text{Admission fee} = TR = (MC \times Q) + CS = (MC \times Q) + 0.5(b_0 - MC)Q$$
$$= 1(24) + 0.5(9 - 1)24 = 24 + 96 = \$120$$

Thus the estimated consumer surplus of the average park guest is $96.

Two-Part Pricing

A variation of block pricing is *two-part pricing*. Two-part pricing is used to enhance a firm's profits by first charging a fixed fee for the right to purchase or use the good or service, then adding a per-unit charge. As in the case of block pricing, two-part pricing is often used by clubs to extract consumer surplus. To see how two-part pricing works, consider Figure 11.4, which illustrates the demand for country club membership.

In Figure 11.4 the per-visit demand to the country club is

$$Q = 26.5 - 0.5P$$

The club's total cost equation is

$$TC = 15 + 5Q$$

If the management of the country club were to charge its members a single price, the profit-maximizing price and output level would be 12 and \$29, respectively. The country club's profit would be (\$24 × 12) – (\$5 × 12) = \$288. At this price–quantity combination, each member of the club would receive consumer surplus (value received but not paid for) of 0.5[(53 – 29) × 12] = \$144.

If, on the other hand, the country club were to use two-part pricing, it could extract the maximum amount of consumer surplus, which is the shaded area in Figure 11.4. In this case, the club would charge an initiation fee of 0.5[(\$53 – \$5) × \$24] = \$576 and impose a per-visit charge of \$5 to cover the cost of services. It is clear that the initiation fee is pure profit and is a substantial improvement over the profit of \$288 earned by charging a single price per visit.

Commodity Bundling

Another form of second-degree price discrimination is *commodity bundling*. Commodity bundling involves combining two or more different products into a single package, which is sold at a single price. Like block pricing, commodity bundling is an attempt to enhance the firm's profits by extracting at least some consumer surplus.

A vacation package offered by a travel agent that includes airfare, hotel accommodations, meals, entertainment, ground transportation, and so on is an example of commodity bundling. Another example of commodity bundling, and one that has elicited considerable attention from the U.S. Department of Justice, is Microsoft's bundling of its Internet Explorer internet web browser with its Windows 98 software package. The federal government's interest stemmed not so much from Microsoft's ability to enhance profits by bundling its products, but from a near monopoly in the market for web browsers. Microsoft was able to ochieve because economies of scale.

To understand how commodity bundling enhances a company's profits, consider the case of a resort hotel that sells weekly vacation packages. Suppose that the package includes room, board, and entertainment. Let us further suppose that the marginal cost to the resort hotel of providing the package is \$1,000.

Management has identified two groups of individuals that would be interested in the vacation package. Although the hotel is not able to identify members of either group, it does know that each group values the components of the package differently. To keep the example simple, assume that

TABLE 11.1 Commodity bundling and vacation packages.

Group	Room and board	Entertainment
1	$2,500	$500
2	$1,800	$750

there are an equal number of members in each group. To further simplify the example, assume that total membership in each group is a single individual. Table 11.1 illustrates the maximum amount that each group will pay for the components of the package.

If the resort hotel could identify the members of each group, it might engage in first-degree price discrimination and charge members of the first group $3,000 and members of the second group $2,550 for the vacation package. Since the marginal cost of providing the service to each group is $1,000, the hotel's profit would be $3,550 per group. Since the hotel is not able to identify members of each group, what price should the hotel charge for the package?

Suppose the hotel decides to price each component of the package separately. If it charges $2,500 for room and board, it would sell only to the first group, and its total revenue would be $2,500. Members of the second group will not be interested because the price is above what the value they attach to room and board. If, on the other hand, the hotel were to charge $1,800 for room and board, it would sell to both groups for a total revenue of $3,600. Clearly, then, the hotel will charge $1,800.

The same scenario holds true for entertainment. If the hotel charges $750, then only members of the second group will purchase entertainment and the hotel will generate revenues of only $750. On the other hand, if the hotel charges $500, both groups will purchase entertainment and generate revenues of $1,000. Thus, whether the hotel charges per item or charges a package price of $1,800 + $500 = $2,300, the profit from each group will be $1,300. Since we have assumed that there is only one individual in each group, the hotel's total profit is $2,600.

Now, although a package price of $2,300 appears to be reasonable from the point of view of the profit-conscious hotel, the story does not end there. As it turns out, the hotel can do even better if it charges a package price of $1,800 + $750 = $2,550. The reason is simple. Management knows that the value of the package to the first group is $2,500 + $500 = $3,000. It also knows that the value to the second group is $1,800 + $750 = $2,550. By bundling room, board, and entertainment and selling the package for $2,550, the hotel will sell both components of the package to members of both groups. At a package price of $2,550, the hotel earns a profit of $1,550, instead of $1,300, from each group. Again, since we have assumed that there is only one person in each group, the hotel's total profit is now $3,100.

TABLE 11.2 Commodity bunding and new car options I.

Group	Power steering	CD stereo system
1	$1,700	$300
2	$1,600	$320
3	$1,500	$340

In the foregoing example, by bundling room, board, and entertainment and charging a single package price, the hotel has enhanced its profits by $250 per group member. The hotel has extracted the entire amount of consumer surplus from members of the second group and some consumer surplus from members of the first group.

Problem 11.3. A car dealership offers power steering and a compact disc stereo system as options in all new models. Suppose that the dealership sells to members of three different groups of new car buyers and that there are five individuals in each group. Table 11.2 illustrates how the members of each group value power steering and a compact disc stereo sound system.

Suppose that the per-unit cost of providing power steering and a CD stereo system is $1,200 and $250, respectively.

a. If the dealership sold each option separately, how much profit would it earn from each group member?
b. If the dealership cannot easily identify the members of each group, how should it price a package consisting of power steering and a CD stereo system? What will be the dealership's profit on each package sold?

Solution

a. If the dealership sells each item separately, it would change $1,500 for power steering, for a profit of $300 per sale. Given that there are five members in each group, the dealership has generated total profits of $4,500. By contrast, if the dealership sells power steering for $1,600, it will earn a profit of $400 per sale. But since only members of the second and third groups will purchase power steering, the dealership's total profit will only be $4,000.

   Similarly, the dealership will sell compact disc stereo systems for $300, for a profit of $50 per sale. Again, since there are five members in each group, the dealership's total profit will be $750. By contrast, if the dealership sells the option for $320 it will earn a profit of $70 per sale. Since, however, only members of the first and second group will opt for the CD stereo system at this price, the dealership's total profit will be $700.
b. If the dealership sells power steering and a CD stereo system at a package price of $1,800, as suggested in the answer to part a, the total

TABLE 11.3 Commodity bundling and new car options II.

Group	Power steering	CD stereo system
1	$2,000	$300
2	$1,800	$350
3	$1,500	$400

profit will be $4,700. However, if the dealership sells the package for $1,840, it will appeal to members of all three groups. In this way, the dealership will extract total consumer surplus from members of the third group, and at least some consumer surplus from the remaining two groups. The dealership's total profit will be $5,850.

Problem 11.4. Suppose that the members of each group in Problem 11.3 valued power steering and a compact disc stereo sound system as in Table 11.3.

The per-unit cost of providing power steering and a CD stereo system remains $1,200 and $250, respectively. How much will the dealership now change for power steering and a CD stereo system as a package? What will be the dealership's profit on each package sold? What is the dealership's total profit?

Solution. In Problem 11.3, we saw that the profit-maximizing price for the package was equivalent to the sum of the prices the third group was willing to pay for each option separately. If we were to follow that practice in this case, the profit on each package sold would be $1,900 – $1,450 = $450, for a total profit of $450 × 15 = $6,750. Suppose, however, that the dealership charged $2,150 for the package, which is the value placed on the package by the second group? The profit on each package sold would be $2,150 – $1,450 = $700, for a total profit of $700 × 10 = $7,000. Finally, if the dealership charged $2,300 for both options, which is the value placed on the package by the first group, the profit on each package would be $850, for a total profit of $850 × 5 = $4,250. Clearly, under the conditions specified in Table 11.3, the dealership will charge a package price of $2,150 and sell only to the first two groups.

THIRD-DEGREE PRICE DISCRIMINATION

In some cases, it is possible for the firm to charge different groups different prices for its goods or services. It is a common practice, for example, for theaters, restaurants, and amusement parks to offer senior citizen, student, and youth discounts. This kind of pricing strategy, which is perceived as altruistic or community spirited, has considerable public relations appeal. In reality, however, this *third-degree price discrimination* in fact results in increased company profits.

Definition: Third-degree price discrimination occurs when firms segment the market for a particular good or service into easily identifiable groups, then charge each group a different price.

For third-degree price discrimination to be effective, a number of conditions must be satisfied. First, the firm must be able to estimate each group's demand function. As we will see, the degree of price variation will depend of differences in each group's price elasticity of demand. In general, groups with higher price elasticities of demand will be charged a lower price.

A second condition that must be satisfied for a firm to engage in third-degree price discrimination is that members of each group must be easily identifiable by some distinguishable characteristic, such as age; or perhaps groups can be identified in terms of the time of the day in which the good or service, such as movie tickets, is purchased.

Finally, for third-degree price discrimination to be successful, it must not be possible for groups purchasing the good or service at a lower price to be able to resell that good or service to groups changed the higher price. If resales are possible, the firm would not be able to sell anything to the group paying the higher price because they would simply buy the good or service from the group eligible for the lower price.

The rationale behind third-degree price discrimination is straightforward. Different individuals or groups of individuals with different demand functions will have different marginal revenue functions. Since the marginal cost of producing the good is the same, regardless of which group purchases the good, the profit-maximizing condition must be $MC = MR_1 = MR_2 = \cdots = MR_n$, where n is the number of identifiable and separable groups. To see why this must be the case, suppose that $MR_1 > MC$. Clearly, in this case, it would pay for the firm to produce one more unit of the good or service and sell it to group 1, since the addition to total revenues would exceed the addition to total cost from producing the good. As more of the good or service is sold to group 1, marginal revenue will fall until $MR_1 = MC$ is established.

The mathematics of this third-degree price discrimination is fairly straightforward. Assume that a firm sells its product in two easily identifiable markets. The total output of the firm is, therefore,

$$Q = Q_1 + Q_2 \tag{11.11}$$

By the law of demand, the quantity sold in each market will vary inversely with the selling price. If the demand function of each group is known, the total revenue earned by the firm selling its product in each market will be

$$TR(Q) = TR_1(Q_1) + TR_2(Q_2) \tag{11.12}$$

where $TR_1 = P_1Q_1$ and $TR_2 = P_2Q_2$. The total cost of producing the good or service is a function of total output, or,

$$TC(Q) = TC(Q_1 + Q_2) \tag{11.13}$$

Note that the marginal cost of producing the good is the same for both markets. By the chain rule,

$$\frac{\partial TC(Q)}{\partial Q_1} = \left(\frac{dTC}{dQ}\right)\left(\frac{\partial Q}{\partial Q_1}\right) = \frac{\partial TC}{dQ} \tag{11.14}$$

since $\partial Q/\partial Q_1 = 1$. Likewise for Q_2,

$$\frac{\partial TC(Q)}{\partial Q_2} = \left(\frac{dTC}{dQ}\right)\left(\frac{\partial Q}{\partial Q_2}\right) = \frac{\partial TC}{dQ} \tag{11.15}$$

since $\partial Q/\partial Q_2 = 1$. Equations (11.14) and (11.15) simply affirm that the marginal cost of producing the good or service remains the same, regardless of the market in which it is sold.

Upon combining Equations (11.11) to (11.15), the firm's profit function may be written

$$\pi(Q_1, Q_2) = TR_1(Q_1) + TR_2(Q_2) - TC(Q_1 + Q_2) \tag{11.16}$$

Equation (11.16) indicates that profit is a function of both Q_1 and Q_2. The objective of the firm is to maximize profit with respect to both Q_1 and Q_2. Taking the first partial derivatives of the profit function with respect to Q_1 and Q_2, and setting the results equal to zero, we obtain

$$\frac{\partial \pi}{\partial Q_1} = \frac{\partial TR_1}{\partial Q_1} - \left(\frac{dTC}{dQ}\right)\left(\frac{\partial Q}{\partial Q_1}\right) = 0 \tag{11.17a}$$

$$\frac{\partial \pi}{\partial Q_2} = \frac{\partial TR_2}{\partial Q_2} - \left(\frac{dTC}{dQ}\right)\left(\frac{\partial Q}{\partial Q_2}\right) = 0 \tag{11.17b}$$

Solving Equations (11.17) simultaneously with respect to Q_1 and Q_2 yields the profit-maximizing unit sales in the two markets. Assuming that the second-order conditions are satisfied, the first-order conditions for profit maximization may be written as

$$MC = MR_1 = MR_2 \tag{11.18}$$

Finally, since $TR_1 = P_1Q_1$ and $TR_2 = P_2Q_2$, then

$$\begin{aligned} MR_1 &= P_1\left(\frac{dQ_1}{dQ_1}\right) + Q_1\left(\frac{dP_1}{dQ_1}\right) \\ &= P_1\left[1 + \left(\frac{dP_1}{dQ_1}\right)\left(\frac{Q_1}{P_1}\right)\right] = P_1\left(1 + \frac{1}{\varepsilon_1}\right) \end{aligned} \tag{11.19}$$

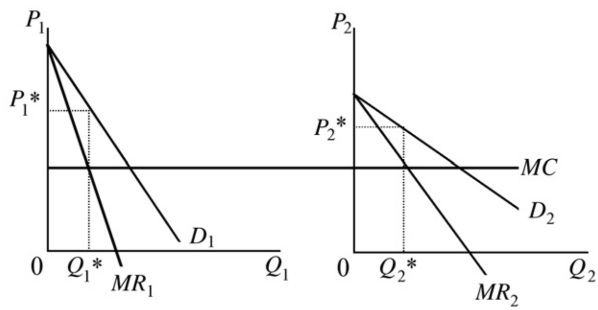

FIGURE 11.5 Third-degree price discrimination.

$$MR_2 = P_2\left(1+\frac{1}{\varepsilon_2}\right) \tag{11.20}$$

where ε_1 and ε_2 are the price elasticities of demand in the two markets. By the profit-maximizing condition in Equations (11.17), it is easy to see that the firm will charge the same price in the two markets only if $\varepsilon_1 = \varepsilon_2$. When $\varepsilon_1 \neq \varepsilon_2$, the prices in the two markets will not be the same. In fact, when $\varepsilon_1 > \varepsilon_2$, the price charged in the first market will be greater than the price charged in the second market. Figure 11.5 illustrates this solution for linear demand curves in the two markets and constant marginal cost.

Problem 11.5. Red Company sells its product in two separable and identifiable markets. The company's total cost equation is

$$TC = 6 + 10Q$$

The demand equations for its product in the two markets are

$$Q_1 = 10 - (0.2)P_1$$

$$Q_2 = 10 - (0.2)P_2$$

where $Q = Q_1 + Q_2$.

a. Assuming that the second-order conditions are satisfied, calculate the profit-maximizing price and output level in each market.
b. Verify that the demand for Red Company's product is less elastic in the market with the higher price.
c. Give the firm's total profit at the profit-maximizing prices and output levels.

Solution

a. This is an example of price discrimination. Solving the demand equations in both markets for price yields

$$P_1 = 50 - 5Q_1$$

$$P_2 = 30 - 2Q_2$$

The corresponding total revenue equations are

$$TR_1 = 50Q_1 - 5Q_1^2$$

$$TR_2 = 30Q_2 - 2Q_2^2$$

Red Company's total profit equation is

$$\pi = TR_1 + TR_2 - TC = 50Q_1 - 5Q_1^2 + 30Q_2 - 2Q_2^2 - 6 - 10(Q_1 + Q_2)$$

Maximizing this expression with respect to Q_1 and Q_2 yields

$$\frac{\partial \pi}{\partial Q_1} = 50 - 10Q_1 - 10 = 40 - 10Q_1 = 0$$

$$Q_1{}^* = 4$$

$$\frac{\partial \pi}{\partial Q_2} = 30 - 4Q_2 - 10 = 20 - 4Q_2 = 0$$

$$Q_2{}^* = 5$$

$$P_1{}^* = 50 - 5(4) = 50 - 20 = 30$$

$$P_2{}^* = 30 - 2(5) = 30 - 10 = 20$$

b. The relationships between the selling price and the price elasticity of demand in the two markets are

$$MR_1 = P_1\left(1 + \frac{1}{\varepsilon_1}\right)$$

$$MR_2 = P_2\left(1 + \frac{1}{\varepsilon_2}\right)$$

where

$$\varepsilon_1 = \left(\frac{dQ_1}{dP_1}\right)\left(\frac{P_1}{Q_1}\right)$$

$$\varepsilon_2 = \left(\frac{dQ_2}{dP_2}\right)\left(\frac{P_2}{Q_2}\right)$$

From the demand equations, $dQ_1/dP_1 = -0.2$ and $dQ_2/dP_2 = -0.5$. Substituting these results into preceding above relationships, we obtain

$$\varepsilon_1 = (-0.2)\left(\frac{30}{4}\right) = \frac{-6}{4} = -1.5$$

$$\varepsilon_2 = (-0.5)\left(\frac{20}{5}\right) = \frac{-10}{5} = -2$$

This verifies that the higher price is charged in the market where the price elasticity of demand is less elastic.

c. The firm's total profit at the profit-maximizing prices and output levels are

$$\pi^* = 50(4) - 5(4)^2 + 30(5) - 2(5)^2 - 6 - 10(4+5)$$
$$= 200 - 80 + 150 - 50 - 6 - 90 = 124$$

Problem 11.6. Copperline Mountain is a world-famous ski resort in Utah. Copperline Resorts operates the resort's ski-lift and grooming operations. When weather conditions are favorable, Copperline's total operating cost, which depends on the number of skiers who use the facilities each year, is given as

$$TC = 10S + 6$$

where S is the total number of skiers (in hundreds of thousands). The management of Copperline Resorts has determined that the demand for ski-lift tickets can be segmented into adult (S_A) and children 12 years old and under (S_C). The demand curve for each group is given as

$$S_A = 10 - 0.2P_A$$

$$S_C = 15 - 0.5P_C$$

where P_A and P_C are the prices charged for adults and children, respectively.

a. Assuming that Copperline Resorts is a profit maximizer, how many skiers will visit Copperline Mountain?
b. What prices should the company charge for adult and child's ski-lift tickets?
c. Assuming that the second-order conditions for profit maximization are satisfied, what is Copperline's total profit?

Solution

a. Total profit is given by the expression

$$\pi = TR - TC = (TR_A + TR_C) - TC$$
$$= P_A S_A + P_C S_C - TC$$
$$= (50 - 5S_A)S_A + (30 - 2S_C)S_C - [10(S_A + S_C) + 6]$$
$$= -6 + 40S_A + 20S_C - 5S_A^2 - 2S_C^2$$

Taking the first partial derivatives with respect to S_A and S_C, setting the results equal to zero, and solving, we write

$$\frac{\partial \pi}{\partial S_A} = 40 - 10S_A = 0$$

$$S_A = 4$$

$$\frac{\partial \pi}{\partial S_C} = 20 - 4S_C = 0$$

$$S_C = 5$$

The total number of skiers that will visit Copperline Mountain is

$$S = S_A = S_C = 4 + 5 = 9\ (\times 10^5)\ \text{skiers}$$

b. Substituting these results into the demand functions yields adult and child's, ski-lift ticket prices.

$$4 = 10 - 0.2P_A$$

$$P_A = \$30$$

$$5 = 15 - 0.5P_C$$

$$P_C = \$20$$

c. Substituting the results from part a into the total profit equation yields

$$\pi = -6 + 40(4) + 20(5) - 5(4)^2 - 2(5)^2$$
$$= -6 + 160 + 100 - 80 - 50 = \$124\ (\times 10^3)$$

Problem 11.7. Suppose that a firm sells its product in two separable markets. The demand equations are

$$Q_1 = 100 - P_1$$

$$Q_2 = 50 - 0.25P_2$$

The firm's total cost equation is

$$TC = 150 + 5Q + 0.5Q^2$$

a. If the firm engages in third-degree price discrimination, how much should it sell, and what price should it charge, in each market?
b. What is the firm's total profit?

Solution

a. Assuming that the firm is a profit maximizer, set $MR = MC$ in each market to determine the output sold and the price charged. Solving the demand equation for P in each market yields

$$P_1 = 100 - Q_1$$

$$P_2 = 200 - 4Q_2$$

The respective total and marginal revenue equations are

$$TR_1 = 100Q_1 - Q_1^2$$

$$TR_2 = 200Q_2 - Q_2^2$$

$$MR_1 = 100 - 2Q_1$$

$$MR_2 = 200 - 8Q_2$$

The firm's marginal cost equation is

$$MC = \frac{dTC}{dQ} = 5 + Q$$

Setting $MR = MC$ for each market yields

$$100 - 2Q_1 = 5 + Q_1$$

$$200 - 8Q_2 = 5 + Q_2$$

$$Q_1^* = 31.67$$

$$Q_2^* = 15$$

$$P_1^* = 100 - 31.67 = \$68.33$$

$$P_2^* = 200 - 4(15) = \$140.00$$

b. The firm's total profit is

$$\pi^* = P_1^*Q_1^* + P_2^*Q_2^* - \left[150 + 5\left(Q_1^* + Q_2^*\right) + 0.5\left(Q_1^* + Q_2^*\right)^2\right]$$
$$= 68.33(31.67) + 140(15) - (150 + 233.35 + 1,089.04) = \$2,791.62$$

Problem 11.8. Suppose that the firm in Problem 11.7 charges a uniform price in the two markets in which it sells its product.

a. Find the uniform price charged, and the quantity sold, in the two markets.
b. What is the firm's total profit?
c. Compare your answers to those obtained in Problem 11.7.

Solution

a. To determine the uniform price charged in each market, first add the two demand equations:

$$Q = Q_1 + Q_2 = 100 - P_1 + 50 - 0.25P_2 = 150 - 1.25P$$

Next, solve this equation for P:

$$P = 120 - 0.8Q$$

The total and marginal revenue equations are

$$TR = PQ = 120Q - 0.8Q^2$$

$$MR = 120 - 1.6Q$$

The profit-maximizing level of output is

$$MR = MC$$

$$120 - 1.6Q = 5 + Q$$

$$Q^* = 44.23$$

That is, the profit-maximizing output of the firm is 44.23 units. The uniform price is determined by substituting this result into the combined demand equation:

$$P^* = 120 - 0.8(44.23) = 120 - 35.38 = \$84.62$$

The amount of output sold in each market is

$$Q_1^* = 100 - 84.62 = 15.38$$

$$Q_2^* = 50 - 0.25(84.62) = 50 - 21.16 = 28.85$$

Note that the combined output of the two markets is equal to the total output Q^* already derived.

b. The firm's total profit is

$$\begin{aligned}\pi^* &= P^*Q^* - (150 + 5Q^* + 0.5Q^{*2})\\ &= 84.62(44.23) - \left[150 + 5(44.23) + 0.5(44.23)^2\right]\\ &= 3{,}742.74 - (150 + 221.15 + 978.15) = \$2{,}393.44\end{aligned}$$

c. The uniform price charged ($84.62) is between the prices charged in the two markets ($68.33 and $140.00) when the firm engaged in third-degree price discrimination. When the firm engaged in uniform pricing, the amount of output sold is lower in the first market (15.38 units compared with 31.67 units) and higher in the second market (28.85 units compared with 15 units). Finally, the firm's total profit with uniform pricing ($2,393.44) is lower than when the firm engaged in third-degree price discrimination ($2,791.62, from Problem 11.7).

When third-degree price discrimination is practiced in foreign trade it is sometimes referred to as *dumping*. This rather derogatory term is often used by domestic producers claiming unfair foreign competition. Defined by the U.S. Department of Commerce as selling at below fair market value, dumping results when a profit-maximizing exporter sells its product at a different, usually lower, price in the foreign market than it does in its home market. Recall that when resale between two markets is not possible, the monopolist will sell its product at a lower price in the market in which demand is more price elastic. In international trade theory, the difference between the home price and the foreign price is called the *dumping margin*.

NONMARGINAL PRICING

Most of the discussion of pricing practices thus far has assumed that management is attempting to optimize some corporate objective. For the most part, we have assumed that management attempts to maximize the firm's profits, but other optimizing behavior has been discussed, such as revenue maximization. In each case, we assumed that the firm was able to calculate its total cost and total revenue equations, and to systematically use that information to achieve the firm's objectives. If the firm's objective is to maximize profit, for example, then management will produce at an output level and charge a price at which marginal revenue equals marginal cost. This is the classic example of marginal pricing.

In reality, however, firms do not know their total revenue and total cost equations, nor are they ever likely to. In fact, because firms do not have this information, and in spite management's protestations to the contrary, most firms are (unwittingly) not profit maximizers. Moreover, even if this information were available, there are other corporate objectives, such as satisficing behavior, that do not readily lend themselves to marginal pricing strategies. Consequently, most firms engage in nonmarginal pricing. The most popular form of nonmarginal pricing is cost-plus pricing.

Definition: Firms determine the profit-maximizing price and output level by equating marginal revenue with marginal cost. When the firm's total revenue and total cost equations are unknown, however, management will often practice nonmarginal pricing. The most popular form of nonmarginal pricing is cost-plus pricing, also known as markup or full-cost pricing.

COST-PLUS PRICING

As we have seen, profit maximization occurs at the price–quantity combination at which where marginal cost equals marginal revenue. In reality, however, many firms are unable or unwilling to devote the resources necessary to accurately estimate the total revenue and total cost equations, or do not know enough about demand and cost conditions to determine the profit-maximizing price and output levels. Instead, many firms adopt rule-of-thumb methods for pricing their goods and services. Perhaps the most commonly used pricing practice is that of *cost-plus pricing*, also known as *mark up* or *full-cost pricing*. The rationale behind cost-plus pricing is straightforward: approximate the average cost of producing a unit of the good or service and then "mark up" the estimated cost per unit to arrive at a selling price.

Definition: Cost-plus pricing is the most popular form of nonmarginal pricing. It is the practice of adding a predetermined "markup" to a firm's estimated per-unit cost of production at the time of setting the selling price.

The firm begins by estimating the average variable cost (AVC) of producing a good or service. To this, the company adds a per-unit allocation for fixed cost. The result is sometimes referred to as the *fully allocated per-unit cost* of production. With the per-unit allocation for fixed cost denoted AFC and the fully allocated, average total cost ATC, the price a firm will charge for its product with the percentage mark up is

$$P = ATC(1+m) \tag{11.21}$$

where m is the percentage markup over the fully allocated per-unit cost of production. Solving Equation (11.21) for m reveals that the mark up may also be expressed as the difference between the selling price and the per-unit cost of production.

$$m = \frac{P - ATC}{ATC} \tag{11.22}$$

The numerator of Equation (11.22) can also be written as $P - AVC - AFC$. The expression $P - AVC$ is sometimes referred to as the *contribution margin per unit*. The marked-up selling price, therefore, may be referred to as the profit contribution per unit plus some allocation to defray overhead costs.

Problem 11.9. Suppose that the Nimrod Corporation has estimated the average variable cost of producing a spool of its best-selling brand of industrial wire, Mithril, at $20. The firm's total fixed cost is $20,000.

a. If Nimrod produces 500 spools of Mithril and its standard pricing practice is to add a 25% markup to its estimated per-spool cost of production, what price should Nimrod charge for its product?
b. Verify that the selling price calculated in part a represents a 25% markup over the estimated per-spool cost of production.

Solution

a. At a production level of 500 spools, Nimrod's per-unit fixed cost allocation is

$$AFC = \frac{20,000}{500} = 40$$

The cost-plus pricing equation is given as

$$P = ATC(1 + m)$$

where m is the percentage markup and ATC is the sum of the average variable cost of production (AVC) and the per-unit fixed cost allocation (AFC). Substituting, we write

$$P = (20 + 40)(1 + 0.25) = 60(1.25) = \$75$$

Nimrod should charge $75 per spool of Mithril. In other words, Nimrod should charge $15 over its estimated per-unit cost of production.

b. The percentage markup is given by the equation

$$m = \frac{(P - ATC)}{ATC}$$

Substituting the relevant data into this equation yields

$$m = \frac{75 - 60}{60} = \frac{15}{60} = 0.25$$

Of course, the advantage of cost-plus pricing is its simplicity. Cost-plus pricing requires less than complete information, and it is easy to use. Care must be exercised, however, when one is using this approach. The usefulness of cost-plus pricing will be significantly reduced unless the appropriate cost concepts are employed. As in the case of break-even analysis, care must be taken to include all relevant costs of production. Cost-plus pricing, which is based only on accounting (explicit) costs, will move the firm further away from an optimal (profit-maximizing) price and output level. Of course, the more appropriate approach would be to calculate total economic costs, which include both explicit and implicit costs of production.

There are two major criticisms of cost-plus pricing. The first criticism involves the assumption of fixed marginal cost, which at fixed input prices is in defiance of the law of diminishing marginal product. It is this assumption that allows us to further assume that marginal cost is approximately equal to the fully allocated per-unit cost of production. If it can be argued, however, that marginal cost is approximately constant over the firm's range of production, this criticism loses much of its sting.

A perhaps more serious criticism of cost-plus pricing is that it is insensitive to demand conditions. It should be noted that, in practice, the size of a firm's markup tends to reflect the price elasticity of demand for of goods of various types. Where the demand for a product is relatively less price elastic, because of, say, the paucity of close substitutes, the markup tends to be higher than when demand is relatively more price elastic. As will be presently demonstrated, to the extent that this observation is correct, the criticism of insensitivity loses some of its bite.

Recall from our discussion of the relationship between the price elasticity of demand and total revenue in Chapter 4, the relationship between marginal revenue, price, and the price elasticity of demand may be expressed as

$$MR = P\left(1 + \frac{1}{\varepsilon_p}\right) \tag{4.15}$$

The first-order condition for profit maximization is $MR = MC$. Replacing MR with MC in Equation (4.15) yields

$$MC = P\left(1 + \frac{1}{\varepsilon_p}\right) \tag{11.23}$$

Solving Equation (11.23) for P yields

$$P = \frac{MC}{1 + 1/\varepsilon_p} \tag{11.24}$$

If we assume that MC is approximately equal to the firm's fully allocated per-unit cost (ATC), Equation (11.24) becomes,

$$P = \frac{ATC}{1 + 1/\varepsilon_p} \tag{11.25}$$

Equating the right-hand side of this result to the right-hand side of Equation (11.21), we obtain

$$\frac{ATC}{1 + 1/\varepsilon_p} = ATC(1 + m)$$

where m is the percentage markup. Solving this expression for the markup yields

$$m = \frac{-1}{\varepsilon_p + 1} \tag{11.26}$$

Equation (11.26) suggests that when demand is price elastic, then the selling price should have a positive markup. Moreover, the greater the price elasticity of demand, the lower will be the markup. Suppose, for example, that $\varepsilon_p = -2.0$. Substituting this value into Equation (11.26), we find that the markup is $m = -1/(-2 + 1) = -1/-1 = 1$, or 100%. On the other hand, if $\varepsilon_p = -5.0$, then $m = -1/(-5 + 1) = -1/-4 = 0.25$, or a 25% markup.

What happens, however, if the demand for the good or service is price inelastic? Suppose, for example, that $\varepsilon_p = -0.8$. Substituting this into Equation (11.26) results in a markup of $m = -1/(-0.8 + 1) = -1/0.2 = -5$. This result suggests that the firm should mark down the price of its product by 500%! Equation (11.26) suggests that if the demand for a product is price inelastic, the firm should sell its output at below the fully allocated per-unit cost of production, a practice that is clearly not observed in the real world. Fortunately, this apparent paradox is easily resolved.

It will be recalled from Chapter 4, and is easily seen from Equation (4.15), that when the demand for a good or service is price inelastic, it marginal revenue must be negative. For the profit-maximizing firm, this suggests that marginal cost is negative, since the first-order condition for profit maximization is $MR = MC$, which is clearly impossible for positive input prices and positive marginal product of factors of production.

Problem 11.10. What is the estimated percentage markup over the fully allocated per-unit cost of production for the following price elasticities of demand?

a. $\varepsilon_p = -11$
b. $\varepsilon_p = -4$
c. $\varepsilon_p = -2.5$
d. $\varepsilon_p = -2.0$
e. $\varepsilon_p = -1.5$

Solution

a. $m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-11+1} = 0.10$ or a 10% mark up

b. $m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-4+1} = 0.333$ or a 33.3% mark up

c. $m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-2.5+1} = 0.667$ or a 66.7% mark up

d. $m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-2.0+1} = 1.0$ or a 100% mark up

e. $m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-1.5+1} = 2.0$ or a 200% mark up

Problem 11.11. What is the percentage markup on the output of a firm operating in a perfectly competitive industry?

Solution. A firm operating in a perfectly competitive industry faces an infinitely elastic demand for its product. Substituting $\varepsilon_p = -\infty$ into Equation (11.26) yields

$$m = \frac{-1}{\varepsilon_p + 1} = \frac{-1}{-\infty + 1} = 0$$

A firm operating in a perfectly competitive industry cannot mark up the selling price of its product. This is as it should be, since such a firm has no market power; that is, the firm is a price taker. The firm must sell its product at the market-determined price.

Problem 11.12. Suppose that a firm's marginal cost of production is constant at $25. Suppose further that the price elasticity of demand (ε_p) for the firm's product is +5.0.

a. Using cost-plus pricing, what price should the firm charge for its product?
b. Suppose that ε_p = –0.5. What price should the firm charge for its product?

Solution

a. The firm's profit-maximizing condition is

$$MR = MC$$

Recall from Chapter 4 that

$$MR = P\left(1 + \frac{1}{\varepsilon_p}\right)$$

Substituting this result into the profit-maximizing condition yields

$$MC = P\left(1 + \frac{1}{\varepsilon_p}\right)$$

Since *MC* is constant, then *MC* = *ATC*. After substituting, and rearranging, we obtain

$$P^* = ATC\frac{\varepsilon_p}{\varepsilon_p + 1} = 25\left(\frac{-5}{-5+1}\right) = 25\left(\frac{-5}{-4}\right) = \$31.25$$

b. If ε_p = –0.5, then

$$P^* = 25\left(\frac{-0.5}{-0.5+1}\right) = 25\left(\frac{-0.5}{0.5}\right) = -\$25.00$$

This result, however, is infeasible, since a firm would never charge a negative price for its product. Recall that a profit-maximizing firm will never produce along the inelastic portion of the demand curve.

MULTIPRODUCT PRICING

We have thus far considered primarily firms that produce and sell only one good or service at a single price. The only exception to this general statement was our discussion of commodity bundling, in which a firm sells a package of goods at a single price. We will now address the issue of pricing strategies of a single firm selling more than one product under alternative scenarios. These scenarios include the optimal pricing of two or more products with interdependent demands, optimal pricing of two or more products with independent demands that are jointly produced in variable proportions, and optimal pricing of two or more products with independent demands that are jointly produced in fixed proportions.

Definition: Multiproduct pricing involves optimal pricing strategies of firms producing and selling more than one good or service.

OPTIMAL PRICING OF TWO OR MORE PRODUCTS WITH INTERDEPENDENT DEMANDS AND INDEPENDENT PRODUCTION

Often a firm will produce two or more goods that are either complements or substitutes for each other. Dell Computer, for example, sells a number of different models of personal computers. These models are, to a degree, substitutes for each other. Personal computers also come with a variety of accessories (mouses, printers, modems, scanners, etc.). These options not only come in different models, and are, therefore, substitutes for each other, but they are also complements to the personal computers.

Because of the interrelationships inherent in the production of some goods and services, it stands to reason that an increase in the price of, say, a Dell personal computer model will lead to a reduction in the quantity demanded of that model and an increase in the demand for substitute models. Moreover, an increase in the price of the Dell personal computer model will lead to a reduction in the demand for complementary accessories. For this reason, a profit-maximizing firm must ascertain the optimal prices and output levels of each product manufactured jointly, rather than pricing each product independently.

The problem may be formally stated as follows. Consider the demand for two products produced by the same firm. If these two products are related, the demand functions may be expressed as

$$Q_1 = f_1(P_1, Q_2) \tag{11.27a}$$

$$Q_2 = f_2(P_2, Q_1) \tag{11.27b}$$

By the law of demand, $\partial Q_1/\partial P_1$ and $\partial Q_2/\partial P_2$ are negative. The signs of $\partial Q_1/\partial Q_2$ and $\partial Q_2/\partial Q_1$ depend on the relationship between Q_1 and Q_2. If the values of these first partial derivatives are positive, then Q_1 and Q_2 are complements. If the values of these first partials are negative, then Q_1 and Q_2 are substitutes.

Upon solving Equation (11.27a) for P_1 and Equation (11.27b) for P_2, and substituting these results into the total revenue equations, we write

$$TR_1(Q_1, Q_2) = P_1Q_1 = h_1(Q_1, Q_2)Q_1 \tag{11.28a}$$

$$TR_2(Q_1, Q_2) = P_2Q_2 = h_2(Q_1, Q_2)Q_2 \tag{11.28b}$$

Since the two goods are independently produced, the total cost functions are

$$TC_1 = TC_1(Q_1) \tag{11.29a}$$

$$TC_2 = TC_2(Q_2) \tag{11.29b}$$

The total profit equation for this firm is, therefore,

$$\begin{aligned} \pi &= TR_1(Q_1, Q_2) + TR_2(Q_1, Q_2) - TC_1(Q_1) - TC_2(Q_2) \\ &= P_1Q_1 + P_2Q_2 + TC_1(Q_1) - TC_2(Q_2) \\ &= h_1(Q_1, Q_2)Q_1 + h_2(Q_1, Q_2)Q_2 - TC_1(Q_1) - TC_2(Q_2) \end{aligned} \tag{11.30}$$

The first-order conditions for profit maximization are

$$\frac{\partial \pi}{\partial Q_1} = \frac{\partial TR_1}{\partial Q_1} + \frac{\partial TR_2}{\partial Q_1} - \frac{\partial TC_1}{\partial Q_1} = 0 \tag{11.31a}$$

$$\frac{\partial \pi}{\partial Q_2} = \frac{\partial TR_2}{\partial Q_2} + \frac{\partial TR_1}{\partial Q_2} - \frac{\partial TC_2}{\partial Q_2} = 0 \tag{11.31b}$$

which may be expressed as

$$MC_1 = \frac{\partial TR_1}{\partial Q_1} + \frac{\partial TR_2}{\partial Q_1} \tag{11.32a}$$

$$MC_2 = \frac{\partial TR_2}{\partial Q_2} + \frac{\partial TR_1}{\partial Q_2} \tag{11.32b}$$

We will assume that the second-order conditions for profit maximization are satisfied.

Equations (11.32) indicate that a firm producing two products with interrelated demands will maximize its profits by producing where marginal cost is equal to the change in total revenue derived from the sale of the product itself, plus the change in total revenue derived from the sale of the related product. If the second term on the right-hand side of Equation (11.31) is positive, then Q_1 and Q_2 are complements. If this term is negative, then Q_1 and Q_2 are substitutes.

Problem 11.13. Gizmo Brothers, Inc., manufactures two types of hi-tech yo-yo: the Exterminator and the Eliminator. Denoting Exterminator output as Q_1 and Eliminator output as Q_2, the company has estimated the following demand equations for its yo-yos:

$$Q_1 = 10 - 0.2P_1 - 0.4Q_2$$

$$Q_2 = 20 - 0.5P_2 - 2Q_1$$

The total cost equations for producing Exterminators and Eliminators are

$$TC_1 = 4 + 2Q_1^2$$

$$TC_2 = 8 + 6Q_2^2$$

a. If Gizmo Brothers is a profit-maximizing firm, how much should it charge for Exterminators and Eliminators? What is the profit-maximizing level of output for Exterminators and Eliminators?
b. What is Gizmo Brothers's profit?

Solution

a. Solving the demand equations for P_1 and P_2, respectively, yields

$$P_1 = 50 - 5Q_1 - 2Q_2$$

$$P_2 = 40 - 2Q_2 - 4Q_1$$

The profit equation is

$$\begin{aligned}\pi &= TR_1(Q_1, Q_2) + TR_2(Q_1, Q_2) - TC_1(Q_1) - TC_2(Q_2)\\ &= P_1Q_1 + P_2Q_2 - TC_1(Q_1) - TC_2(Q_2)\end{aligned}$$

Substitution yields

$$\begin{aligned}\pi &= (50 - 5Q_1 - 2Q_2)Q_1 + (40 - 2Q_2 - 4Q_1)Q_2 - (4 + 2Q_1^2) - (8 + 6Q_2^2)\\ &= 50Q_1 + 40Q_2 - 6Q_1Q_2 - 7Q_1^2 - 8Q_2^2 - 12\end{aligned}$$

The first-order conditions for profit maximization are

$$\frac{\partial \pi}{\partial Q_1} = 50 - 14Q_1 - 6Q_2 = 0$$

$$\frac{\partial \pi}{\partial Q_2} = 40 - 6Q_1 - 16Q_2 = 0$$

Recall from Chapter 2 that the second-order conditions for profit maximization are

$$\frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} < 0$$

$$\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right) - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2 > 0$$

The appropriate second partial derivatives are

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -14 < 0$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} = -16 < 0$$

$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = -6$$

$$(-14)(-16) - (6)^2 = 244 - 36 = 208 > 0$$

Thus, the second-order conditions for profit maximization are satisfied. Solving the first-order conditions for Q_1 and Q_2 we obtain

$$14Q_1 + 6Q_2 = 50$$

$$6Q_1 + 16Q_2 = 40$$

which may be solved simultaneously to yield

$$Q_1{}^* = 2.979$$

$$Q_2{}^* = 1.383$$

Upon substituting these results into the price equations, we have

$$P_1{}^* = 50 - 5(2.979) - 2(1.383) = \$32.34$$

$$P_2{}^* = 40 - 2(1.383) - 4(2.979) = \$25.32$$

b. Gizmo Brothers's profit is

$$\pi = 50(2.979) + 40(1.383) - 6(2.979)(1.383) - 7(2.979)^2 - 8(1.383)^2 - 12$$
$$= \$90.17$$

OPTIMAL PRICING OF TWO OR MORE PRODUCTS WITH INDEPENDENT DEMANDS JOINTLY PRODUCED IN VARIABLE PROPORTIONS

Let us now suppose that a firm sells two goods with independent demands that are jointly produced in variable proportions. An example of this might be a consumer electronics company that produces automobile taillight bulbs and flashlight bulbs on the same assembly line. In this case, the demand functions are given by the expressions

$$Q_1 = f_1(P_1) \tag{11.33a}$$

$$Q_2 = f_2(P_2) \tag{11.33b}$$

where $\partial Q_1/\partial P_1$ and $\partial Q_2/\partial P_2$ are negative. The total cost function is given by the expression

$$TC = TC(Q_1, Q_2) \tag{11.34}$$

The firm's total profit function is

$$\pi = TR_1(Q_1) + TR_2(Q_2) - TC(Q_1, Q_2) \tag{11.35}$$

Solving the demand equations for P_1 and P_2 and substituting the results into Equation (11.35) yields

$$\begin{aligned} \pi &= P_1Q_1 + P_2Q_2 - TC(Q_1, Q_2) \\ &= h_1(Q_1)Q_1 + h_2(Q_2)Q_2 - TC(Q_1, Q_2) \end{aligned} \tag{11.36}$$

The first-order conditions for profit maximization are

$$\frac{\partial \pi}{\partial Q_1} = \frac{\partial TR_1}{\partial Q_1} - \frac{\partial TC_1}{\partial Q_1} = 0 \tag{11.37a}$$

$$\frac{\partial \pi}{\partial Q_2} = \frac{\partial TR_2}{\partial Q_2} - \frac{\partial TC_2}{\partial Q_2} = 0 \tag{11.37b}$$

which may be written as

$$MR_1 = MC_1 \tag{11.38a}$$

$$MR_2 = MC_2 \tag{11.38b}$$

We will assume that the second-order conditions for profit maximization are satisfied.

Equations (11.38) indicate that a profit-maximizing firm jointly producing two goods with independent demands that are jointly produced in variable proportions will equate the marginal revenue generated from the sale of each good to the marginal cost of producing each product.

Problem 11.14. Suppose Gizmo Brothers also produces Tommy Gunn action figures for boys ages 7 to 12, and Bonzey, a toy bone for pet dogs. Except for the molding phase, both products are made on the same assembly line. Denoting Tommy Gunn as Q_1 and Bonzey as Q_2, the company has estimated the following demand equations:

$$Q_1 = 10 - 0.5P_1$$

$$Q_2 = 20 - 0.2P_2$$

The total cost equation for producing the two products is

$$TC = Q_1^2 + 2Q_1Q_2 + 3Q_2^2 + 10$$

a. As before, Gizmo Brothers is a profit-maximizing firm. Give the profit-maximizing levels of output for Tommy Gunn and for Bonzey. How much should the firm charge for Tommy Gunn and Bonzey?
b. What is Gizmo Brothers's profit?

Solution

a. Solving the demand equations for P_1 and P_2, respectively, yields

$$P_1 = 20 - 2Q_1$$

$$P_2 = 100 - 5Q_2$$

Gizmo Brothers's profit equation is

$$\pi = TR_1(Q_1) + TR_2(Q_2) - TC_1(Q_1, Q_2) = P_1Q_1 + P_2Q_2 - TC_1(Q_1, Q_2)$$

Substituting the demand equations into the profit equation yield

$$\begin{aligned}\pi &= (20 - 2Q_1)Q_1 + (100 - 5Q_2)Q_2 - (Q_1^2 + 2Q_1Q_2 + 3Q_2^2 + 10)\\ &= -10 + 20Q_1 + 100Q_2 - 3Q_1^2 - 8Q_2^2 - 2Q_1Q_2\end{aligned}$$

The first-order conditions for profit maximization are

$$\frac{\partial \pi}{\partial Q_1} = 20 - 6Q_1 - 2Q_2 = 0$$

$$\frac{\partial \pi}{\partial Q_2} = 100 - 16Q_2 - 2Q_1 = 0$$

The second-order conditions for profit maximization are

$$\frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} < 0$$

$$\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right) - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2 > 0$$

The appropriate second-partial derivatives are

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -6 < 0$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} = -16 < 0$$

$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = -2$$

$$(-6)(-16) - (-2)^2 = 96 - 4 = 92 > 0$$

Thus, the second-order conditions for profit maximization are satisfied. Solving the first-order conditions for Q_1 and Q_2 yields

$$6Q_1 + 2Q_2 = 20$$

$$2Q_1 + 16Q_2 = 100$$

which may be solved simultaneously to yield

$$Q_1{}^* = 1.304$$

$$Q_2{}^* = 6.087$$

Substituting these results into the price equations yields

$$P_1{}^* = 20 - 2(1.304) = \$17.39$$

$$P_2{}^* = 100 - 2(6.087) = \$69.66$$

b. Gizmo Brothers's profit is

$$\pi = 20(1.304) + 100(6.087) - 2(1.304)(6.087) - 3(1.304)^2 - 8(6.087)^2 - 10$$
$$= \$88.17$$

OPTIMAL PRICING OF TWO OR MORE PRODUCTS WITH INDEPENDENT DEMANDS JOINTLY PRODUCED IN FIXED PROPORTIONS

Now, let us assume that a firm jointly produces two goods in fixed proportions but with independent demands. In many cases, the second product is a by-product of the first, such as beef and hides. With joint production in fixed proportions, it is conceptually impossible to consider two separate products, since the production of one good automatically determines the quantity produced of the other.

Suppose that the demand functions for two goods produced jointly are given as Equations (11.33). The total cost equation is given as Equation (11.13).

$$TC(Q) = TC(Q_1 + Q_2) \tag{11.13}$$

The analysis differs, however, in that Q_1 and Q_2 are in direct proportion to each other, that is,

$$Q_2 = kQ_1 \tag{11.39}$$

where the constant $k > 0$. Solving Equation (11.33) for P_1 and P_2 yields

$$P_1 = h_1(Q_1) \tag{11.40a}$$

$$P_2 = h_2(Q_2) \tag{11.40b}$$

Substituting Equation (11.39) into Equations (11.13) and (11.40b) yields

$$P_1 = h_1(Q_1)$$

$$P_2 = h_2(Q_1) \tag{11.41}$$

$$TC(Q) = TC(Q_1) \tag{11.42}$$

Substituting Equations (11.39), (11.40a), (11.41), and (11.42) into Equation (11.36) yields the firm's profit equation:

$$\begin{aligned} \pi &= P_1Q_1 + P_2(kQ_1) - TC(Q_1) \\ &= h_1(Q_1)Q_1 + h_2(Q_1)(kQ_1) - TC(Q_1) \end{aligned} \tag{11.43}$$

Stated another way, the firm's total profit function is

$$\pi(Q_1) = TR_1(Q_1) + TR_2(Q_1) - TC(Q_1) \tag{11.44}$$

Equation (11.44) indicates that total profit is a function of the single decision variable, Q_1. Equation (11.44) may also be written

$$\pi(Q_2) = TR_1(Q_2) + TR_2(Q_2) - TC(Q_2) \tag{11.45}$$

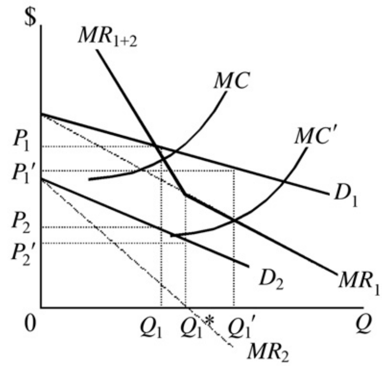

FIGURE 11.6 Optimal pricing of two goods jointly produced in fixed proportions with independent demands.

From Equation (11.44), the first-order condition for profit maximization is

$$\frac{d\pi}{dQ_1} = \frac{dTR_1}{dQ_1} + \frac{dTR_2}{dQ_1} - \frac{dTC_1}{dQ_1} = 0 \tag{11.46}$$

Equation (11.46) may be rewritten

$$\frac{dTR_1}{dQ_1} + \frac{dTR_2}{dQ_1} = \frac{dTC_1}{dQ_1}$$

$$MR_1(Q_1) + MR_2(Q_1) = MC(Q_1) \tag{11.47}$$

Equation (11.47) says that a profit-maximizing firm that jointly produces two goods in fixed proportions with independent demands will equate the sum of the marginal revenues of both products expressed in terms of one of the products with the marginal cost of jointly producing both products expressed in terms of the same product. This situation is depicted diagrammatically in Figure 11.6.

In Figure 11.6 the marginal cost curve is labeled MC. According to Equation (11.47) the firm should produce Q_1 units where marginal cost is equal to the sum of MR_1 and MR_2. The amount of Q_2 produced is proportional to Q_1. At that output level the firm charges P_1 for Q_1 and P_2 for Q_2. It should be noted that beyond output level Q_1^* in Figure 11.6, MR_2 becomes negative and MR_{1+2} becomes simply MR_1.

Suppose that marginal cost increases to MC'. In this case, the firm should produce Q_1', but still only sell Q_1^* units. Any output in excess of Q_1^* should be disposed of, since the firm's marginal revenue beyond Q_1^* is negative. The amount of Q_2 produced will be in fixed proportion to Q_1'. The price of Q_1^* is P_2' and the price of Q_2 is P_1'.

Problem 11.15. Suppose that a firm produces two units of Q_2 for each unit of Q_1. Suppose further that the demand equations for these two goods are

$$Q_1 = 10 - 0.5P_1$$

$$Q_2 = 20 - 0.2P_2$$

The total cost of production is

$$TC = 10 + 5Q^2$$

a. What are the profit-maximizing output levels and prices for Q_1 and Q_2?
b. At the profit-maximizing output levels, what is the firm's total profit?

Solution

a. Solving the demand equations for P_1 and P_2 yields

$$P_1 = 20 - 2Q_1$$

$$P_2 = 100 - 5Q_2$$

The firm's total profit equation is

$$\begin{aligned}\pi &= P_1Q_1 + P_2Q_2 - TC(Q_1 + Q_2)\\ &= (20 - 2Q_1)Q_1 + (100 - 5Q_2)Q_2 - (10 + 5Q^2)\\ &= 20Q_1 - 2Q_1^2 + 100Q_2 - 5Q_2^2 - 10 - 5(Q_1 + Q_2)^2\end{aligned}$$

Since $Q_2 = 2Q_1$, this may be rewritten as

$$\begin{aligned}\pi &= 20Q_1 - 2Q_1^2 + 100(2Q_1) - 5(2Q_1)^2 - 10 - 5(Q_1 + 2Q_1)^2\\ &= -10 - 220Q_1 - 67Q_1^2\end{aligned}$$

The first-order condition for profit maximization is

$$\frac{d\pi}{dQ_1} = 220 - 134Q_1 = 0$$

The second-order condition for profit maximization is

$$\frac{d^2\pi}{dQ_1^2} < 0$$

Since $d^2\pi/dQ_1^2 = -137$ the second-order condition is satisfied. Solving the first-order condition for Q_1 yields

$$Q_1{}^* = 1.64$$

The profit-maximizing level of Q_2 is

$$Q_2{}^* = 2Q_1{}^* = 3.28$$

Substituting these results into the price equations yield

$$P_1{}^* = 20 - 2(1.64) = \$16.72$$

$$P_2{}^* = 100 - 5(3.28) = \$83.60$$

b. The firm's total profit is

$$\pi = 220(1.64) - 67(1.64)^2 - 10 = 360.80 - 180.20 - 10 = \$170.60$$

Problem 11.16. Suppose that a firm jointly produces two goods. Good B is a by-product of the production of good A. The demand equations for the two goods are

$$Q_A = 200 - 10P_A$$

$$Q_B = 120 - 5P_B$$

The firm's total cost equation is

$$TC = 500 + 15Q + 0.05Q^2$$

a. What is the profit-maximizing price for each product?
b. What is the firm's total profit?

Solution

a. Solving the demand equation for price yields

$$P_A = 20 - 0.1Q_A$$

$$P_B = 24 - 0.2Q_B$$

The respective total and marginal revenue equations are

$$TR_A = 20Q_A - 0.1Q_A^2$$

$$MR_A = 20 - 0.2Q_A$$

$$TR_B = 24Q_B - 0.2Q_B^2$$

$$MR_B = 24 - 0.4Q_B$$

The firm's marginal revenue equation is

$$MR = MR_A + MR_B = 20 - 0.2Q_A + 24 - 0.4Q_B = 44 - 0.6Q$$

The firm's marginal cost equation is

$$MC = \frac{dTC}{dQ} = 15 + 0.1Q$$

The profit-maximizing rate of output is

$$MR = MC$$

$$44 - 0.6Q = 15 + 0.1Q$$

$$Q^* = 41.43$$

The profit-maximizing prices for the two goods are

$$P_A^* = 20 - 0.1(41.43) = 20 - 4.14 = \$15.86$$

$$P_B^* = 20 - 0.2(41.43) = 24 - 8.29 = \$15.71$$

b. The firm's total profit is

$$\begin{aligned}\pi^* &= P_A^*Q^* + P_B^*Q^* - (500 + 15Q^* + 0.05Q^{*2})\\ &= 15.86(41.43) + 15.71(41.43) - \left[500 - 15(41.43) + 0.05(41.43)^2\right]\\ &= \$1,343.57\end{aligned}$$

PEAK-LOAD PRICING

In many markets the demand for a service is higher at certain times than at others. The demand for electric power, for example, is higher during the day than at night, and during summer and winter than during spring and fall. The demand for theater tickets is greater at night and on the weekends or for midweek matinees. Toll bridges have greater traffic during rush hours than at other times of the day. The demand for airline travel is greater during holiday seasons than at other times. During such "peak" periods it becomes difficult, if not impossible, to satisfy the demands of all customers. Thus the profit-maximizing firm will charge a higher price for the product during "peak" periods and a lower price during "off-peak" periods. This kind of pricing scheme is known as *peak-load pricing*.

Definition: Peak-load pricing is the practice of charging a higher price for a service when demand is high and capacity is fully utilized and a lower price when demand is low and capacity is underutilized.

Figure 11.7 illustrates an example of peak-load pricing for a profit-maximizing firm. Here the marginal cost of providing a service is assumed to be constant until capacity is reached at a peak output level of O_p. At the peak output level the marginal cost curve becomes vertical. This reflects the fact that to satisfy additional demand at O_p, the firm must increase its capacity, by building a new bridge, installing a new hydroelectric generator, or other high-cost measure.

The short-run production function is typically defined in terms of a time interval over which certain factors of production are "fixed." Strictly speaking, this assertion is incorrect. In principle, virtually any factor may be varied if the derived benefits are great enough. It is certainly the case, however, that some factors of production are more easily varied that others. It is clearly easier and less expensive to hire an additional worker at a moment's

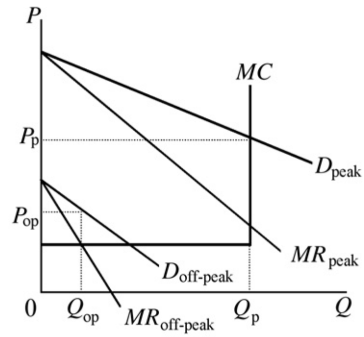

FIGURE 11.7 Peak-load pricing.

notice than to build a new bridge. Thus, it is reasonable to assume that the short-run marginal cost of expanding bridge traffic or increasing hydroelectric capacity is infinite. For that reason, the marginal cost curve at Q_p is assumed to be vertical.

To maximize profits subject to capacity limitations, the firm will charge different prices at different times. Off-peak prices are determined by equating marginal revenue to marginal operating costs. Peak prices, on the other hand, are determined by equating marginal revenue to the marginal cost of increasing capacity. In Figure 11.7, for example, $MR = MC$ for off-peak users at output level Q_{op}. At that output level the firm will charge off-peak users a price of P_{op}. On the other hand, the profit-maximizing level of output for peak users is at the firm's capacity, which in Figure 11.7 occurs at output level Q_p. At that output level the marginal cost curve of producing the service becomes vertical. The profit-maximizing price at that output level is P_p.

Peak-load pricing suggests that users of, say, congested bridges during rush hours, ought to be charged a higher toll than users during non–rush hour periods when there is excess capacity. Since peak-period demand strains capacity, the cost of additional capital investment ought to be borne by peak-period users. This tends to run contrary to the common practice on trains and toll bridges of offering multiple-use discounts to commuters traveling during rush hour, such as lower per-ride prices for, say, monthly tickets on commuter railways.

Problem 11.17. The Gotham Bridge and Tunnel Authority (GBTA) has estimated the following demand equations for peak and off-peak automobile users of the Frog's Neck Bridge:

$$\text{Peak: } T_p = 10 - 0.02Q_p$$

$$\text{Off-peack: } T_{op} = 5 - 0.05Q_{op}$$

where T is the toll charged for a one-minute trip (Q) across the bridge. The marginal cost of operating the bridge has been estimated at \$2 per automobile bridge crossing. The peak capacity of the Frog's Neck Bridge has been estimated a 50 automobiles per minute. What toll should the GBTA charge peak and off-peak users of the bridge?

Solution. This is a problem of peak-load pricing. If the GBTA is a profit maximizer, then off-peak drivers should be charged a price consistent with the first-order condition for profit maximization, $MC = MR$. The total revenue equation for off-peak users of the bridge is given as

$$TR_{\text{op}} = T_{\text{op}}Q_{\text{op}} = (5 - 0.05Q_{\text{op}})Q_{\text{op}} = 5Q - 0.05Q_{\text{op}}^2$$

The marginal revenue equation is

$$MR_{\text{op}} = \frac{dTR}{dQ_{\text{op}}} = 5 - 0.1Q_{\text{op}}$$

Equating marginal revenue to marginal cost yields

$$MR_{\text{op}} = MC_{\text{op}}$$

$$5 - 0.1Q_{\text{op}} = 2$$

$$Q_{\text{op}} = 30$$

Substituting this result into the off-peak demand equation yields the toll charged to off-peak automobile users of the bridge:

$$T_{\text{op}} = 5 - 0.05Q_{\text{op}} = 5 - 0.05(30) = \$3.50$$

At a bridge capacity of 50 automobiles per minute, the marginal cost curve is vertical. Substituting bridge capacity into the peak demand equation yields the toll that should be charged to peak automobile users of the bridge:

$$T_p = 10 - 0.02(50) = \$9$$

Peak users of the bridge should be charged \$9 per crossing.

TRANSFER PRICING

In recent years, the growth of large, conglomerate corporations producing a multitude of products has been accompanied by the parallel development of semiautonomous profit centers or subsidiaries. The creation of these "companies within a company" was an attempt to control rising production costs that accompanied the burgeoning managerial and administrative superstructure necessary to coordinate the activities of multiple corporate divisions.

Often the output of a division or subsidiary of a parent company is used as a productive input in the manufacture of the output of another division. A subsidiary of a large, multinational firm, for example, might assemble automobiles, while another subsidiary manufactures automobile bodies. Still another subsidiary might produce air and oil filters, while yet another produces electronic ignition systems, all of which are used in the production of automobiles.

Transfer pricing concerns itself with the correct pricing of intermediate products that are produced and sold between divisions of a parent company. For example, what price should one division of a company that produces, say, ignition systems, charge another division that assembles automobiles. The optimal pricing of intermediary goods is important because the organizational objective of each division is to maximize profit. What is more, the price charged for the output of one division that is used as an input in the production of another division affects not only each division's profits but also profits of the parent company as a whole.

Definition: Transfer pricing involves the optimal pricing of the output of one subsidiary of a parent company that is sold as an intermediate good to another subsidiary of the same parent company.

The literature dealing with transfer pricing typically focuses on three possible scenarios. In the first scenario, there is no external market for the output of the division or subsidiary producing the intermediate good. In other words, the division producing the final product is the sole customer for the output of the division producing the intermediate good. In the second scenario there exists a perfectly competitive external market for the intermediate good. In the third scenario the division or subsidiary operates in an industry that may be characterized as imperfectly competitive.

TRANSFER PRICING WITH NO EXTERNAL MARKET

Assume that a parent company comprises two subsidiary companies. One subsidiary sells its output, Q_1, exclusively to the other subsidiary that is used in the production of Q_2, for final sale in an external market. Assume further that there exists no other demand for Q_1; that is, there is no external market for the intermediate good. Finally assume that one unit of Q_1 is used to produce one unit of Q_2.

Since the parent company comprises only two subsidiaries, the marginal cost of producing Q_2 for final sale must include the marginal cost of producing Q_1. The rationale for this is straightforward. Although the company has been divided into separate profit centers, in the final analysis the company is in the business of producing and selling Q_2 for final sale. The

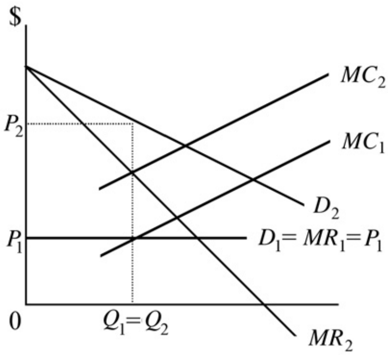

FIGURE 11.8 Transfer pricing with no external market for the intermediate good.

marginal cost of producing Q_2 must, of course, include the marginal cost of producing Q_1. If we assume that the parent company is a profit maximizer, it will produce at the output level where $MR_2 = MC_2$. This situation is illustrated in Figure 11.8.

Since Q_1 and Q_2 are used on a one-to-one basis, the output level that maximizes profit of Q_2 will be the same output level as Q_1. The selling price of Q_2 is P_2. Since the output level of Q_1 has been determined by the output level of Q_2, the profit-maximizing price for Q_1 must be P_1 (i.e., where $MC_1 = MR_1$). Thus, the correct transfer price for the intermediate good Q_1 must be P_1. It should be noted that any increase in the marginal cost of producing Q_1 will result in an increase in the marginal cost of producing Q_2, which will further result in an increase in the selling price of Q_2 and an increase in the transfer price (i.e., the price of the intermediate good that is sold between divisions).

On the other hand, suppose that the marginal cost curve of producing Q_1 remains unchanged, but the marginal cost curve of producing Q_2 shifts upward, perhaps because of an increase in factor or intermediate goods prices purchased in the external market. The result will be an increase in P_2, a decline in the output of Q_2 and Q_1, and, assuming an upward sloping marginal cost curve for Q_1, a fall in the transfer price. When the marginal cost of producing Q_1 is constant, the transfer price remains unaffected by the additional increase in the marginal cost of producing Q_2.

Problem 11.18. Parallax Corporation produces refractive telescopes for amateur astronomers. The demand equation for Parallax telescopes was estimated by the operations research department as

$$Q_T = 2{,}000 - 20P_T$$

Parallax's total cost equation was estimated as

$$TC_T = 100 + 2Q_T^2$$

Although the company procures most of its components from outside vendors, each Parallax telescope requires three highly polished lenses that are manufactured on site. Because these components are manufactured to exact specifications, there is no outside market for Parallax lenses. The total cost equation for producing Parallax lenses is

$$TC_L = 200 + 0.025Q_L^2$$

Because of the rapid growth of the company in the 1980s, Parallax management decided to divide the company into two separate profit centers to control costs—the telescope division and the lens division.

a. What is the profit-maximizing price and quantity for Parallax telescopes?
b. What is Parallax's total profit?
c. What transfer price should the lens division charge the telescope division?

Solution

a. Solving the demand equation for P_T yields

$$P_T = 100 - 0.05Q_T$$

The corresponding total revenue equation for Parallax telescopes is

$$TR_T = P_TQ_T = (100 - 0.05Q_T)Q_T = 100Q_T - 0.05Q_T^2$$

The total profit equation for Parallax telescopes is

$$\begin{aligned}\pi_T &= TR_T - TC_T = (100Q_T - 0.05Q_T^2) - (100 + 2Q_T^2)\\ &= -100 + 100Q_T - 2.05Q_T^2\end{aligned}$$

The first-order condition for profit maximization is $d\pi/dQ_T = 0$. Taking the first derivative of the profit equation yields

$$\frac{d\pi}{dQ} = 100 - 4.1Q_T = 0$$

Solving, we have

$$Q_T{}^* = 24.39$$

The second-order condition for profit maximization is $d\pi/dQ_T < 0$. After taking the second derivative of the profit equation, we obtain

$$\frac{d^2\pi}{dQ^2} = -4.1 < 0$$

which guarantees that this output level represents a local maximum.

The profit-maximizing price, therefore, is

$$P_T^* = 100 - 0.05(24.39) = \$98.78$$

b. Parallax's profit at the profit-maximizing price and quantity is

$$\pi = -100 + 100(24.39) - 2.05(24.39)^2 = \$1,119.51$$

c. Since there is no external market for Parallax lenses, the transfer price is equal to the marginal cost of producing the lenses at the profit-maximizing output level. Parallax's marginal cost equation for producing lenses is

$$MC_L = \frac{dTC_1}{dQ_L} = 0.05Q_L$$

Since Parallax needs three lenses for every telescope produced, the total number of lenses required by the telescope division is 73.17 lenses (3×23.39 telescopes). The marginal cost of producing these lenses, therefore, is

$$MC_L = 0.05(70.17) = \$3.51 = P_L$$

The transfer price of the lenses, therefore, is \$3.51 per lens.

TRANSFER PRICING WITH A PERFECTLY COMPETITIVE EXTERNAL MARKET

We will now consider the situation in which there exists an external market for the intermediate good. That is, the division or subsidiary producing the final product has the option of purchasing the intermediate good either from a subsidiary of its own parent company or from an outside vendor. If the intermediate good is purchased from within, what will its transfer price be? The answer to this question will depend on whether the external market for the intermediate good is or is not perfectly competitive. We will begin by assuming that there exists a perfectly competitive external market for the intermediate good produced by the subsidiary.

Since both divisions are assumed to be profit maximizers, it stands to reason that the division producing the final good will pay no more for the intermediate good than it would pay in the perfectly competitive external market. Similarly, the division producing the intermediate good will sell its output for nothing less than the perfectly competitive external market price. Thus, the transfer price for the intermediate good is the perfectly competitive price in the external market. This situation is depicted in Figure 11.9, where the price for the intermediate good is the same price depicted in Figure 11.8.

It should be noted that because the price of the intermediate good in Figure 11.9 is assumed to be the same price depicted in Figure 11.8, the mar-

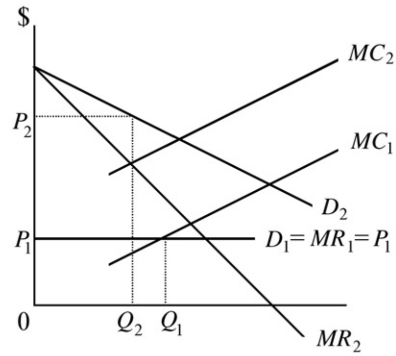

FIGURE 11.9 Transfer pricing with a perfectly competitive external market for intermediate good.

ginal cost of producing the final good remains unchanged. Thus, the profit-maximizing price and the quantity for the final good remain unchanged at P_2 and Q_2. Unlike the situation depicted in Figure 11.8, the amount of output produced by the intermediate good division no longer needs to equal the output of the final good division. Moreover, the transfer price for the intermediate good is the perfectly competitive price in the external market. In the preceding case, with no external market for the intermediate good, the transfer price was determined by the level of output that maximized profits for the final goods division.

In the situation depicted in Figure 11.9, the marginal cost of producing the intermediate good is lower than that depicted in Figure 11.8. The profit-maximizing, intermediate good division will produce at an output level at which $MC_1 = MR_1$. This occurs at an output level that is greater than Q_2. The division producing the intermediate good will sell Q_2 to the division producing the final good and will sell the surplus output of $Q_1 - Q_2$ in the external market at the perfectly competitive price, P_1.

Problem 11.19. Suppose that in the Parallax telescope example of Problem 11.18 the lenses produced by the subsidiary are of a standard variety produced by a perfectly competitive firm. Suppose further that the market-determined price of these lenses is $4.

a. Find the profit-maximizing price and quantity for Parallax telescopes. What is Parallax's total profit?
b. What transfer price should the lens division charge the telescope division?
c. How many lenses will the lens division produce? Will the number of lenses produced be sufficient to satisfy the requirements of the telescope division? If not, what should the telescope division do? If the lens division produces more lenses than the telescope division requires, what should the overproducing division do?

Solution

a. Since there is no change in the demand for Parallax telescopes, there is no change in the firm's total revenue function. However, since lenses are now \$0.49 more expensive than before, Parallax must spend \$1.47 more to produce each telescope. Parallax's total cost equation for telescopes is now

$$TC_T = 100 - 2Q_T^2 + 1.47Q_T$$

Parallax's total profit equation is

$$\pi = TR - TC_T = (100Q_T - 0.05Q_T^2) - (100 + 2Q_T^2 + 1.47Q_T)$$
$$= -100 + 100Q_T - 2.05Q_T^2 - 1.47Q_T = -100 + 98.53Q_T - 2.05Q_T^2$$

The first-order condition for profit maximization is $d\pi/dQ_T = 0$. Taking the first derivative of the profit equation yields

$$\frac{d\pi}{dQ} = 98.53 - 4.1Q_T = 0$$

Solving, we have

$$Q_T{}^* = 24.03$$

The second-order condition for profit maximization is $d\pi/dQ_T < 0$. Taking the second derivative of the profit equation yields

$$\frac{d^2\pi}{dQ^2} = -4.1 < 0$$

which guarantees that this output level represents a local maximum. The profit-maximizing price, therefore, is

$$P_T{}^* = 100 - 0.05(24.03) = \$98.80$$

Parallax's total profit at the profit-maximizing price and quantity is

$$\pi = -100 + 98.53(24.03) - 2.05(24.03)^2 = \$1{,}083.93$$

b. The transfer price for lenses is the price set in the perfectly competitive market (i.e., $P_L = \$4$).

c. The lens division will maximize profit by setting the marginal cost of producing lenses equal to the marginal revenue of selling lenses, that is,

$$MC_L = MR_L$$

Since lens production takes place in a perfectly competitive industry, the marginal revenue from selling lenses is \$4 per lens. The marginal cost equation of lens production is

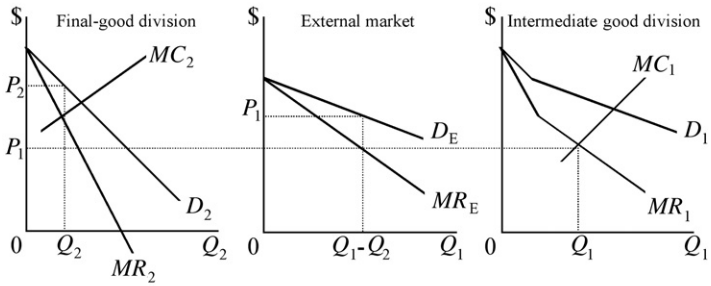

FIGURE 11.10 Transfer pricing with an imperfectly competitive external market for the intermediate good.

$$MC_L = \frac{dTC_L}{dQ_L} = 0.05Q_L$$

Substitution yields

$$4 = 0.05Q_L$$

$$Q_L{}^* = 80$$

The lens division should produce 80 lenses. Since the telescope division needs 72.09 lenses (3 × 24.03 telescopes), the lens division should sell the remaining 7.91 lenses in the external market.

TRANSFER PRICING WITH AN IMPERFECTLY COMPETITIVE EXTERNAL MARKET

Finally, let us consider an imperfectly competitive external market for the intermediate good. In this case, the price charged by the intermediate good division to the final good division will differ from the price of the intermediate good in the imperfectly competitive external market. The prices charged internally and externally by the intermediate good division become a matter of third-degree price discrimination. Consider Figure 11.10.

In Figure 11.10, the intermediate good division faces a downward-sloping demand curve for its output. The total demand for Q_1 includes the demand for the intermediate good by the final-good division and the demand by the external market. This demand curve is labeled D_1. Once again, the marginal cost of producing the final good Q_2 includes the marginal cost of producing Q_1. The profit-maximizing level of output for the intermediate good division is Q_1. The corresponding $MR_1 = MC_1$ will determine the selling price of the intermediate good to the final good division. This will be the transfer price of the intermediate good.

The amount of Q_1 that will be sold to the final good division will be determined by the profit-maximizing level of output Q_2, since $Q_1 = Q_2$. This leaves $Q_1 - Q_2$ units of Q_1 available for sale in the external market. The intermediate good division will maximize its profits by charging a price in the external market such that $MC_1 = MR_E$. In Figure 11.10, the intermediate good division engages in third-degree price discrimination by charging more in the external market than it charges the final good division.

OTHER PRICING PRACTICES

This chapter has so far focused on the pricing behavior of profit-maximizing firms operating under somewhat unique circumstances. In each case, the firm's pricing practices were predicated on subtle economic concepts. It was also assumed that management had complete information about the realities of the market in which the firm operated. In practice however, a firm's pricing practices are much looser in the sense that they are based less on detailed mathematical analysis than on perception, custom, and intuition. The remainder of this chapter is devoted to a review of five of these alternative pricing practices-*price leadership*, *price skimming*, *penetration pricing*, *prestige pricing*, and *psychological pricing*.

PRICE LEADERSHIP

Price leadership is a phenomenon that is likely to be observed in oligopolistic industries. It was noted in Chapter 10 that oligopolistic industries are characterized by the interdependence of managerial decisions between and among the firms in the industry. Firms in oligopolistic industries are keenly aware that the pricing and output decisions of any individual firm will provoke a reaction by competing firms. A consequence of this interdependence is relatively infrequent price changes.

Definition: Price leadership occurs when a dominant company in an industry establishes the selling price of a product for the rest of the firms in the industry. Two forms of price leadership are barometric price leadership and dominant price leadership.

Barometric Price Leadership

We saw in our discussion of the kinked demand curve that in oligopolistic industries, marginal cost may fluctuate within a fairly narrow range without evoking a price change. The reason for this is the discontinuity in the firm's marginal revenue curve. As a result, prices are relatively stable at the "kink" in the demand curve. What happens, however, when cost conditions for the typical firm in the industry increase significantly because of some exogenous shock? How will the increased cost of production manifest itself in the selling price of the product when, for example, the United Auto Workers negotiate higher wages and benefits for union workers in all firms in the U.S. automobile industry, or OPEC production cutbacks result in higher energy prices?

Definition: Barometric price leadership occurs when a price change by one firm in an oligopolistic industry, usually in response to perceived changes in macroeconomic or market conditions, is quickly followed by price changes by other firms in the industry.

In an oligopolistic industry characterized by firms of roughly the same size, price changes may sometimes be explained by *barometric price leadership*. In this case, a typical firm in the industry initiates, say, a price increase based on management's belief that changes in macroeconomic or market conditions will have a uniform impact on all other firms in the industry. If other firms believe that the firm's interpretations of economic events are correct, they will quickly follow suit. If they disagree, the firm initiating the price increase will be forced to reevaluate its decision and may modify or repeal the price increase. If the price increase is modified, the evaluation process begins again. Ultimately, member firms in the industry will form a consensus and a new, stable, price will be established.

An example of this type of price leadership can be seen in the commercial banking industry. Based on its reading of macroeconomic conditions, a leading money-center commercial bank, such as Citibank, may announce its decision to raise or lower the prime rate (the interest rate on loans to its best customers). If the rest of the industry agrees with Citibank's interpretation of macroeconomic conditions, other money-center commercial banks will quickly follow suit. If not, they will not raise their prime rates and Citibank will quietly lower its prime rate to a level consistent with the sentiments of the industry.

Dominant Price Leadership

Some industries are characterized by a single, dominant firm and many smaller competitors. The dominant firm may be the industry leader because of its leadership in product innovation, or because of economies of scale. If the firm is large enough or efficient enough, it may be able to force smaller competitors out of business by undercutting their prices, or it may simply buy them out. Such behavior, however, often incurs the wrath of the U.S. Department of Justice, which is charged with enforcing federal antitrust legislation.

Definition: Dominant price leadership occurs when one firm in the industry is able to establish the industry price as a result of its profit-maximizing behavior. Once a price has been established by the dominant firm, the remaining firms in the industry become price takers and face a perfectly elastic demand curve for their output.

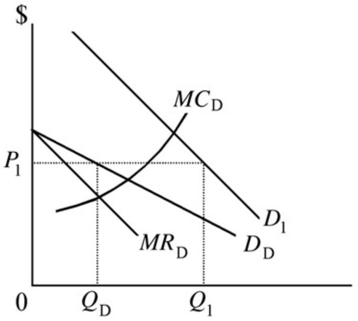

FIGURE 11.11 Dominant price leadership.

Industries dominated by a single large firm are characterized by price stability. The reason for this is that the dominant firm establishes the selling price of the product, and the smaller firms quickly adjust their price and output decisions accordingly. This situation is illustrated in Figure 11.11, which indicates that the dominant firm in the industry will behave like a monopolist by producing at output level Q_D, where its marginal cost is equal to its marginal revenue, $MC_D = MR_D$. The profit-maximizing price P_I will then serve as the industry standard. The amount of output provided by the rest of the industry will be $Q_I - Q_D$.

What is interesting about this analysis is that once the industry price has been established by the dominant firm, the remaining firms take on the appearance of a perfectly competitive industry in which the demand curve for their product is perfectly elastic. The other words, other firms in the industry are price takers. If entry and exit into and from the industry are relatively easy, the existence of above-normal profits will attract new firms into the industry, while below-normal profits will provide an incentive for firms to leave. It is speculative whether the influx or outflow of firms into the industry weakens or strengthens the market power of the dominant firm. In large part, this will depend on the circumstances explaining the firm's rise to industry dominance, and whether those factors are sufficient to maintain the firm's preeminent position.

PRICE SKIMMING

If a firm is first to market with a new product, it may engage in a form of first-degree price discrimination called price skimming. As with first-degree price discrimination, price skimming is an attempt to extract consumer surplus. During the interval between the firm's introduction of a new product and the competition's development of their own versions of the new product, the innovating firm is a virtual monopoly. If the innovating firm wants to extract consumer surplus, however, it must act fast.

Definition: Price skimming is an attempt by a firm that introduces a new product to extract consumer surplus through differential pricing before competitors develop their own versions of the new product.

The firm begins by initially charging a very high price for its product. Consumers willing and able to pay this price will buy first. Before competitors have a chance to sell their versions of the new product, the innovating firm will lower its price just a bit to attract the next, lower tier of consumers. This process is continued until the price charged equals the marginal cost of production. Pricing in this manner will enable the innovating firm to extract consumer surplus to enhance profits. Of course, for this pricing scheme to be successful the firm must have knowledge of the demand curve of the product. Management is unlikely to have such knowledge, however, because of the novel nature of the product. The firm could, of course, conduct consumer surveys, market experiments, and so on, to develop information regarding the demand for the product, but care would have to be taken not to "tip off" the competition.

PENETRATION PRICING

Penetration pricing occurs when a firm entering a new market charges a price that is below the prevailing market price to gain a foothold in the industry. A form of penetration pricing is *dumping*. Dumping is defined by the U.S. Department of Commerce as selling a product at less than fair market value. The most egregious form of this kind of pricing behavior is *predatory dumping*, the attempt by a foreign producer to gain control of the market in another country by selling a product there at less than fair market value with the goal of driving out domestic producers.

Definition: Penetration pricing is the practice of charging a price that is lower than the prevailing market price to gain a foothold in the industry.

PRESTIGE PRICING

Prestige pricing is essentially an attempt by firms to increase sales of certain products by capitalizing on snob appeal. Many consumers derive a degree of personal identity from the ostentatious display of certain brand-name items. For them, an enhanced personal image from the conspicuous consumption of upscale products is as valuable, and sometimes more so, than the usefulness or quality of the product itself. The mere fact that a product sells for a higher price often conveys the impression of higher quality, which may or may not be supported by reality. Prestige pricing is an attempt by some firms to exploit this perception by charging higher prices because of the increased prestige that they believe ownership of their products confers. An often cited example is the luxury automobile market, where higher priced automobiles are perceived to be superior to lower priced automobiles of similar quality.

Definition: Prestige pricing is the practice of charging a higher price for a product to exploit the belief by some consumers that a higher price means better quality, which in turn confers on the owner greater prestige.

PSYCHOLOGICAL PRICING

Finally, psychological pricing is a marketing ploy designed to create the illusion in the mind of the consumer that a product is being sold at a significantly lower price when, in fact, the price differential is inconsequential. Retailer sale of a product for $4.99 instead of $5.00 is psychological pricing. Retailers who engage in psychological pricing are attempting to exploit consumers' initial impressions or their lack of familiarity with the product. The effects of psychological pricing tend to be transitory, however, as initial impressions wear off or the consumer becomes more knowledgeable about the product.

Definition: Psychological pricing is a marketing ploy designed to create the illusion in the mind of the consumer that a product is being sold at a significantly lower price when, in fact, the price differential is inconsequential.

CHAPTER REVIEW

Earlier chapters discussed output and pricing decisions under some very simplistic assumptions. We assumed, for example, that profit-maximizing firms produce a single good or service, that production takes place in a single location, that these firms sell their products in a well-defined market, that managements have perfect information about production, revenue, and cost functions, and that the firms sell their output at a uniform price to all customers. In reality, these assumptions are rarely valid. For that reason, we considered alternative pricing practices, which in some cases are derivatives of the more general cases already encountered.

In *price discrimination*, a firm sells identical products in two or more markets at different prices. Economists have identified three degrees of price discrimination. *First-degree price discrimination* occurs when a firm charges each buyer a different price based on what he or she is willing to pay. In practice, first-degree price discrimination is virtually impossible.

In *second-degree price discrimination*, often referred to as volume discounting, different prices are charged for different blocks of units, or different products are bundled and sold at a package price. An example of second-degree price discrimination is block pricing, in which there are different prices for different blocks of goods and services. Second-degree price discrimination requires that a firm be able to closely monitor the level of services consumed by individual buyers.

In *third-degree price discrimination*, which is by far the most frequently practiced type of price discrimination, firms segment the market for a particular good or service into easily identifiable groups and then charge each group a different price. Such market segregation may be based on such factors as geography, age, product use, or income. For third-degree price discrimination to be successful, firms must be able to prevent resale of the good or service across segregated markets.

Cost-plus pricing, also known as *markup* or *full-cost pricing*, is an example of *nonmarginal pricing*. Firms that engage in nonmarginal pricing are unable or unwilling to devote the resources required to accurately estimate the total revenue and total cost equations, or do not know enough about demand and cost conditions to determine the profit-maximizing price and output levels. In cost-plus pricing, a firm sets the selling price of its product as a markup above its *fully allocated per-unit cost* of production. One criticism of cost-plus pricing is that it is insensitive to demand conditions. In practice, however, the size of a firm's markup tends to be inversely related to the price elasticity of demand for a good or service.

Multi product pricing involves optimal pricing strategies of firms producing and selling more than one good or service. Firms that independently produce two products with interrelated demands will maximize profits by producing at a level at which marginal cost is equal to the change in total revenue derived from the sale of the product itself, plus the change in total revenue derived from the sale of the related product. A profit-maximizing firm selling two goods with independent demands that are jointly produced in variable proportions will equate the marginal revenue generated from the sale of each good to the marginal cost of producing each product. Finally, a profit-maximizing firm that jointly produces two goods in fixed proportions with independent demands will equate the sum of the marginal revenues of both products expressed in terms of one of the products with the marginal cost of jointly producing both products expressed in terms of the same product.

Peak-load pricing occurs when a profit-maximizing firm charges a one price for a service when capacity is fully utilized and a lower price when capacity is underutilized. Off-peak prices are determined by equating marginal revenue to marginal operating costs. Peak prices, on the other hand, are determined by equating marginal revenue to the marginal cost of increasing capacity.

Price leadership appears when an oligopolist establishes a price that is quickly adopted by other firms in the industry. There are two types of price leadership: *barometric price leadership* and *dominant price leadership*.

Barometric price leadership exists when a price change by one firm in an oligopolistic industry, usually in response to perceived changes in macroeconomic or market conditions, is quickly followed by price changes by other firms in the industry.

Dominant price leadership exists when the largest firm in the industry establishes the industry price as a result of its profit-maximizing behavior. Once the industry price has been established, the remaining firms become price takers in the sense that they face a perfectly elastic demand curve for their output.

Other important pricing practices include transfer pricing, price skimming, penetration pricing, prestige pricing, and psychological pricing.

Transfer pricing is a method of correctly pricing a product as it is transferred from one stage of production to another.

Price skimming is the practice of taking advantage of weak or nonexistent competition to change a higher price for a new product than is justified by economic analysis. While competitors are trying to catch up, the firm may have monopoly pricing power.

Penetration pricing is found when a firm entering a new market charges less than its competitors to gain a foothold in the industry.

Prestige pricing is the setting of a high price for a product in the belief that demand will be higher because of the prestige that ownership bestows on the buyer.

Finally, *psychological pricing* is a marketing ploy designed to create the illusion in the mind of the consumer that a product is being sold at a significantly lower price when, in fact, the price differential is inconsequential. A retailer that sells a product for $4.99 instead of $5.00 is engaging in psychological pricing. The effect of psychological pricing tends to be transitory.

KEY TERMS AND CONCEPTS

Barometric price leadership A price change by one firm in an oligopolistic industry, usually in response to perceived changes in macroeconomic or market conditions, quickly followed by price changes by other firms in the industry.

Block pricing A form of second-degree price discrimination. It involves charging different prices for different "blocks" of goods and services to enhance profits by extracting at least some consumer surplus.

Commodity bundling Like block pricing, a form or second-degree price discrimination. Commodity bundling involves the combining of two or more different products into a single package, which is sold at a single price. Like block pricing, commodity bundling is an attempt to enhance profits by extracting at least some consumer surplus.

Consumer surplus The value of benefits received per unit of output consumed minus the product's selling price.

Cost-plus pricing The most popular form of nonmarginal pricing, cost-plus pricing is the practice of adding a predetermined "markup" to a firm's estimated per-unit cost of production at the time of setting the selling price of its product. Cost-plus pricing is given by the expression $P = ATC(1 + m)$, where m is the percentage markup and ATC is the fully allocated per-unit cost of production. The percentage markup may also be expressed as $m = (P - ATC)/ATC$.

Differential pricing Another term for price discrimination. It involves charging different prices to different groups, for different prices for different blocks of goods or services.

Dominant price leadership Establishment of the industry price by the dominant firm in the industry, as a result of its profit-maximizing behavior. Once the industry price has been established, the remaining firms in the industry become price takers and face a perfectly elastic demand curve for their output.

Dumping Third-degree price discrimination practiced in foreign trade. An exporting company that sells its product at a different, usually lower, price in the foreign market than it does in the home market is practicing dumping.

Dumping margin The difference between the price charged for a product sold by a firm in a foreign market and the price charged in the domestic market.

First-degree price discrimination The changing of a different price for each unit purchased. The price charged for any unit, which is based on the seller's knowledge of the individual buyer's demand curve, reflects the consumer's valuation of each unit purchased. The purpose of first-degree price discrimination is to maximize profits by extracting from each consumer the full amount of consumer surplus.

Full-cost pricing Another term for cost-plus pricing.

Fully allocated per-unit cost The sum of the estimated average variable cost of producing a good or service and a per-unit allocation for fixed cost. It is an approximation of average total cost.

Markup pricing Another term for cost-plus pricing.

Multiproduct pricing Optimal pricing strategies of a firm producing and selling more than one good under a number of alternative scenarios, including pricing of two or more goods with interdependent demands, pricing of two or more goods with independent demands produced in variable proportions, pricing of two or more goods with independent demands jointly produced in fixed proportions, and pricing of two or more goods given capacity limitations.

Nonmarginal pricing The profit maximizing price and output level are determined by equating marginal cost with marginal revenue. Management will often practice nonmarginal pricing, however, when the firm's total cost and total revenue equations are difficult or impossible to estimate. The most popular form of nonmarginal pricing is cost-plus pricing, also known as markup on full-cost pricing.

Peak-load pricing The practice of charging one price for a service when demand is high and capacity is fully utilized and a lower price for the service when demand is low and capacity is underutilized.

Penetration pricing The practice of charging less than the prevailing market price to gain a foothold in the industry; a strategy sometimes selected by firms entering a new market.

Prestige pricing The practice of charging a high price for a product to exploit the belief by some consumers that a high price tag means better quality, which confers upon the owner greater prestige.

Price discrimination The management, by a profit-maximizing firm, to charge different individuals or groups different prices for the same good or service.

Price leadership Seen when a dominant company in an industry establishes the selling price of a product for the rest of the firms in the industry. Two forms of price leadership are barometric price leadership and dominant price leadership.

Price skimming An attempt by a firm that introduces a new product to extract consumer surplus through differential pricing before the firm's competitiors develop their own versions of the new product.

Psychological pricing A marketing ploy designed to create the illusion in the mind of the consumer that a product is being sold at a significantly lower price when, in fact, the price reduction is inconsequential. Retailer sale of a product for $4.99 instead of $5.00 represents psychological pricing.

Relationship between the markup and the price elasticity of demand The size of a firm's markup tends to be inversely related to the price elasticity of demand for a good or service. When the demand for a product is low, the markup tends to be high, and vice versa. This relationship may be expressed as $m = -1/(\varepsilon_p + 1)$.

Second-degree price discrimination Similar in principle to first-degree price discrimination, it involves products in "blocks" or "bundles" rather than one unit at a time.

Third-degree price discrimination Segmenting the market for a particular good or service into easily identifiable groups, with a different price for each group.

Transfer pricing The optimal pricing of the output of one subsidiary of a parent company that is sold as an intermediate good to another subsidiary of the same parent company.

Two-part pricing A variation of second-degree price discrimination, two-part pricing is an attempt to enhance a firm's profits by charging a fixed fee for the right to purchase a good or service, plus a per-unit charge.

The per-unit charge is set equal to the marginal cost of providing the product, while the fixed fee is used to extract maximum consumer surplus, which is pure profit.

Volume discounting A form of second-degree price discrimination.

CHAPTER QUESTIONS

11.1 Explain each of the following pricing practices.

a. First-degree price discrimination

b. Second-degree price discrimination

c. Third-degree price discrimination

11.2 What is consumer surplus?

11.3 An important objective of firms engaged that practice price discrimination is the extraction of consumer surplus. Do you agree? Explain.

11.4 First-degree price discrimination is a relatively common practice, especially by firms dealing directly with the public, such as restaurants and retail outlets. Do you agree with this statement? If not, then why not?

11.5 What is the difference between block pricing and commodity bundling?

11.6 Sales of frankfurter rolls in packages of eight or beer in six-packs are examples of what pricing practice? What is the objective of the firm?

11.7 Explain the use of block pricing by amusement parks. Why do some amusement parks engage in block pricing while other, usually older, amusement parks do not?

11.8 The pricing of private goods is fundamentally different from the pricing of public goods because of the properties of excludability and depletability. Explain.

11.9 Explain how block pricing by amusement parts is similar to block pricing by cable television companies.

11.10 What is two-part pricing? Provide examples.

11.11 Explain how the practice of commodity bundling may give to a firm an unfair competitive advantage over its rivals.

11.12 Explain cost-plus (markup) pricing. Markup pricing suffers from what theoretical weakness? What are the advantages and disadvantages of markup pricing?

11.13 The more price elastic is the demand for a good or service, the higher will be the price markup over the marginal cost of production. Do you agree with this statement? Explain.

11.14 A firm producing two goods with interrelated demands, such as personal computers and modems, will maximize profits by equating the marginal revenue generated from the sale of each good separately to the marginal cost of producing each good. Do you agree with this statement? Explain.

11.15 A firm producing in variable proportions two goods with independent demands, such as automobile taillight and flashlight bulbs, will maximize profits by equating the marginal cost of producing each good separately to the combined marginal revenue generated from the sale of both goods. Do you agree with this statement? Explain.

11.16 A firm producing two goods with independent demands, which are produced in fixed proportions, will maximize profits by equating the sum of the marginal revenues generated from the sale of both goods, expressed in terms of one of the goods, to the marginal cost of jointly producing both goods. Do you agree? Explain.

11.17 Identify situations in which peak-load pricing may be appropriate. What is the distinguishing characteristic of short-run production functions in these situations?

11.18 Peak-load pricing suggests that users of commuter railroads be charged higher fares during off-peak hours to compensate the company for lost revenues arising from fewer riders. Do you agree with this statement? Explain.

11.19 Suppose a firm that produces a product for sale in the market also produces a vital component of that good for which there is no outside market. How should the firm "price" this component?

11.20 Explain each of the following pricing practices.

a. Barometric price leadership
b. Dominant price leadership
c. Price skimming
d. Penetration pricing
e. Prestige pricing
f. Psychological pricing

CHAPTER EXERCISES

11.1 Assume that an individual's demand for a product is

$$Q = 20 - 0.5P$$

Suppose that the market price of the product $10.

a. Approximate the value of this individual's consumer surplus for $\triangle Q = 1$.
b. What is value of consumer surplus as $\triangle Q \to 0$?

11.2 An amusement park has estimated the following demand equation for the average park guest

$$Q = 16 - 2P$$

where Q represents the number or rides per guest, and P the price per ride. The total cost of providing a ride is characterized by the equation

$$TC = 2 + 0.5Q$$

a. How much should the park charge on a per-ride basis to maximize its profit? What is the amusement park's total profit per customer?
b. Suppose that the amusement park decides to charge a one-time admission fee. What admission fee will maximize the park's profit? What is the estimated average profit per park guest?

11.3 A firm sells its product in two separable and identifiable markets. The firm's total cost of production is

$$TC = 5 + 5Q$$

The demand equations for its product in the two markets are

$$Q_1 = 10 - \frac{P_1}{2}$$

$$Q_2 = 20 - \frac{P_2}{5}$$

where $Q = Q_1 + Q_2$.

a. Calculate the firm's profit-maximizing price and output level in each market.
b. Verify that the demand for the product is less elastic in the market with the higher price.
c. Find the firm's total profit at the profit-maximizing prices and output levels.

11.4 Ned Bayward practices third-degree price discrimination when selling barrels of Eastfarthing Leaf in the isolated villages of Toadmorton and Forlorn. The reason for this is that the residents of Toadmorton have a particular preference for Eastfarthing Leaf, while the people in Forlorn can either take it or leave it. Ned's total cost of producing Eastfarthing Leaf is given by the equation

$$TC = 10 + 0.5Q^2$$

The respective demand equations in Forlorn and Toadmorton are

$$Q_1 = 50 - \frac{P_1}{4.5}$$

$$Q_2 = 75 - \frac{P_2}{7.5}$$

where $Q = Q_1 + Q_2$.

a. Calculate Ned's profit-maximizing price and output level in each market.
b. Verify that the demand for Eastfarthing Leaf is less elastic in the Toadmorton than in Forlorn. What does your answer imply about Ned's pricing policy?
c. Find the firm's total profit at the profit-maximizing prices and output levels.

11.5 Suppose a company has estimated the average variable cost of producing its product to be \$10. The firms total fixed cost is \$100,000.

a. If the company produces 1,000 units and its standard pricing practice is to add a 35% markup, what price should the company charge?
b. Verify that the selling price calculated in part a represents a 35% markup over the estimated average cost of production.

11.6 What is the estimated percentage markup over the fully allocated per-unit cost of production for the following price elasticities of demand?

a. $\varepsilon_p = -10$
b. $\varepsilon_p = -6$
c. $\varepsilon_p = -3$
d. $\varepsilon_p = -2.3$
e. $\varepsilon_p = -1.8$

11.7 A company produces two products, *I* and *F*. The demand equation for *F* is

$$Q_F = 1{,}500 - 15P_F$$

The total cost equation is

$$TC_F = 100 + 2Q_F^2$$

The company produces product *I* exclusively as an intermediate good in the production of product *F*. The total cost equation for producing good *I* is

$$TC_1 = 50 - 0.02Q_I^2$$

The company is divided into two semiautonomous profit centers: *I* division and the *F* division.

a. What is the profit-maximizing price and quantity for *F* division?
b. What is *F* division's total profit?
c. What transfer price should *I* division charge *F* division?

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12

Capital Budgeting

Much of the preceding discussion was concerned with the manner in which firms organize factors of production during a given period to maximize total economic profits. Clearly this was short-term analysis, with the focus of managerial decision making primarily on day-to-day, operational matters. Whereas short-run profit maximization is certainly important, senior management must also cast an eye to the future well-being of the firm and its shareholders. As a result, senior management must always question whether the current product line is adequate to sustain and enhance the firm's future profitability, or whether current production capacity is sufficient to meet future demand. If production capacity is deemed to be inadequate to meet future needs, the firm must examine its investment options to ascertain its most profitable, risk-adjusted course of action.

This chapter will concentrate on long-term, strategic considerations, focusing primarily on the firm's investment opportunities. The discussions in the preceding chapters have dealt almost entirely with per-period profit maximization. That analysis was fundamentally static. By contrast, investment is fundamentally dynamic, since it involves streams of expenditures and revenues over time. But, $1 received or expended today is worth more than $1 received or expended tomorrow because the $1 may be invested and earn a rate of return. Thus, an essential element of any investment decision is the proper evaluation of alternative investment opportunities involving alternative initial outlays, expected net returns, and time horizons.

In this chapter we evaluate alternative investment opportunities with different characteristics. *Capital budgeting* refers to the process of evaluating the comparative net revenues (expenditures on assets less expected revenues) from alternative investment projects. Since every investment opportunity involves expenditures (cash outflows) and revenues (cash inflows) that are spread out over a number of time periods, capital budgeting is an especially critical element of effective management decision making. Capital budgeting techniques are used to evaluate the potential profitability of possible new product lines, to plan for the replacement of damaged or worn-out (depreciated) plant and equipment, to expand existing production capacity, to engage in research and development, to institute or expand existing worker and management training programs, and evaluate the effectiveness of a major advertising campaign.

Definition: Capital budgeting is the process whereby senior management analyzes the comparative net revenues from alternative investment projects.

CATEGORIES OF CAPITAL BUDGETING PROJECTS

There are several types of capital budgeting decision, including whether to expand facilities, invest in new or improved products, replace worn-out plant and equipment or replace usable equipment with more efficient units; other capital budgeting decisions involve whether to lease or purchase plant and equipment, produce components for a product, or contract components to a vendor. In general, capital budgeting projects may be classified into one of several major categories. We shall discuss *capital expansion*, *replacement*, *new product lines*, *mandated investments*, and *miscellaneous investments*.

CAPITAL EXPANSION

Projected permanent increases in the demand for a firm's output will often lead management to commit significant financial resources to expanding existing production capacity. For companies engaged in the delivery of services (e.g., banking and finance, consulting, the legal profession and other more cerebral activities), capital expansion might take the form of an increase in the number of branch offices, more extensive communications and computing facilities, more intensive and expansive personnel training, and so on.

REPLACEMENT

Over time a firm's plant and equipment may depreciate, be damaged or destroyed, or become obsolete. At a very fundamental level, the replacement of a firm's capital stock is necessary if for no other reason than to maintain existing output levels to meet existing product demand. More subtly, replacement may be necessary to minimize the firm's cash outflows arising from, say, maintenance costs, which is a necessary condition for maximizing the firm's net revenues and shareholder value.

NEW OR IMPROVED PRODUCT LINES

One of the most important roles of management is to keep abreast of changing consumer preferences. Very often, this will require the firm to introduce new or improved goods and services to satisfy often fickle consumer tastes. Once senior management has decided that a change in consumer preferences is likely to continue into the foreseeable future, investment in plant and equipment may be necessary to bring new or improved products to market.

MANDATED INVESTMENTS

The primary obligation of senior management is to satisfy the firm's investors. This obligation often defines the firm's organizational objective, which is usually the maximization of shareholder returns. This concern for the return on shareholders' investment is often tempered, however, by societal or other considerations. These considerations, which often involve quality-of-life issues, such as safety in the workplace or a cleaner environment may entail the construction of access ramps for the disabled or the installation of workplace safety equipment and waste disposal facilities, such as "scrubbers" to treat industrial effluents before they are discharged into the environment.

Mandated capital expenditures often are not undertaken voluntarily because of the obvious negative impact on shareholder returns and company's market share, especially if the firm attempts to pass the increased cost of production along to the customer. In such cases, municipal, state, and federal regulators often step in to mandate such investment expenditures by all firms in an industry, thereby mitigating the competitive disadvantage to any single firm.

MISCELLANEOUS INVESTMENTS

"Miscellaneous investments" is a catchall for capital budgeting projects not easily pigeonholed into any of the foregoing categories. Such capital budgeting projects include the construction of employee parking lots, training and personnel development programs, the purchase of executive jets, or any management decision involving an analysis of cash outflows and inflows that do not easily fall into traditional capital budgeting classifications.

TIME VALUE OF MONEY

At its core, capital budgeting recognizes that $1 received today does not have the same value as $1 received tomorrow. Why not? From a psychological perspective it could be argued that, other things being equal, most people prefer the current consumption and enjoyment of a good or a service to consumption at some future date. There are, however, more practical reasons to conclude that $1 today is worth more than $1 tomorrow. If that $1 were deposited into a savings account paying a certain 5% annual interest rate, the value of that deposit would be worth $1.05 a year later. Thus, receiving $1 today is worth $1.05 a year from now.

Definition: The time value of money reflects the understanding that a dollar received today is worth more than a dollar received tomorrow.

In capital budgeting, future cash inflows and outflows of different capital investment projects are expressed as a single value at a common point in time for purposes of comparison. In most cases, future cash flows are expressed as a single value at the moment of undertaking the project.

CASH FLOWS

FUTURE VALUE WITH DISCRETE (ANNUAL) COMPOUNDING

The *future value* of an investment refers to the final accumulated value of a sum of money at some future time period, usually denoted as $t = n$. The future value of an investment will depend, of course, not only on the rate of return on that investment, i, but also how often that rate of return is calculated. The frequency of calculation of the rate of return is called *compounding*.

Definition: Future value is the final accumulated value of a sum of money at some future time period.

Definition: Compounding refers to the frequency that the rate of return on an investment is calculated.

Suppose, for example, that on May 1, 2000 ($t = 0$) an investor deposits $500 into a certificate of deposit that pays an annually compounded nominal (market) interest rate of 5%. Assume further that the investor plans to make no additional deposits. How much will the certificate of deposit be worth on April 30, 2005 ($t = 5$). It is often useful to visualize such problems with a *cash flow diagram*. Figure 12.1 presents the cash flow diagram for this problem.

Note that in Figure 12.1 the downward-pointing arrow at $t = 0$ represents an outflow of $500 as funds are deposited into the certificate of deposit. When the certificate of deposit matures on April 30, 2005, the future value

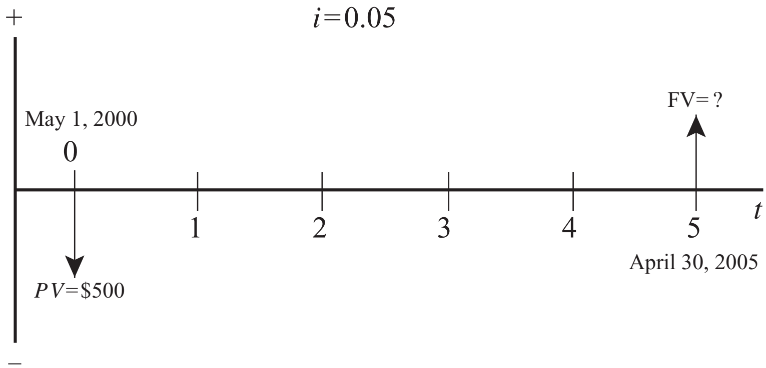

FIGURE 12.1 Future value cash flow diagram.

of the initial investment will be returned in the form of a cash inflow, which is illustrated as the upward-pointing arrow at $t = 5$.

Definition: A cash flow diagram illustrates the cash inflows and cash outflows expected to arise from a given investment.

We define the *present value* as the value of a sum of money at some initial time period, usually denoted as $t = 0$. In Figure 12.1, the present value of the certificate of deposit on May 31, 2000, is \$500. What will be the future value of this investment at the end of 5 years if the interest rate earned is 5% ($i = 0.05$) annually. Assume that interest is paid on the last day of each period (April 30) and that interest earnings are reinvested.

Definition: Present value is value of a sum of money at some initial time period.

It is easily seen that the accumulated value of the certificate of deposit at the end of the first year (May 1, 2000, to April 30, 2001) is

$$FV_1 = PV_0 + PV_0 i = PV_0(1+i) = \$500(1.05) = \$525 \qquad (12.1)$$

where PV_0 represents the present value of the investment at the beginning of the first year and FV_1 refers to the future value of the certificate of deposit at the end of the first year.

Suppose further that FV_1 (principal and earned interest) is reinvested in the certificate of deposit for a second year at the same interest rate, $i = 0.05$. The future value of the certificate of deposit at the end of the second year ($t = 2$) is

$$FV_2 = FV_1 + FV_1 i = FV_1(1+i) = \$525(1.05) = \$551.25 \qquad (12.2)$$

Note that Equation (12.1) may be substituted into Equation (12.2) to yield

$$FV_2 = PV_0(1+i)(1+i) = PV_0(1+i)^2 = \$500(1.05)^2 = \$551.25 \qquad (12.3)$$

If we continue to assume that principal and accumulated interest are reinvested at the prevailing rate of interest, then the future value of the certificate of deposit at the end of the third year ($t = 3$) is

$$FV_3 = FV_2 + FV_2 i = FV_2(1+i) = \$551.25(1.05) = \$578.81 \qquad (12.4)$$

Once again, substituting Equation (12.3) into Equation (12.4) we get

$$\begin{aligned} FV_3 &= FV_2(1+i) = PV_0(1+i)^2(1+i) \\ &= PV_0(1+i)^3 = 500(1.05)^3 = \$578.81 \end{aligned} \qquad (12.5)$$

Repeating this procedure, we find that at the end of 5 years, the value of the certificate of deposit is

$$FV_5 = PV_0(1+i)^5 = \$500(1.05)^5 = \$500(1.2763) = \$638.14 \qquad (12.6)$$

The step-by-step calculation of the future value of \$500, compounded annually for 5 years at $i = 0.05$, is illustrated in Figure 12.2, where the downward-pointing arrow at $t = 0$ indicates a cash outflow (–) following the purchase of the certificate of deposit. In $t = 5$ the upward-pointing arrow represents the cash inflow of \$638.14 as cash is received when the certificate of deposit matures.

If we generalize the foregoing calculations, the future value of an initial investment for n periods is

$$FV_n = PV_0(1+i)^n \qquad (12.7)$$

Problem 12.1. Adam borrows \$10,000 at an interest rate of 6% compounded annually from National Security Bank to buy a new car. If Adam agrees to a lump-sum repayment of the principal and interest, how much must he repay in 3 years?

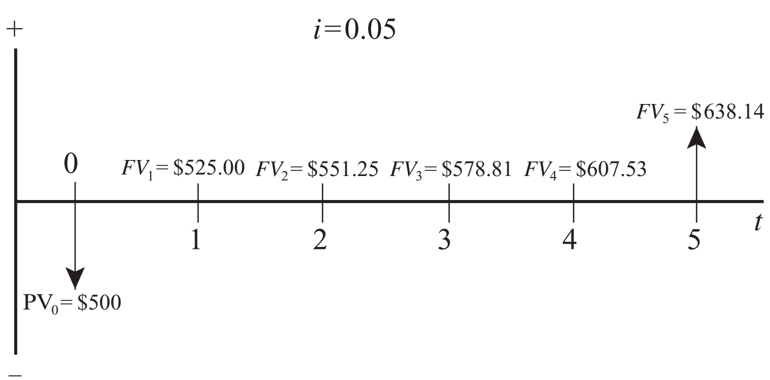

FIGURE 12.2 Future value cash flow diagram.

Solution. Substituting the information provided into Equation (12.7) yields

$$FV_3 = PV_0(1+i)^n = \$10,000(1.06)^3 = \$10,000(1.191) = \$11,910.16$$

This solution is summarized in the cash flow diagram of Figure 12.3.

Note, again, that at $t = 0$ the upward-pointing arrow indicates that the loan of \$10,000 represents a cash inflow (+). At $t = 3$ the downward-pointing arrow represents the loan repayment and interest payment, which is a cash outflow (−).

FUTURE VALUE WITH DISCRETE (MORE FREQUENT) COMPOUNDING

The discussion thus far has focused on the future value of a cash amount at some interest rate compounded annually. In most cases, however, interest will be compounded more frequently, say semiannually, quarterly, or monthly. Most bonds, for example, pay interest semiannually; stocks typically pay dividends quarterly, and most mortgages, automobile loans, and student loans require monthly payments.

Consider, again, the example illustrated in Figure 12.1. Suppose that the certificate of deposit pays 5%, which is compounded semiannually. To begin with, the student should note that the number of compounding periods has been doubled from 5 to 10. Since compounding will occur every 6 months instead of every 12 months, the periodic interest rate is now 2.5% semiannually instead of 5% per year. With these adjustments, Equation (12.6) may be rewritten as

$$FV_5 = PV_0\left(1+\frac{i}{2}\right)^{5\times 2} = \$500(1.025)^{10} = \$640.04 \tag{12.8}$$

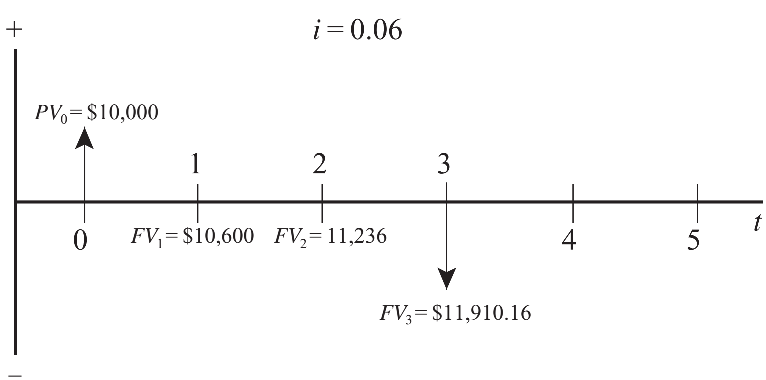

FIGURE 12.3 Diagrammatic solution to problem 12.1.

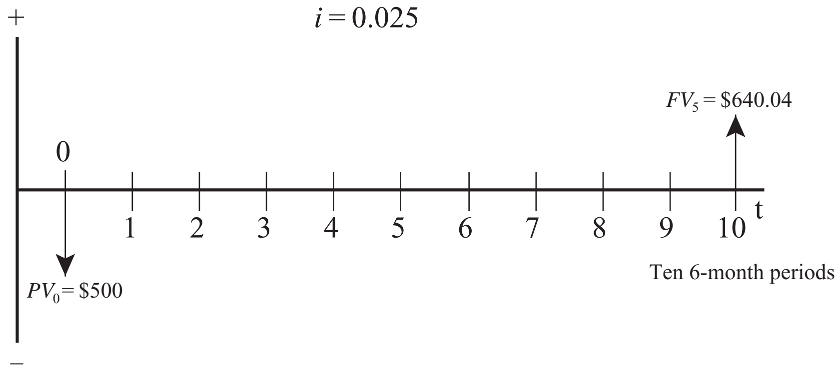

FIGURE 12.4 Future value cash flow diagram with discrete compounding.

This solution is illustrated in Figure 12.4.

The future value calculations just considered are examples of discrete compounding, that is, interest rate compounding that occurs at specific time intervals. In general, for more frequent compounding over n periods Equation (12.7) may be rewritten as

$$FV_n = PV_0\left(1+\frac{i}{m}\right)^{mn} \tag{12.9}$$

where i is the nominal (market) interest, n is the number of years, and m is the number of times that compounding takes place per year.

Problem 12.2. Suppose that in Problem 12.1 Adam had borrowed \$10,000 from National Security Bank to buy a new car and agreed to repay the loan in 3 years at an annual interest rate of 6% compounded monthly. What is the total amount that Adam must repay?

Solution. Substituting the information provided into Equation (12.7) yields

$$FV_n = PV_0\left(1+\frac{i}{m}\right)^{mn}$$

$$FV_3 = \$10,000\left(1+\frac{0.06}{12}\right)^{12\times 3} = \$10,000(1.005)^{36} = \$11,966.81$$

Clearly more frequent compounding results in higher interest payments for Adam of \$56.65 than if interest is compounded annually.

Problem 12.3. Suppose that Sergeant Garcia deposits \$100,000 in a time deposit that pays 10% interest per year compounded annually. How much will Sergeant Garcia receive when the deposit matures after 5 years? How would your answer have been different for interest compounded quarterly?

Solution. The future value of Sergeant Garcia's deposit of \$100,000 that pays an interest rate of 10% compounded annually is

$$FV_n = PV_0(1+i)^n = \$100,000(1.1)^5 = \$100,000(1.61051) = \$161,051$$

The future value of Sergeant Garcia's deposit when compounded quarterly is

$$FV_n = PV_0\left(1+\frac{i}{m}\right)^{mn} = \$100,000\left(1+\frac{0.1}{4}\right)^{5\times 4}$$
$$= \$100,000(1.025)^{20} = \$163.861.64$$

FUTURE VALUE WITH CONTINUOUS COMPOUNDING

Referring to Equation (12.9), what happens as the number of compounding periods becomes infinitely large, that is, as $m \to \infty$? This is the case of continuous compounding. To understand the effect of continuous compounding on the future value of a particular sum, recall from Chapter 2 the definition of the natural logarithm of base e:

$$e = \lim_{h\to\infty}\left(1+\frac{1}{h}\right)^h = 2.71829\ldots \quad (2.50)$$

Setting $1/h = i/m$ and substituting into Equation (12.9), we obtain

$$FV_n = PV_0\left(1+\frac{i}{m}\right)^{mn} = PV_0\left(1+\frac{1}{h}\right)^{mn} \quad (12.10)$$

where i is the interest rate and $1/h$ is the interest rate per compounding period. Substituting $m = hi$ into Equation (12.10) yields

$$FV_n = PV_0\left[\left(1+\frac{1}{h}\right)^h\right]^{in} \quad (12.11)$$

From Equation (2.50), the limit of Equation (12.11) as h approaches infinity is

$$FV_n = \lim_{h\to\infty} PV_0\left[\left(1+\frac{1}{h}\right)^h\right]^i = PV_0\left[\lim_{h\to\infty}\left(1+\frac{1}{h}\right)^h\right]^{in} = PV_0 e^{in} \quad (12.12)$$

Problem 12.4. Suppose that Adam borrows \$10,000 from National Security Bank and agrees to repay the loan in 3 years at an interest rate of 6% per year, compounded continuously. How much must Adam repay the bank at the end of 3 years?

Solution. Substituting the information provided into Equation (12.12) yields

$$FV_n = PV_0 e^{in}$$

$$FV_3 = \$10,000e^{0.06\times 3} = \$10,000(2.71829\ldots)^{0.18}$$
$$= \$10,000(1.1972) = \$11,972.17$$

This result indicates that Adam will have to pay $5.36 more in interest than if interest is compounded monthly, and $62.01 more than if interest is compounded annually.

FUTURE VALUE OF AN ORDINARY (DEFERRED) ANNUITY (FVOA)

Thus far we have considered the future value of a single cash amount that earns an interest rate of i for n years compounded m times per year. Suppose, however, that an individual wanted to make regular and periodic investment over the life of the investment? Such investments are referred to as annuities. For example, suppose that a person wanted to invest $500 into an interest-bearing account at the end of each of the next 5 years, with an interest rate of 5% per year compounded annually. What is the future value of these investments at the end of the fifth year? This situation, which is illustrated in Figure 12.5, is referred to as an ordinary (deferred) annuity.

In the case of an ordinary annuity, note carefully that the fixed payments are made at the *end* of each period. Thus, the first deposit is made at the end of $t = 0$, which means that no interest will be earned until the start of $t = 1$. Moreover, the final annuity payment is not made until the *end* of $t = 5$. Since the account matures at the end of 5 years, no interest will be earned on the final deposit. At first, the reader may find such an arrangement peculiar. After all, what type of investment requires that the first deposit be made at the end of the first year and the last deposit made at maturity? The confusion quickly disappears, however, when it is recalled that banks often

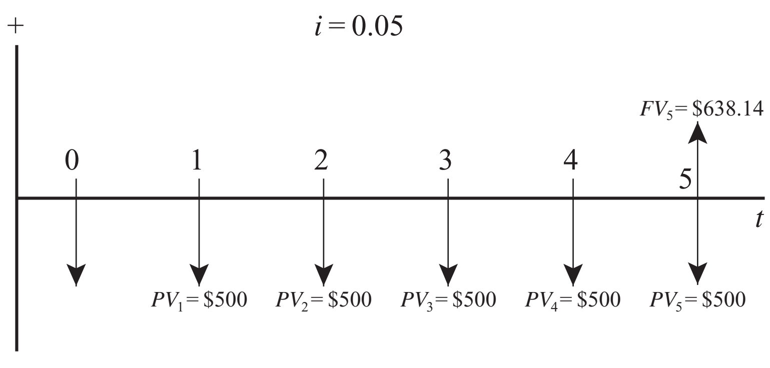

FIGURE 12.5 Future value of an ordinary annuity cash flow diagram.

make loans in which the first repayment is not made until the end of the first month, with the final payment made at maturity. Such an arrangement is called an amortized loan and represents an investment for the bank.

Definition: An annuity is a series of equal payments, which are made at fixed intervals for a specified number of periods.

Definition: An ordinary (deferred) annuity is an annuity in which the fixed payments are made at the end of each period.

Definition: The future value of an ordinary annuity (*FVOA*) is the future value of an annuity in which the fixed payments are made at the end of each period.

For the situation depicted in Figure 12.5, the future value of the ordinary annuity may be determined by calculating the sum of the future value of each of five separate investments, that is,

$$FVOA_5 = PV_1(1+i)^4 + PV_2(1+i)^3 + PV_3(1+i)^2 + PV_4(1+i)^1 + PV_5(1+i)^0$$

Substituting the annuity value and interest rate from Figure 12.5 into the foregoing expression yields

$$\begin{aligned} FVOA_5 &= \$500(1.05)^4 + \$500(1.05)^3 + \$500(1.05)^2 + \$500(1.05)^1 + \$500(1) \\ &= \$500(1.21551) + \$500(1.15763) + \$500(1.1025) \\ &\quad + \$500(1.05) + \$500(1) \\ &= \$607.76 + \$578.81 + \$551.25 + \$525.00 + \$500 = \$2,762.82 \end{aligned}$$

In general, if we denote the constant annuity payment as A, the future value of fixed annuity payments for n periods in which the first payment is made at the end of $t = 0$ is

$$\begin{aligned} FVOA_n &= A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i)^0 \\ &= A \sum_{t=1 \to n} (1+i)^{n-1} \end{aligned} \qquad (12.13)$$

By the algebra of a sum of a geometric progression (see Chapter 2), Equation (12.12) may be rewritten as

$$FVOA_n = \frac{A\left[(1+i)^n - 1\right]}{i} \qquad (12.14)$$

Applying Equation (12.14) to the information provided yields the same outcome as before:

$$\begin{aligned} FVOA_5 &= \frac{\$500\left[(1.05)^5 - 1\right]}{0.05} = \frac{\$500(0.2763)}{0.05} \\ &= \frac{\$138.14}{0.05} = \$2,762.82 \end{aligned}$$

The value $[(1 + i)^n - 1]/i$ is referred to as the *future value interest factor for an annuity* ($FVIFA_{i,n}$).

FUTURE VALUE OF AN ANNUITY DUE (FVAD)

Suppose that in our example the five payments of $500 had commenced at the beginning of the first year (i.e., at $t = 0$) rather than at the beginning of the second year ($t = 1$). This is the same thing as saying that an investor has decided to deposit $500 immediately, and another $500 each year for the next 4 years. This arrangement is similar to many savings programs. How much will the investor withdraw at the end of the fifth year? This sort of arrangement, which is referred is to as an annuity due, is depicted in Figure 12.6.

Definition: An annuity due is an annuity in which the fixed payments are made at the beginning of each period.

Definition: The future value of an annuity due is the future value of an annuity in which the fixed payments are made at the beginning of each period.

In the case of an annuity due, the fixed payments are made at the *beginning* of each period. Thus, the first deposit is made at the beginning of $t =$ 0 and begins to earn interest immediately. Moreover, in the case depicted in Figure 12.6, the final annuity payment is not made until the beginning of $t = 5$. As with an ordinary annuity, the future value of an annuity due may be determined by calculating the sum of the future value of each of five separate investments. The difference, of course, is that the future value of an annuity due is equal to the future value of an ordinary annuity compounded for one additional period.

$$FVAD_n = FVOA_n(1+i) = A\left\{\frac{\left[(1+i)^n - 1\right]}{i}\right\}(1+i) \qquad (12.15)$$

By using the information depicted in Figure 12.6, we find the future value of an annuity due:

$$FVAD_5 = FVOA_5(1.05) = \$2,762.82(1.05) = \$2,900.96$$

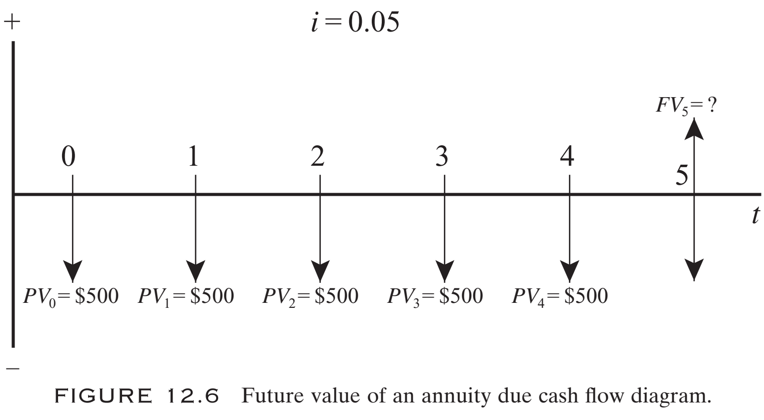

FIGURE 12.6 Future value of an annuity due cash flow diagram.

Compare this amount with the future value of an ordinary annuity of \$2,762.82. The difference of \$138.14 is attributable to allowing each payment of \$500 to compound for one additional period.

Problem 12.5. Andrew's father, Tom, is thinking about putting away a few dollars away to help pay for his son's college tuition. Tom would like to invest \$1,000 a year for 10 years into a certificate of deposit. He is reasonably certain of earning an annual interest rate of 5% per annum over the life of the investment. Tom is uncertain, however, whether to open the certificate of deposit immediately, or wait until the end of the year to make his first deposit. Tom realizes that by waiting a year before making his deposit that he will lose a full year of interest compounding. On the other hand, he has the opportunity of earning \$100 in interest the first year by lending \$1,000 to his Uncle Ned. Assume that Uncle Ned is not a deadbeat and will repays the loan with interest. Suppose that Tom plans to deposit the \$100 in interest earned in a regular savings account earning 4% interest annually for 9 years. Should Tom make the loan to Uncle Ned and deposit the initial \$1,000 at the end of the year, or should he open the certificate of deposit immediately?

Solution. If Tom decides to deposit \$1,000 into a certificate of deposit immediately, the future value of an annuity due is

$$FVAD_n = A\left\{\frac{\left[(1+i)^n - 1\right]}{i}\right\}(1+i) = \$1{,}000\left\{\frac{\left[(1.05)^{10} - 1\right]}{0.05}\right\}(1.05) = \$13{,}209$$

On the other hand, if Tom decides to wait a year before making the first deposit, the future value of an ordinary annuity is

$$FVOA_n = \frac{A\left[(1+i)^n - 1\right]}{i} = \frac{\$1{,}000[(1.05^{10} - 1)]}{0.05} = \$1{,}000(12.578) = \$12{,}578$$

To this amount must be added the future value of \$100 received from Uncle Ned compounded annually for 9 years at an interest rate of 4%. This amount is given as

$$FV_n = PV_0(1+i)^n = \$100(1.04)^9 = \$100(1.4233) = \$142.33$$

Adding this amount to the future value of an ordinary annuity yields

$$\$12{,}578 + \$143.33 = \$12{,}721.33$$

Since he can make \$13,209 by opening the certificate of deposit immediately, Tom will not make the loan to Uncle Ned.

FUTURE VALUE OF AN UNEVEN CASH FLOW

The two preceding sections were devoted to calculating the future value of an annuity, which is sometimes referred to as an even cash flow. In this

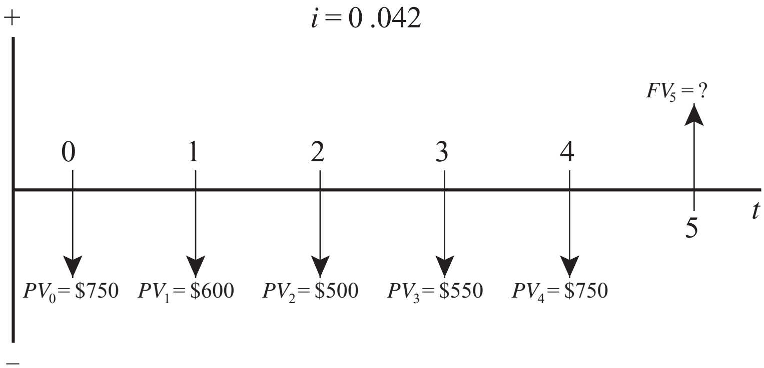

FIGURE 12.7 Uneven cash flow.

section we will discuss the calculation of an uneven cash flow. The future value of an uneven cash flow is determined by compounding each payment and summing the future values. The future value of an uneven cash flow is sometimes referred to as the terminal value. Figure 12.7 illustrates an uneven cash flow diagram.

The future value of the uneven cash flow may be calculated by repeatedly applying Equation (12.7). In particular, the present value of an uneven cash flow is

$$\begin{aligned} FV_n &= PV_0(1+i)^n + PV_1(1+i)^{n-1} + \ldots + PV_{t-1}(1+i)^{n-t} \\ &= \sum_{t=0\to n} PV_t(1+i)^{n-t} \end{aligned} \tag{12.16}$$

The future value of the uneven cash flow depicted in Figure 12.7 at $t = 5$ is

$$\begin{aligned} FV_5 &= 750(1.042)^5 + 600(1.042)^4 + 500(1.042)^3 + 550(1.042)^2 + 750(1.042)^1 \\ &= \$3752.98 \end{aligned}$$

This solution is illustrated in Figure 12.8.

PRESENT VALUE WITH DISCRETE (ANNUAL) COMPOUNDING

So far we have considered the answer to the question: What will be the value of a payment, or series of payments, at the end of a given period of time? We would now like to turn this around a bit. Suppose that we were interested in determining the value an immediate payment, or series of payments, required to grow to a specified value at some time in the future. Suppose, for example, that Adam wanted to know how much he needed to invest in a certificate of deposit today at 5% interest such that the value of

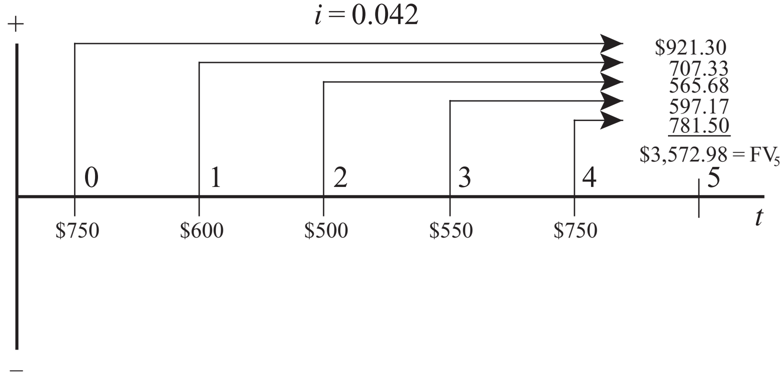

FIGURE 12.8 Future value of an uneven cash flow.

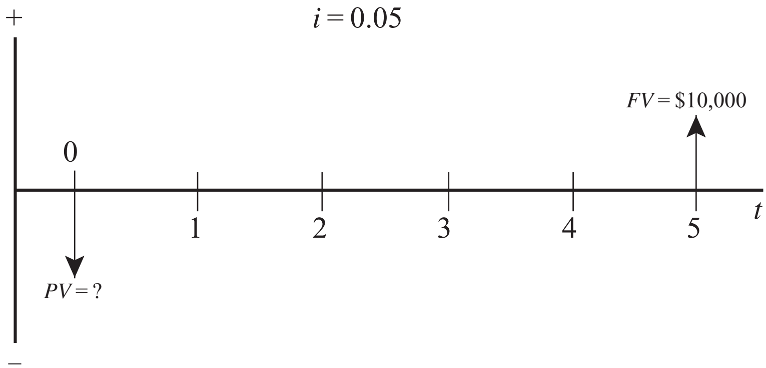

FIGURE 12.9 Present value with discrete (annual) compounding.

the investment in 5 years would be \$10,000. This amount is referred to as the *present value* of the investment. The cash flow diagram for this problem is depicted in Figure 12.9.

Definition: Present value is the value today of an investment, or series of investments, that will grow to some future specified amount at a designated rate of interest.

To calculate the present value of a lump-sum investment for n periods at an interest rate of i, consider again Equation (12.7).

$$FV_n = PV_0(1+i)^n \tag{12.7}$$

Solving Equation (12.7) for PV_0 yields

$$PV_0 = \frac{FV_n}{(1+i)^n} \tag{12.17}$$

The present value of an investment, or series of investments, at some designated rate of interest is often referred to as *discounted cash flow*. The rate of interest that is used to discount a cash flow is called the *discount rate*.

Definition: A discounted cash flow is the present value of an investment or series of investments.

Definition: The discount rate is the rate of interest that is used to discount a cash flow.

Substituting the foregoing information into Equation (12.17), we find that the present value of an investment earning 5% interest compounded annually that will be worth $10,000 in 5 years is

$$PV_0 = \frac{\$10{,}000}{(1.05)^5} = \frac{\$10{,}000}{1.276} = \$7{,}835.26$$

That is, if Adam invests $7,835.26 into a certificate of deposit that earns 5% compounded annually, the value of the investment in 5 years will be $10,000.

Problem 12.6. How much should Turin Turambar invest in a certificate of deposit today for that investment to be worth $500 in 7 years if the interest rate is 18% per year, compounded annually?

Solution. Substituting this information into Equation (12.17), we find that the present value of the investment is

$$PV_0 = \frac{FV_n}{(1+i)^n} = \frac{\$500}{(1.18)^7} = \frac{\$500}{3.185} = \$156.96$$

PRESENT VALUE WITH DISCRETE (MORE FREQUENT) AND CONTINUOUS COMPOUNDING

To calculate the present value of a lump-sum investment for n periods at an interest rate of i compounded m times per period, consider again Equation (12.9).

$$FV_n = PV_0\left(1+\frac{i}{m}\right)^{mn} \tag{12.9}$$

Solving Equation (12.9) for PV_0 yields

$$PV_0 = \frac{FV_n}{(1+i/m)^{mn}} \tag{12.18}$$

To calculate the present value of a lump-sum investment for n periods at an interest rate of i compounded continuously, consider again Equation (12.12).

$$FV_n = PV_0 e^{in} \tag{12.12}$$

Solving Equation (12.11) for PV_0 yields

$$PV_0 = \frac{FV_n}{e^{in}} \tag{12.19}$$

To continue with our example, suppose that Adam wanted to know how much he needed to invest in a certificate of deposit today at a 5% interest rate compounded quarterly so that the final value of the investment, 5 years from now, would be $10,000. Substituting this information into Equation (12.18), we get

$$PV_0 = \frac{FV_n}{(1+i/m)^{mn}} = \frac{\$10,000}{(1+0.05/4)^{4\times 5}} = \$7,800.09$$

How much would the initial investment be if interest were compounded continuously? Substituting this information into Equation (12.19) we obtain

$$PV_0 = \frac{FV_n}{e^{in}} = \frac{\$10,000}{e^{0.05\times 5}} = \$7,788.01$$

Compare the present value of $10,000 with annual compounding ($7,835.26) with the present values where interest is compounded quarterly ($7,800.09) and continuously ($7,788.01). Clearly, the more frequent the compounding, the smaller is the required initial investment.

Problem 12.7. If the prevailing interest rate on a time deposit is 8% per year, how much would Sergeant Garcia have to deposit today to receive $200,000 at the end of 5 years if the interest rate were compounded quarterly, monthly, and continuously?

Solution. To receive $200,000 in 5 years on a time deposit that pays 8% compounded quarterly, Sergeant Garcia will have to invest

$$PV_0 = \frac{FV_n}{(1+i/m)^{mn}} = \frac{\$200,000}{(1+0.08/4)^{4\times 5}} = \$134,594.27$$

If interest is compounded monthly, Sergeant Garcia will have to invest

$$PV_0 = \frac{FV_n}{(1+i/m)^{mn}} = \frac{\$200,000}{(1+0.08/12)^{12\times 5}} = \$134,424.09$$

Finally, if interest is compounded continuously Sergeant Garcia will have to invest

$$PV_0 = \frac{FV_n}{e^{in}} = \frac{\$200,000}{e^{0.08\times 5}} = \frac{\$200,000}{e^{0.4}} = \$134,064.01$$

Note, once again, that the more frequent the compounding, the smaller the present value, or the amount to be invested at $t = 0$.

PRESENT VALUE OF AN ORDINARY ANNUITY (PVOA)

Earlier, we introduced the concept of an annuity as a series of fixed payments made at fixed intervals for a specified period of time. An ordinary (deferred) annuity was defined as a series of payments made at the end of each period. Another way to evaluate ordinary annuities is to calculate their present values. In general, the present value of an ordinary annuity may be calculated by using Equation (12.20):

$$\begin{aligned} PVOA_n &= \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \ldots + \frac{A}{(1+i)^n} \\ &= A \sum_{t=1 \to n} \left(\frac{1}{1+i}\right)^t \end{aligned} \qquad (12.20)$$

Consider, again, the situation depicted in Figure 12.5, where an investor deposits \$500 at the end of each of the next 5 years and earns 5% per year compounded annually. What is the present value of these investments at the *beginning* of $t = 1$, which is the same thing as the end of $t = 0$? Substituting the information provided into Equation (12.20), the present value of the annuity due is

$$PVOA_5 = \frac{\$500}{(1.05)^1} + \frac{\$500}{(1.05)^2} + \frac{\$500}{(1.05)^3} + \frac{\$500}{(1.05)^4} + \frac{\$500}{(1.05)^5} = \$2,164.73$$

The cash flow diagram for this problem is illustrated in Figure 12.10.

Problem 12.8. Suppose that an individual invests \$2,500 at the end of each of the next 6 years and earns an annual interest rate of 8%. Calculate the present value of this series of annuity payments.

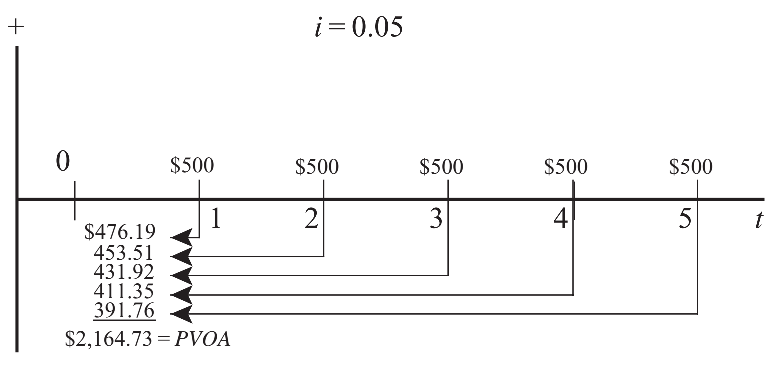

FIGURE 12.10 Present value of an ordinary annuity cash flow diagram.

Solution. Substituting the information provided into Equation (12.20) yields

$$PVOA_n = A\sum_{t=1\to n}\left(\frac{1}{1+i}\right)^t$$

$$\begin{aligned} PVOA_6 &= \$2,500\sum_{t=1\to 6}\left(\frac{1}{1+0.08}\right)^t \\ &= \frac{\$2,500}{(1.08)^1}+\frac{\$2,500}{(1.08)^2}+\frac{\$2,500}{(1.08)^3}+\frac{\$2,500}{(1.08)^4}+\frac{\$2,500}{(1.02)^5}+\frac{\$2,500}{(1.02)^6} \\ &= \$11,557.20 \end{aligned}$$

PRESENT VALUE OF AN ANNUITY DUE (PVAD)

As we saw, an annuity due is an annuity in which the fixed payments are made at the beginning of each period. In general, the present value of an annuity due may be calculated by using Equation (12.21).

$$\begin{aligned} PVAD_n &= \frac{A}{(1+n)^{n-1}}+\frac{A}{(1+n)^{n-2}}+\ldots+\frac{A}{(1+n)^0} \\ &= A\sum_{t=0\to n}\left(\frac{1}{1+i}\right)^{n-t} \end{aligned} \tag{12.21}$$

By substituting the information provided in Figure 12.7 into Equation (12.21), we find that the present value of an annuity due is

$$\begin{aligned} PVAD_5 &= \frac{\$500}{(1.05)^4}+\frac{\$500}{(1.05)^3}+\frac{\$500}{(1.05)^2}+\frac{\$500}{(1.05)^1}+\frac{\$500}{(1.05)^0} \\ &= \$411.35+\$431.92+\$453.51+\$476.19+\$500 = \$2,272,97 \end{aligned}$$

The cash flow diagram for this problem is illustrated in Figure 12.11.

Problem 12.9. Suppose that an individual invests $2,500 at the beginning of each of the next 6 years and earns an annual interest rate of 8%. Calculate the value of this series of annuity payments. How does this result compare with the solution to Problem 12.8?

Solution. Substituting the information provided into Equation (12.21) yields

$$PVAD_n = A\sum_{t=0\to n}\left(\frac{1}{1+i}\right)^{n-t}$$

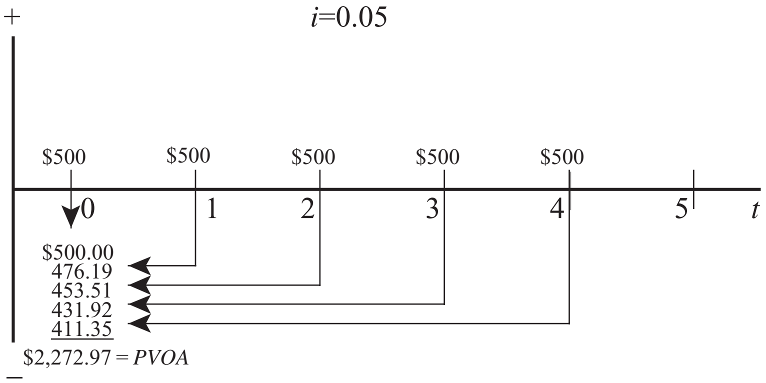

FIGURE 12.11 Present value of an annuity due cash flow diagram.

$$PVAD_6 = \$2,500 \sum_{t=1 \to 6} \left(\frac{1}{1.08}\right)^{n-t}$$
$$= \frac{\$2,500}{(1.08)^5} + \frac{\$2,500}{(1.08)^4} + \frac{\$2,500}{(1.08)^3} + \frac{\$2,500}{(1.08)^2} + \frac{\$2,500}{(1.08)^1} + \frac{\$2,500}{(1.08)^0}$$
$$= \$12,481.78$$

The present value of an annuity due in this problem is less than the present value of an ordinary annuity calculated in Problem 12.8 because

$$PVOA_6 - PVAD_6 = \frac{\$2,500}{(1.02)^6} - \frac{\$2,500}{(1.08)^0}$$

where $\$2,500/(1.08)^6 < \$2,500/(1.08)^0$. In other words, since compounding takes place for one less period, $PVOA_6 > PVAD_6$.

AMORTIZED LOANS

Amortized loans represent one of the most useful applications of the future value of an ordinary annuity. These loans are repaid in equal periodic installments. Once again, consider the example in which Adam borrows $10,000 from National Security Bank to buy a new car. Suppose that Adam agrees to repay the loan in 3 years at an interest rate of 6% per year, compounded annually. Adam further agrees to repay the loan in equal annual installments, with the first installment due at the end of the first year. How can he determine the amount of his yearly debt service (principal and interest) payments? This problem is depicted in Figure 12.12.

To determine the amount of Adam's monthly payments, consider again Equation (12.20).

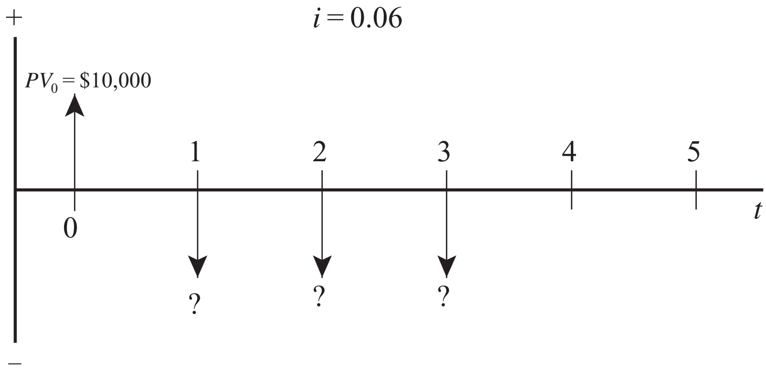

FIGURE 12.12 Amortized loan cash flow diagram.

$$PVOA_n = A\sum_{t=1\to n}\left(\frac{1}{1+i}\right)^t \tag{12.20}$$

In this case, we know that $PVOA_3 = \$10{,}000$ and $i = 0.06$. The task at hand is to determine the amount of Adam's yearly payments, A. Solving Equation (12.20) for A we obtain

$$A = \frac{PVOA_n}{\Sigma_{t=1\to n}[1/(1+i)]^t} \tag{12.22}$$

Substituting the information provided in the problem into Equation (12.22) and solving yields

$$A = \frac{PVOA_n}{\Sigma_{t=1\to n}[1/(1+i)]^t}$$
$$= \frac{\$10{,}000}{(1/1.06)+(1/1.06)^2+(1/1.06)^3} = \$3{,}741.11$$

Thus, Adam must pay National Security Bank \$3,741.11 at the end of each of the next three years. Each payment consists of interest due and partial repayment of principal. This series of repayments is referred to as an *amortization schedule*. The reader should verify that the largest interest component of the amortization schedule is paid in at the end of the first year; thereafter, as the amount of the principal outstanding declines, the payments are correspondingly less.

Problem 12.10. Suppose that Andrew borrows \$250,000 at 3% to purchase a new home. Andrew agrees to repay the loan in 10 equal annual installments, with the first payment due at the end of the first year.

a. What is the amount of Andrew's mortgage payments?
b. What is the total amount of interest paid?

Solution

a. Substituting the information provided into Equation (12.22) yields

$$A = \frac{PVOA_n}{\Sigma_{t=1\to n}[1/(1+i)]^t}$$
$$= \frac{\$250,000}{\Sigma_{t=1\to 10}[1/(1+1.03)]^t} = \frac{\$250,000}{8.5302} = \$29,307.63$$

b. Andrew will make total mortgage payments of 10(29,307.63) = \$293,076.27. Thus, the total amount of interest paid will be \$293,076.27 – \$250,000 = \$43,076.27.

METHODS FOR EVALUATING CAPITAL INVESTMENT PROJECTS

Now that the fundamental techniques for assessing the time value of money have been established, we turn our attention to some of the most commonly used methods of assessing the returns on capital investment projects. There are five standard methods for ranking capital investment projects. Each method ranks capital investment projects from the most preferred to the least preferred based on the project's net rate of return (i.e., the rate of return from the investment over and above the total cost of financing the project). The cost to the firm of acquiring funds to finance a capital investment project is commonly referred to as its *cost of capital*.

The five most commonly used methods for ranking capital investment projects are the *payback period*, the *discounted payback period*, the *net present value* (*NPV*) method, the *internal rate of return* (*IRR*), and the *modified rate of return* (*MIRR*). We will, illustrate each method by using the hypothetical cash flows (CF_t) for projects *A* and *B* summarized in Table 12.1. To keep the analyses manageable, we will assume that cash flows have been adjusted to reflect inflation, taxes, depreciation, and salvage values.

TABLE 12.1 Net Cash Flows (CF_t) for Projects *A* and *B*

Year, *t*	Project *A*	Project *B*
0	–\$25,000	–\$25,000
1	10,000	3,000
2	8,000	5,000
3	6,000	7,000
4	5,000	9,000
5	4,000	11,000

PAYBACK PERIOD METHOD

The payback period of a capital investment project is the number of periods required to recover the original investment. In general, the shorter the payback period, the more preferred the capital investment project. Using the payback period method to evaluate alternative investment opportunities is perhaps the oldest technique for evaluating capital budgeting projects.

Definition: The payback period is the number of periods required to recover the original investment.

We can see that for project *A* by the end of year 3 cumulative cash flows are \$24,000, or 96% of the original investment has been recovered. By the end of year 4 cumulative cash flows are \$29,000, or 116% of the original investment has been recovered. Since only an additional \$1,000 cash flow was required in year 4 to fully cover the original \$25,000 investment, then the total number of years required to recover the original investment (P_A) was 3 years plus \$1,000/\$5,000 years, or 3.2 years. The payback period for Project *B* (P_B) is 4 years plus \$1,000/\$11,000 years, or 4.09 years In general, the expression for calculating the payback period is

$$P_j = (F-1) + \frac{(-CF_0 - \Sigma_{t=1\rightarrow F-1} CF_t)}{CF_F} \tag{12.23}$$

where P_k is the payback period of investment j, $(F - 1)$ is the year before full recovery of the original investment, CF_0 is the original investment, which is a cash outflow (–), $\Sigma_{t=1\rightarrow F-1} CF_t$ is the sum of all cash flows up to and including the year before full recovery of the original investment, and CF_F is the cash flow in the year of full recovery. Substituting the information in Table 12.1 into Equation (12.23) we obtain

$$P_A = 3 + \frac{-(-\$25,000) - \$24,000}{\$5,000} = 3 + \frac{\$1,000}{\$5,000} = 3.20 \text{ years}$$

$$P_B = 4 + \frac{-(-\$25,000) - \$24,000}{\$11,000} = 4 + \frac{\$1,000}{\$11,000} = 4.09 \text{ years}$$

Assuming that these projects are *mutually exclusive*, investment project *A* is preferred to project *B* because project *A* has a shorter payback period. Projects are said to be mutually exclusive if the acceptance of one project means that all other potential projects are rejected. Projects are said to be *independent* if the cash flows from alternative projects are unrelated to each other.

Definition: Projects are mutually exclusive if acceptance of one project means rejection of all other projects.

Definition: Projects are independent if their cash flows are unrelated.

Problem 12.11. The chief financial analyst of Valaquenta Microprocessors, Inc. has been asked to analyze two proposed capital investment projects, projects A and B. Each project has an initial cost of $10,000. The projects cash flows, which have been adjusted to reflect inflation, taxes, depreciation, and salvage values, are as follows:

Which project should be selected according to the payback period method?

Solution. From the information in Table 12.2, by the end of year 2, the year before full recovery, the cumulative cash flow for project A is $9,500, or 95% of the original investment. By the end of year 3, the year of full recovery, cumulative cash flows are $11,000, or 110 percent of the original investment. The cumulative cash flow for project B by the end of year 2 is $8,000, or 80% of the original investment. By the end of year 3 the cumulative cash flow for Project B is $12,000, or 120% of the original investment. Substituting the rest of the information in the table into Equation (12.23), we see that the payback periods for projects A and B are

$$P_j = (F-1) + \frac{-CF_0 - \Sigma_{t=1\to F-1} CF_t}{CF_F}$$

$$P_A = (3-1) + \frac{-CF_0 - \Sigma_{t=1\to 3-1} CF_t}{CF_3}$$

$$= 2 + \frac{-(-10,000) - 7,500 - 2,000}{1,500} = 2 + \frac{500}{1,500} = 2.33 \text{ years}$$

$$P_B = (3-1) + \frac{-CF_0 - \Sigma_{t=1\to 3-1} CF_t}{CF_3}$$

$$= 2 + \frac{-(-10,000) - 4,000 - 4,000}{4,000} = 2 + \frac{2,000}{4,000} = 2.50 \text{ years}$$

Thus, project A is preferred to project B because of its shorter payback period.

TABLE 12.2 Net Cash Flows (CF_t) for Projects A and B

Year, t	Project A	Project B
0	–$10,000	–$10,000
1	7,500	4,000
2	2,000	4,000
3	1,500	4,000
4	1,000	4,000

DISCOUNTED PAYBACK PERIOD METHOD

A variation on the payback period method is the discounted payback period method. The rationale behind the second method is the same as that for the first except that we consider the present value of the projects' cash flows. The projects are discounted to the present using the investor's *cost of capital*. The cost of capital is also referred to as the *discount rate*, the *required rate of return*, the *hurdle rate*, and the *cutoff rate*. The cost of capital is the opportunity cost of finance capital. It is the minimum rate of return required by an investor to justify the commitment of resources to a project.

Definition: The cost of acquiring funds to finance a capital investment project. It is the minimum rate of return that must be earned to justify a capital investment. The cost of capital is the rate of return that an investor must earn on financial assets committed to a project.

Definition: The discounted payback is the number of periods required to recover the original investment where the projects' cash flows are discounted using the cost of capital.

Suppose that the initial cost of a project is $25,000 and that cost of capital (k) is 10%. To determine each project's discounted cash flow (DCF_t), simply divide each period's cash flow by $(1 + k)^t$. The discounted cash flows for projects A and B are summarized in Table 12.3.

Following the procedure already outlined, we see that for project A by the end of year 4 cumulative cash flows are $23,625.44, or 94.5% recovery of the original investment. By the end of year 5 cumulative cash flows are $26, 109.13, or 104.4% recovery of the original investment. Since only an additional $1,374.56 cash flow was required in year 4 to fully cover the original $25,000 investment, the total number of years required to recover the original investment (P_A) was 4 plus $1,374.56/$2,483.69 years, or 4.55 years. Similarly, the payback period for project B (P_B) is 4 plus $6,735.18 years, or 4.99 years. As before, project A is preferred to project B. In general, the expression for calculating the discounted payback period is

TABLE 12.3 Discounted Net Cash Flows (DCF_t) for Projects A and B

Year, t	Project A	Project B
0	–$25,000.00	–$25,000.00
1	9,090.91	2,727.27
2	6,611.57	4,132.23
3	4,507.89	5,259.20
4	3,415.07	6,146.12
5	2,483.69	6,830.13

$$P_j = (F-1) + \frac{-CF_0 - \Sigma_{t=1 \to F-1} DCF_t}{CF_F} = (F-1) + \frac{-CF_0 - \Sigma_{t=1 \to F-1}\left[CF_t/(1+k)^t\right]}{CF_F} \quad (12.24)$$

where $\Sigma_{t=1 \to F-1} DCF_t = \Sigma_{t=1 \to F-1}[CF_t/(1+k)^t]$ is the sum of all discounted cash flows up to and including the year before full recovery of the original investment. Substituting the information in Table 12.3 into Equation (12.24) we obtain

$$P_A = (5-1) + \frac{-(-\$25,000) - \$10,000/(1.10) - \$8,000/(1.10)^2 - \$6,000/(1.10)^3 - \$5,000/(1.10)^4 - \$4,000/(1.10)^5}{\$2,483.69}$$

$$= 4 + \frac{\$1,374.56}{\$2,483.69} = 4.55 \text{ years}$$

$$P_B = (5-1) + \frac{-(-\$25,000) - \$3,000/(1.10) - \$5,000/(1.10)^2 - \$7,000/(1.10)^3 - \$9,000/(1.10)^4 - \$11,000/(1.10)^5}{\$2,483.69}$$

$$= 4 + \frac{\$6,735.18}{\$6,830.13} = 4.99 \text{ years}$$

Since these projects are assumed to be mutually exclusive, then once again project A is preferred to project B because of its shorter discounted payback period. It should be noted that although the payback and discounted payback methods result in the same project rankings here, this is not always the case.

An important drawback of both the payback and discounted payback methods is that they ignore cash flows after the payback period. Suppose, for example that project A generated no additional cash flows after year 5, but project B continued to generate cash flows that increased to, say, $2,000 for each of the next 5 years. Or, suppose project B generates no cash flows for the first 4 years and then generates a cash flow of $100,000 in the fifth year. Because of these deficiencies, other ranking methodologies, such as net present value, internal rate of return, and modified internal rate of return, are more commonly used to rank investment projects. Nevertheless, the payback and discounted period methods are useful because they fell how long funds will be tied up in a project. The shorter the payback period, the greater a project's liquidity.

NET PRESENT VALUE (NPV) METHOD FOR EQUAL-LIVED PROJECTS

The net present value method of evaluating and ranking capital projects was developed in response to the perceived shortcomings of the payback period and discounted payback period approaches. The net present value of a capital project is calculated by subtracting the present value of all cash outflows from the present value of all cash inflows. If the net present value of a project is negative, it is rejected. If the net present value of a project is positive, it is a candidate for further consideration for adoption. Equal-lived projects (i.e., two or more projects that are expected to be in service for the same length of time, with positive net present values) are then ranked from highest to lowest. In general, higher net-present-valued projects are preferred to projects with lower net present values.[1]

Definition: The net present value of a capital project is the difference between the net present value of cash inflows and cash outflows. Projects with higher net present values are preferred to projects with lower net present values.[1]

The net present value of a project is calculated as

$$\begin{aligned} NPV &= CF_0 + \frac{CF_1}{(1+k)^1} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_n}{(1+k)^n} \\ &= \frac{\Sigma_{t=0\to n} CF_t}{(1+k)^t} \end{aligned} \tag{12.25}$$

where CF_t is the expected net cash flow in period t, k is the cost of capital, and n is the life of the project. Net cash flows are defined as the difference between cash inflows (revenues), R_t, and cash outflows, O_t. Equation (12.25) may thus be rewritten as

$$\begin{aligned} NPV &= \frac{\Sigma_{t=0\to n} R_t}{(1+k)^t} - \frac{\Sigma_{t=0\to n} O_t}{(1+k)^t} \\ &= \frac{\Sigma_{t=0\to n} (R_t - O_t)}{(1+k)^t} \end{aligned} \tag{12.26}$$

[1] The discussion thus far has ignored the possible impact of inflation on the time value of money. In the absence of inflation, the real discount rate and the nominal discount rate, which includes an inflation premium, are one and the same. The same can be said of the relationship between real and nominal expected cash flows. When the expected inflation rate is positive, however, then projected cash flows will increase at the rate of inflation. If the inflation rate is also included in the market cost of capital then inflation-adjusted NPV is identical to the inflation-free NPV, which is calculated using Equation (12.25). On the other hand, if the cost of capital includes an inflation premium, but the cash flows do not, then the calculated NPV will have a downward bias. For more information on the effects of inflation on the capital budgeting process see J.C. VanHorne, "A Note on Biases in Capital Budgeting Introduced by Inflation," *Journal of Financial* and *Quantitative Analysis*, January 1971, pp. 653–658; P.L. Cooley, R.L. Rosenfeldt, and I.K. Chew, "Capital Budgeting Procedures under Inflation, "*Financial Management*, Winter 1975, pp. 18–27; and P.L. Cooley, R.L. Rosenfeldt, and I.K. Chew, "Capital Budgeting Procedures under Inflation: Cooley, Rosenfeldt and Chew vs. Findlay and Frankle," *Financial Management*, Autumn 1974, pp. 83–90.

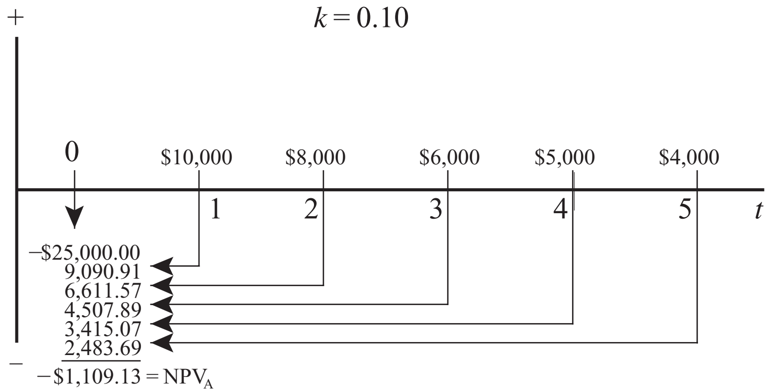

FIGURE 12.13 Net present value calculations for project *A*.

TABLE 12.4 Net Present Value (*NPV*) for Projects *A* and *B*

Year, *t*	Project *A*	Project *B*
0	−$25,000.00	−$25,000.00
1	9,090.91	2,727.27
2	6,611.57	4,132.23
3	4,507.89	5,259.20
4	3,415.07	6,146.12
5	2,483.69	6,830.13
Σ	$1,109.13	$94.95

Consider again the cash flows for projects *A* and *B* summarized in Table 12.1. Also assume that the cost of capital (k) is 10%. To determine the net present value of each project, simply divide the cash flow for each period by $(1 + k)^t$. The calculation for the net present value of project *A* (NPV_A) is illustrated in Figure 12.13 as $1,109.13. It can just as easily be illustrated that the net present value of project *B* is $94.95.

Table 12.4 compares the net present values of projects *A* and *B*. If the two are independent, then both investments should be undertaken. On the other hand, if projects *A* and *B* are mutually exclusive, then project *A* will be preferred to project *B* because its net present value is greater.

A positive net present value indicates that the project is generating cash flows in excess of what is required to cover the cost of capital and to provide a positive rate of return to investors. Finally, if the net present value is negative, the present value of cash inflows is not sufficient to cover the present value of cash outflows. A project should not be undertaken if its net present value is negative.

Problem 12.12. Illuvatar International pays the top corporate income tax rate of 38%. The company is planning to build a new processing plant to manufacture silmarils on the outskirts of Valmar, the ancient capital of Valinor. The new plant will require an immediate cash outlay of $3 million but is expected to generate annual profits of $1 million. According to the Valinor Uniform Tax Code, Illuvatar may deduct $500,000 in taxes annually as depreciation. The life of the new plant is 5 years. Assuming that the annual interest rate is 10%, should Illuvatar build the new processing plant? Explain.

Solution. According to the information provided, Illuvatar's taxable return is $R_t = \pi_t - D_t$, where π_t represents profits and D_t is the amount of depreciation that may be deducted in period t for tax purposes. Illuvatar's taxable rate of return is

$$R_t = \$1{,}000{,}000 - \$500{,}000 = \$500{,}000$$

Illuvatar's annual tax (T_t) is given as $T_t = \tau R_t$, where τ is the tax rate. Illuvatar's annual tax is, therefore,

$$T_t = 0.38(500{,}000) = \$190{,}000$$

Illuvatar's after tax income flow (π_t*) is given as

$$\pi_t{}^* = \pi_t - T_t = \$1{,}000{,}000 - \$190{,}000 = \$810{,}000$$

At an interest rate of 10%, the net present value of the after tax income flow is given as

$$NPV = \frac{\Sigma_{t=1\to5}\pi_t{}^*}{(1+i)^5} - \frac{\Sigma_{t=0\to0}O_t}{(1+i)^0}$$

where O_0 = $3,000,000, the initial cash outlay. Substituting into this expression, we obtain

$$NPV = \frac{810{,}000}{(1.10)} + \frac{810{,}000}{(1.10)^2} + \frac{810{,}000}{(1.10)^3} + \frac{810{,}000}{(1.10)^4} + \frac{810{,}000}{(1.10)^5} - 3{,}000{,}000$$
$$= \$70{,}537.29$$

Because the net present value is positive, Illuvatar should build the new processing plant.

Problem 12.13. Senior management of Bayside Biotechtronics is considering two mutually exclusive investment projects. The projected net cash flows for projects *A* and *B* are summarized in Table 12.5. If the discount rate (cost of capital) is expected to be 12%, which project should be undertaken?

TABLE 12.5 Net Cash Flows (CF_t) for Projects *A* and *B*

Year, *t*	Project *A*	Project *B*
0	–$25,000	–$19,000
1	7,000	6,000
2	8,000	6,000
3	9,000	6,000
4	9,000	6,000
5	5,000	6,000

Solution

a. The net present value of project *A* and project *B* are calculated as

$$NPV_A = \frac{CF_0}{(1+k)^0} + \frac{CF_1}{(1+k)^1} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_n}{(1+k)^5}$$
$$= \frac{-25{,}000}{(1.12)^0} + \frac{7{,}000}{(1.12)^1} + \frac{8{,}000}{(1.12)^2} + \frac{9{,}000}{(1.12)^3} + \frac{9{,}000}{(1.12)^4} + \frac{5{,}000}{(1.12)^5}$$
$$= \$2{,}590.36$$

$$NPV_B = \frac{-19{,}000}{(1.12)^0} + \frac{6{,}000}{(1.12)^1} + \frac{6{,}000}{(1.12)^2} + \frac{6{,}000}{(1.12)^3} + \frac{6{,}000}{(1.12)^4} + \frac{6{,}000}{(1.12)^5}$$
$$= \$2{,}628.66$$

Since $NPV_B > NPV_A$, project *B* should be adopted by Bayside.

Sometimes, mutually exclusive investment projects involve only cash outflows. When this occurs, the investment project with the *lowest* absolute net present value should be selected, as Problem 12.14 illustrates.

Problem 12.14. Finn MacCool, CEO of Quicken Trees Enterprises, is considering two equal-lived psalter dispensers for installation in the employee's recreation room. The projected cash outflows for the two dispensers are summarized in Table 12.6. If the cost of capital is 10% per year and dispense *A* and *B* have salvage values after 5 years of $200 and $350, respectively, which dispenser should be installed?

Solution. The net present values of dispenser *A* and dispenser *B* are calculated as

$$NPV_A = \frac{CF_0}{(1+k)^0} + \frac{CF_1}{(1+k)^1} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_5}{(1+k)^5}$$
$$= \frac{-2{,}500}{(1.10)^0} - \frac{900}{(1.10)^1} - \frac{900}{(1.10)^2} - \frac{900}{(1.10)^3} - \frac{900}{(1.10)^4} - \frac{900}{(1.10)^5} + \frac{200}{(1.10)^5}$$
$$= -\$5{,}787.53$$

TABLE 12.6 Net Cash Flows (CF_t) for Dispensers *A* and *B*

Year, *t*	Dispenser *A*	Dispenser *B*
0	–$2,500	–$3,500
1	–900	–700
2	–900	–700
3	–900	–700
4	–900	–700
5	–900	–700

$$NPV_B = \frac{-3,500}{(1.10)^0} - \frac{700}{(1.10)^1} - \frac{700}{(1.10)^2} - \frac{700}{(1.10)^3} - \frac{700}{(1.10)^4} - \frac{700}{(1.10)^5} + \frac{350}{(1.10)^5}$$
$$= -\$5,936.23$$

Since $|NPV_A| < |NPV_B|$, Finn MacCool will install dispenser *A*.

Problem 12.15. Suppose that an investment opportunity, which requires an initial outlay of $50,000, is expected to yield a return of $150,000 after 20 years.

a. Will the investment be profitable if the cost of capital is 6%?
b. Will the investment be profitable if the cost of capital is 5.5%?
c. At what cost of capital will the investor be indifferent to the investment?

Solution

a. The net present value of the investment with a cost of capital of 6% is given as

$$NPV = \frac{150,000}{(1.06)^{20}} - 50,000 = \frac{150,000}{3.21} - 50,000 = -\$3,229.29$$

Since the net present value is negative, we conclude that the investment opportunity is not profitable.

b. The net present value of the investment with a cost of capital of 5.5% is

$$NPV = \frac{150,000}{(1.055)^{20}} - 50,000 = \frac{150,000}{2.92} - 50,000 = \$1,409.34$$

Since the net present value is positive, we can conclude that the investment opportunity is profitable.

c. The investor will be indifferent to the investment if the net present value is zero. Substituting $NPV = 0$ into the expression and solving for the discount rate yields

$$0 = \frac{150{,}000}{(1+k)^{20}} - 50{,}000$$

$$50{,}000(1+k)^{20} = 150{,}000$$

$$(1+k)^{20} = 3$$

$$1+k = 1.05646$$

$$k = 0.05647$$

That is, the investor will be indifferent to the investment at a cost of capital of approximately 5.65%.

NET PRESENT VALUE (NPV) METHOD FOR UNEQUAL-LIVED PROJECTS

Whereas comparing alternative investment projects with equal lives is a fairly straightforward affair, how do we compare projects that have different lives? Since net present value comparisons involve future cash flows, an appropriate analysis of alternative capital projects must be compared over the same number of years. Unless capital projects are compared over an equivalent number of years, there will be a bias against shorter lived capital projects involving net cash outflows, and a bias in favor of longer lived capital projects involving net cash inflows. To avoid this time and cash flow bias when one is evaluating projects with different lives, it is necessary to modify the net present value calculations to make the projects comparable.

A fair comparison of alternative capital projects requires that net present values be calculated over equivalent time periods. One way to do this is to compare alternative capital projects over the least common multiple of their lives. To accomplish this, the cash flows of each project must be duplicated up to the least common multiple of lives for each project. By artificially "stretching out" the lives of some or all of the prospective projects until all projects have the same life span, we can reduce the evaluation of capital investment projects with unequal lives to a straightforward application of the net present value approach to evaluating projects discussed in the preceding section. In problem 12.16, for example, project *A* has a life expectancy of 2 years, while project *B* has a life expectancy of 3 years. To compare these two projects by means of the net present value approach, project *A* will be replicated three times and project *B* will be replicated twice. In this way, both projects will have a 6-year life span.

Problem 12.16. Brian Borumha of Cashel Company, a leading Celtic oil producer, is considering two mutually exclusive projects, each involving drilling operations in the North Sea. The projected net cash flows for each project are summarized in Table 12.7. Determine which project should be adopted if the cost of capital is 8%.

TABLE 12.7 Net Cash Flows (CF_t) for Projects *A* and *B* ($ millions)

Year, *t*	Project *A*	Project *B*
0	–$2,000	–$5,000
1	1,000	1,000
2	1,500	2,500
3		3,000

Solution. Since the projects have different lives, they must be compared over the least common multiple of years, which in this case is 6 years.

$$\begin{aligned} NPV_A &= \frac{CF_0}{(1+k)^0} + \frac{CF_1}{(1+k)^1} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_6}{(1+k)^6} \\ &= \frac{-\$2,000}{(1.08)^0} + \frac{\$1,000}{(1.08)^1} + \frac{\$1,500}{(1.08)^2} - \frac{\$2,000}{(1.08)^2} + \frac{1,000}{(1.08)^3} + \frac{1,500}{(1.08)^4} \\ &\quad - \frac{2,000}{(1.08)^4} + \frac{1,000}{(1.08)^5} + \frac{1,500}{(1.08)^6} \\ &= \$549.41 \end{aligned}$$

$$\begin{aligned} NPV_B &= \frac{-5,000}{(1.08)^0} + \frac{1,000}{(1.08)^1} + \frac{2,500}{(1.08)^2} + \frac{3,000}{(1.08)^3} - \frac{5,000}{(1.08)^3} \\ &\quad + \frac{1,000}{(1.08)^4} + \frac{2,500}{(1.08)^5} + \frac{3,000}{(1.08)^6} \\ &= \$808.61 \end{aligned}$$

Since $NPV_B > NPV_A$, Brian Borumha will select project *B* over project *A*.

INTERNAL RATE OF RETURN (IRR) METHOD AND THE HURDLE RATE

Yet another method of evaluating a capital investment project is by calculating the *internal rate of return (IRR)*. Before discussing the methodology of calculating a project's internal rate of return, it is important to understand the rationale underlying this approach. Consider, for example, the case of an investor who is considering purchasing a 12-year, 10% annual coupon, $1,000 par-value corporate bond for $1,150.70. Before deciding whether the investor should purchase this bond, consider the following definitions.

Coupon bonds are debt obligations of private companies or public agencies in which the issuer of the bond promises to pay the bearer of the bond a series of fixed dollar interest payments at regular intervals for a specified period of time. Upon maturity, the issuer agrees to repay the bearer the par value of the bond. The *par value of a bond* is the face value of the bond, which is the amount originally borrowed by the issuer. Thus, a corporation that issues a $1,000 coupon bond is obligated to pay the bearer of the bond fixed dollar payments at regular intervals. In the present example, the issuer of the bond promises to pay the bearer of the bond $100 per year for the next 12 years plus the face value of the bond at maturity. Parenthetically, the term "coupon bond" comes from the fact that at one time a number of small, dated coupons indicating the amount of interest due to the owner were attached to the bonds. A bond owner would literally clip a coupon from the bond on each payment date and either cash or deposit the coupon at a bank or mail it to the corporation's paying agent, who would then send the owner a check in the amount of the interest.

Definition: Coupon bonds are debt obligations in which the issuer of the bond promises to pay the bearer of the bond fixed dollar interest payments at regular intervals for a specified period of time, with reimbursement of the face value at the end of the period.

Definition: The par value of a bond is the face value of the bond. It is the amount originally borrowed by the issuer.

Why would an investor consider purchasing a bond for an amount in excess of its par value? The reason is simple. In the present example, when the bond was first issued the prevailing rate of interest paid on bonds with equivalent risk and maturity characteristics was 10%. If the bond holder wanted to sell the bond before maturity, the market price would reflect the prevailing rate of interest.

If current market interest rates are higher than the coupon interest rate, the bearer will have to sell the bond at a discount from par value. Otherwise, no one would be willing to buy such a bond. On the other hand, if prevailing interest rates are lower than the coupon interest rate, then the bearer will be able to sell the bond at a premium. The size of the discount or premium reflects the term to maturity and the differential between the prevailing market interest rate and the coupon rate on bonds with similar risk characteristics. Since the market value of the bond in the present example is greater than its par value, prevailing market rates must be lower than the coupon interest rate.

Returning to our example, should the investor purchase this bond? The decision to buy or not to buy this bond will be based upon the rate of return the investor will earn on the bond if held to maturity. This rate of return is called the bond's *yield to maturity* (*YTM*). If the bond's *YTM* is greater than the prevailing market rate of interest, the investor will purchase the bond. If the *YTM* is less than the market rate, the investor will not purchase. If the *YTM* is the same as the market rate, other things being equal, the investor will be indifferent between purchasing this bond and a newly issued bond.

Definition: Yield to maturity is the rate of return earned on a bond that is held to maturity.

Calculating the bond's *YTM* involves finding the rate of interest that equates the bond's offer price, in this case \$1,150.70, to the net present value of the bond's cash inflows. Denoting the value price of the bond as V_B, the interest payment as *PMT*, and the face value of the bond as *M*, the yield to maturity can be found by solving Equation (12.27) for *YTM*.

$$V_B = \frac{PMT}{(1+YTM)^1} + \frac{PMT}{(1+YTM)^2} + \ldots + \frac{PMT}{(1+YTM)^n} + \frac{M}{(1+YTM)^n} = \frac{\Sigma_{t=1\to n} PMT}{(1+YTM)^t} + \frac{M}{(1+YTM)^n} \quad (12.27)$$

Substituting the information provided into Equation (12.27) yields

$$\$1,150.72 = \frac{\$100}{(1+YTM)^1} + \frac{\$100}{(1+YTM)^2} + \ldots + \frac{\$100}{(1+YTM)^n} + \frac{\$1,000}{(1+YTM)^n}$$

Unfortunately, finding the *YTM* that satisfies this expression is easier said than done. Different values of *YTM* could be tried until a solution is found, but this brute force approach is tedious and time-consuming. Fortunately, financial calculators are available that make the process of finding solution values to such problems a trivial procedure. As it turns out, the yield to maturity in this example is *YTM** = 0.08, or an 8% yield to maturity. The solution to this problem is illustrated in Figure 12.14.

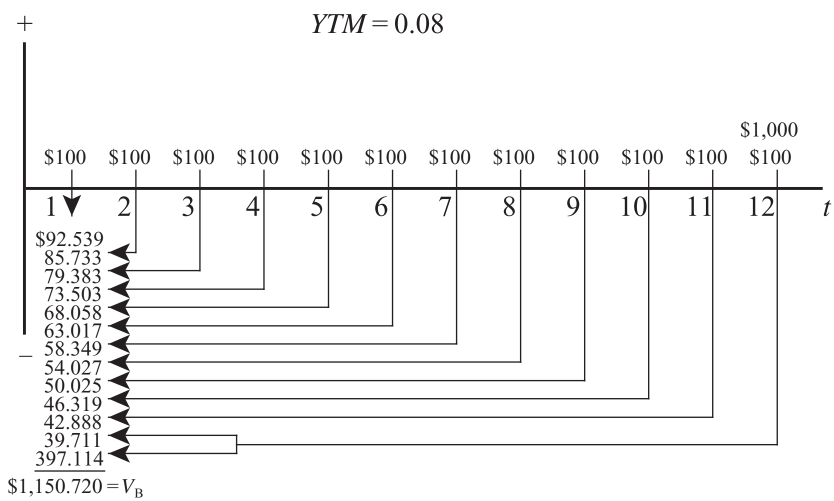

FIGURE 12.14 Yield to maturity.

Thus, the investor will compare the *YTM* to the rate of return on bonds of equivalent risk characteristics before deciding whether to purchase the bond. Parenthetically, the efficient markets hypothesis suggests that the *YTM* on this coupon bond will be the same as the prevailing market interest rate.

We now return to the internal rate of return method for evaluating capital projects, introduced earlier. As we will see shortly, the methodology for determining the yield to maturity on a bond is the same as that used for calculating the internal rate of return. The internal rate of return is the discount rate that equates the present value of a project's expected cash inflows with the project's expected cash outflows. The internal rate of return may be calculated from Equation (12.28).

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)^1} + \frac{CF_2}{(1+IRR)^2} + \ldots + \frac{CF_n}{(1+IRR)^n} = \frac{\Sigma_{t=1\to n} CF_t}{(1+IRR)^t} = 0 \quad (12.28)$$

Consider, again, the information presented in Table 12.1 for project *A*. This problem is illustrated in Figure 12.15.

To determine the discount rate for which *NPV* is zero, substitute the information provided for project *A* in Table 12.1 into Equation (12.27), which yields

$$NPV = -\$25,000 + \frac{\$10,000}{(1+IRR)^1} + \frac{\$8,000}{(1+IRR)^2} + \frac{\$6,000}{(1+IRR)^3} + \frac{\$5,000}{(1+IRR)^4} + \frac{\$4,000}{(1+IRR)^5} = 0$$

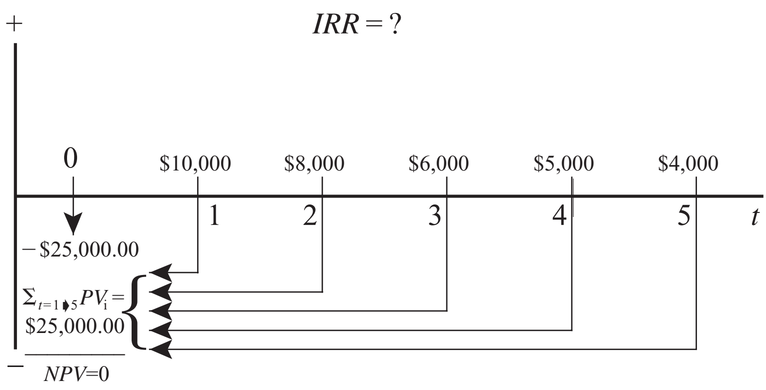

FIGURE 12.15 Internal rate of return is the discount rate for which the net present value of a project is equal to zero.

Of course, finding *IRR* is no easier than solving for *YTM*, as discussed earlier. Once again, a financial calculator comes to the rescue. The internal rate of return for projects *A* and *B* are $IRR_A = 12.05\%$ and $IRR_B = 10.12\%$. Whether these projects are accepted or rejected depends on the cost of capital, which is sometimes referred to as the *hurdle rate*, *required rate of return*, or *cutoff rate*. The somewhat colorful expression “hurdle rate” is meant to express the notion that a company can increase its shareholder value by investing in projects that earn a rate of return that exceeds (hurdles over) the cost of capital used to finance the project.

Definition: The internal rate of return is the discount rate that equates the present value of a project’s expected cash inflows with the project’s expected cash outflows.

Definition: The hurdle rate is the cost of capital of a project that must be exceeded by the internal rate of return if the project is to be accepted. Often referred to as the required rate of return or the cutoff rate.

Another way to look at the internal rate of return is that it is the maximum rate of interest that an investor will pay to finance a capital investment project. Alternatively, the internal rate of return is the minimum acceptable rate of return on an investment. Thus, if the internal rate of return is greater than the cost of capital (hurdle rate), a project will be accepted. If the internal rate of return is less than the hurdle rate, a project will be rejected. Finally, if the internal rate of return is equal to the cost of capital, the investor will be indifferent to the project. Of course, the investor would like to earn as much as possible in excess of the internal rate of return.

Suppose that an investor is considering investing in either project *A* or project *B*. If the two projects are independent and the internal rate of return exceeds the hurdle rate, both projects will be accepted. On the other hand, if the projects are mutually exclusive, project *A* will be preferred to project *B* because of its higher internal rate of return. The *NPV* and *IRR* will always result in the same accept and reject decisions for independent projects. This is because, by definition, when *NPV* is positive, then *IRR* will exceed the cost of funds to finance the project. On the other hand, the *NPV* and *IRR* methods can result in conflicting accept/reject decisions for mutually exclusive projects. A comparison of the *NPV* and *IRR* methods of evaluating capital investment projects will be the subject of the next section.

Problem 12.17. Consider, again, Bayside Biotechtronics. The projected net cash flows for projects *A* and *B* are summarized in Table 12.8.

a. Calculate the internal rate of return for both projects.
b. If the cost of capital for financing the projects (hurdle rate) is 17%, which project should be considered?
c. Verify that if the hurdle rate is 1% lower, $NPV_A > 0$
d. Verify that if the hurdle rate is 1% higher, $NPV_B < 0$.

TABLE 12.8 Net Cash Flows CF_t for Projects A and B

Year, t	Project A	Project B
0	–$25,000	–$19,000
1	7,000	6,000
2	8,000	6,000
3	9,000	6,000
4	9,000	6,000
5	5,000	6,000

Solution

a. To determine the internal rate of return for projects A and B, substitute the information provided in the table into the Equation (12.27) and solve for IRR.

$$NPV_A = CF_0 + \frac{CF_1}{(1+IRR_A)^1} + \frac{CF_2}{(1+IRR_A)^2} + \ldots + \frac{CF_5}{(1+IRR_A)^5}$$
$$= -\$25,000 + \frac{\$7,000}{(1+IRR_A)^1} + \frac{\$8,000}{(1+IRR_A)^2} + \frac{\$9,000}{(1+IRR_A)^3}$$
$$+ \frac{\$9,000}{(1+IRR_A)^4} + \frac{\$5,000}{(1+IRR_A)^5} = 0$$

$$NPV_B = -\$19,000 + \frac{\$6,000}{(1+IRR_B)^1} + \frac{\$6,000}{(1+IRR_B)^2} + \frac{\$6,000}{(1+IRR_B)^3}$$
$$+ \frac{\$6,000}{(1+IRR_B)^4} + \frac{\$6,000}{(1+IRR_B)^5} = 0$$

Since calculating IRR_A and IRR_B by trial and error is time-consuming and tedious, the solution values were obtained by using a financial calculator. The internal rates of return for projects A and B are

$$IRR_A = 16.168\%$$

$$IRR_B = 17.448\%$$

b. The internal rate of return is less than the hurdle rate for project A and greater than the hurdle rate for project B. Thus, project A is rejected and project B is accepted.

c. Substituting into Equation (12.28), we write

$$NPV_A = \frac{\Sigma_{t=1\to n} CF_t}{(1.15168)^t}$$

$$= -\$25,000 + \frac{\$7,000}{(1.15168)^1} + \frac{\$8,000}{(1.15168)^2} + \frac{\$9,000}{(1.15168)^3}$$

$$+ \frac{\$9,000}{(1.15168)^4} + \frac{\$5,000}{(1.15168)^5} = \$584.85$$

d. $NPV_A = \frac{\Sigma_{t=1\to n} CF_t}{(1.17168)^t} = -\$563.64$

COMPARING THE *NPV* AND *IRR* METHODS

Consider, once again, the cash flows for projects *A* and *B* presented in Table 12.1. Table 12.9 summarizes the net present values for the cash flows of project *A* and *B* for different costs of capital. The data summarized in Table 12.9 are illustrated in Figure 12.16. A diagram that plots the relationship between the net present value of a project and alternative costs of capital is called a *net present value profile*.

Definition: A net present value profile is a diagram that shows the relationship between the net present value of a project and alternative costs of capital.

When the cost of capital is zero, the project's net present value is simply the sum the project's net cash flows. In the present example, the net present values for projects *A* and *B* when $k = 0.00\%$ are \$8,000 and \$10,000, respectively. The student will also readily observe from Equation (12.28) that as the cost of capital increases, the net present value of the project declines, which gives rise to the downward-sloping curves in Figure 12.16.

TABLE 12.9 Net Present Value Profiles for Projects *A* and *B*

Cost of capital	Project *A*	Project *B*
0.00	\$8,000	\$10,000
0.02	6,389	7,621
0.04	4,908	5,465
0.05	4,211	4,462
0.05875	**3,623**	**3,623**
0.06	3,541	3,506
0.08	2,278	1,723
0.10	1,109	96
0.12	24	–1,392
0.14	–985	–2,755

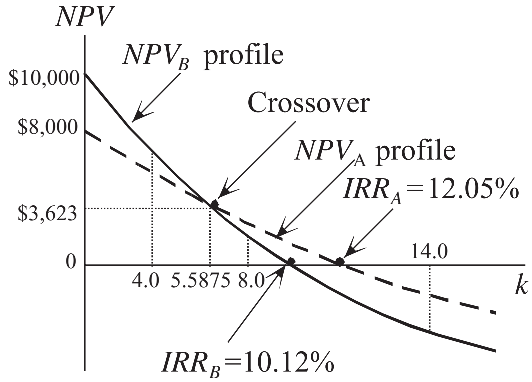

FIGURE 12.16 Internal rates of return and crossover rate.

In one earlier discussion, the internal rate of return was defined as the discount rate at which the *NPV* of a project is zero. For projects *A* and *B*, the internal rates of return (not shown in Table 12.9) are 12.05 and 10.12%, respectively. These values are illustrated in Figure 12.16 at the points at which the net present value profiles for projects *A* and *B* intersect the horizontal axis.

The student will note that when the cost of capital is 5.875%, the net present values of projects *A* and *B* are the same. Additionally, when the cost of capital is less than 5.875% $NPV_A < NPV_B$, and when the cost of capital is greater than 5.875% $NPV_A > NPV_B$. This is illustrated in Figure 12.14 at the point of intersection of the present value profiles of project *A* and *B*. For obvious reasons, the cost of capital at which the *NPV*s of two projects are equal is called the *crossover rate*.

Definition: The crossover rate is the cost of capital at which the net present values of two projects are equal. Diagrammatically, this is the cost of capital at which the net present value profiles of two projects intersect.

An examination of Figure 12.16 also reveals that the marginal change in NPV_B given a change in the cost of capital is greater than that for NPV_A (i.e., $\partial NPV_B/\partial k > \partial NPV_A/\partial k$). In other words, the slope of the net present value profile for project *B* is steeper than the net present value profile for project *A*. The reason for this is that project *B* is more sensitive to changes in the cost of capital than project *A*.

Given the cost of capital, the sensitivity of *NPV* to changes in the cost of capital will depend on the timing of the project's cash flows. To see this, consider once again the cash flows summarized in Table 12.1. Note that these cash flows are received more quickly in the case of project *A* than for project *B*. Referring to Table 12.9, when the cost of capital is doubled from 5.0% to 10.0%, NPV_A falls from $4,211 to $1,109, or a decline of 73.7%. For project *B*, NPV_B falls from $4,462 to $96, or a drop of 97.8%. The reason for the discrepancy is the discounting factor $1/(1 + k)^n$, which will be greater for cash flows received in the distant future than for cash flows received in the near future. Thus, the net present value of projects that receive greater cash flows in the distant future will decline at a faster rate than for projects receiving most of their cash in the early years.

NPV AND *IRR* METHODS FOR INDEPENDENT PROJECTS

It was noted earlier that when the cost of capital is less than *IRR* for both projects, then the *NPV* and *IRR* methods will always result in the same accept and reject decisions. This can be seen in Figure 12.16. If the cost of capital is less than 10.12%, and projects *A* and *B* are independent, both projects will be accepted. If the cost of capital is between 10.12 and 12.05%, project *A* will be accepted and project *B* will be rejected. Finally, If the cost of capital is greater than 12.05%, then both projects will be rejected.

NPV AND *IRR* METHODS FOR MUTUALLY EXCLUSIVE PROJECTS

We noted earlier that if the projects are mutually exclusive (the acceptance of one project means the rejection of the other), the *NPV* and *IRR* methods can result in conflicting accept/reject decisions. To see this, consider again Figure 12.16. If the cost of capital is *greater than the crossover rate*, but less than *IRR* for both projects, in this case 10.12%, then $NPV_A > NPV_B$ and $IRR_A > IRR_B$, in which case both the *IRR* and *NPV* methods indicate that project *A* is preferred to project *B*.

On the other hand, if the cost of capital is *less than the crossover rate*, then although IRR_A is still less than IRR_B, $NPV_B > NPV_A$. Thus, the net present value method indicates that project *B* should be preferred to project *A* and the internal rate of return method ranks project *B* higher than project *A*. In other words, when the cost of capital is less than the crossover rate, a conflict arises between the *NPV* and *IRR* methods. Two questions immediately present themselves:

1. Why do the net present value profiles intersect?
2. When an accept/reject conflict exists because the cost of capital is less than the crossover rate, which method should be used to rank mutually exclusive projects?

The net present value profiles of two projects may intersect for two reasons: differences in project sizes and cash flow timing differences. As noted earlier, the effect of discounting will be greater for cash flows received in the distant future than for cash flows received in the near future. The net present value of projects in which most of the cash flows are received in the distant future will decline at a faster rate than the decline in the net present value for projects in which most of the cash flows are generated in the near future. Thus, if the *NPV* for one project (project *B* in Figure 12.16) is greater than the *NPV* for another project (project *A* in Figure 12.16) when $t = 0$ *and* most of the cash flows for the first project are received in the distant future in comparison to the second project, the net present value profiles of the two projects may intersect.

When the net present value profiles intersect and the cost of capital is less than the crossover rate, which method should be used for selecting a capital investment project? The answer depends on the rate at which the firm reinvests the net cash inflows over the life of the project. The *NPV* method implicitly assumes that net cash inflows are reinvested at the cost of capital. The *IRR* method assumes that net cash inflows are reinvested at the internal rate of return. So, which of these assumptions is more realistic? It may be demonstrated (see Brigham, Gapenski, and Erhardt 1998, Chapter 11) that the best assumption is that a project's net cash inflows are reinvested at the firm's cost of capital. Thus, for ranking mutually exclusive capital investment projects, the *NPV* method is preferred to the *IRR* method.

Problem 12.18. Consider, again, the net cash flows for projects *A* and *B* in Bayside Biotechtronics, summarized in Table 12.10.

a. Illustrate the net present value profiles for projects *A* and *B*.
b. What is the crossover rate for the two projects?
c. Assuming that projects *A* and *B* are mutually exclusive, which project should be selected if the cost of capital is greater than the crossover rate? Which project should be selected if the cost of capital is less than the crossover rate?

Solution

a. A financial calculator was used to find the net present values for projects *A* and *B* for various interest rates are summarized in Table 12.11.

   To determine the crossover rate, using Equation (12.25) to equate the net present value of project *A* with the net present value of project *B* and solve for the cost of capital, *k*.

TABLE 12.10 Net Cash Flows (CF_t) for Projects *A* and *B*

Year, t	Project *A*	Project *B*
0	–$25,000	–$19,000
1	7,000	6,000
2	8,000	6,000
3	9,000	6,000
4	9,000	6,000
5	5,000	6,000

TABLE 12.11 Net Present Value Profiles for Projects *A* and *B*

Cost of capital	Project *A*	Project *B*
0.00	$13,000	$11,000
0.04	8,931	7,711
0.06	7,145	6,274
0.08	5,503	4,956
0.10	3,989	3,745
0.1172	**2,780**	**2,780**
0.12	2,590	2,629
0.14	1,296	1,598
0.16	97	646
0.18	–1,017	–237

$$NPV_A = NPV_B$$

$$\frac{-\$25,000}{(1+k)^0}+\frac{\$7,000}{(1+k)^1}+\frac{\$8,000}{(1+k)^2}+\frac{\$9,000}{(1+k)^3}+\frac{\$9,000}{(1+k)^4}+\frac{\$9,000}{(1+k)^5}=$$
$$\frac{-\$19,000}{(1+k)^0}+\frac{\$6,000}{(1+k)^1}+\frac{\$6,000}{(1+k)^2}+\frac{\$6,000}{(1+k)^3}+\frac{\$6,000}{(1+k)^4}+\frac{\$6,000}{(1+k)^5}$$

Bringing all the terms in this expression to the left-hand side of the equation, we get

$$\frac{-\$6,000}{(1+k)^0}+\frac{\$1,000}{(1+k)^1}+\frac{\$2,000}{(1+k)^2}+\frac{\$3,000}{(1+k)^3}+\frac{\$3,000}{(1+k)^4}-\frac{\$3,000}{(1+k)^5}=0$$

The value for k in this expression may be found using the *IRR* function of a financial calculator. Solving for k yields a crossover rate of 11.72%.

Last, the internal rates of return for projects *A* and *B* may be calculated from Equation (12.28).

$$NPV_A = CF_0 + \frac{CF_1}{(1+IRR)^1}+\frac{CF_2}{(1+IRR)^2}+\ldots+\frac{CF_n}{(1+IRR)^5}$$
$$=\frac{-\$25,000}{(1+IRR)^0}+\frac{\$7,000}{(1+IRR)^1}+\frac{\$8,000}{(1+IRR)^2}+\frac{\$9,000}{(1+IRR)^3}$$
$$+\frac{\$9,000}{(1+IRR)^4}+\frac{\$9,000}{(1+IRR)^5}=0$$

Solving with a financial calculator yields

$$IRR_A = 16.17\%$$

Similarly for project B,

$$NPV_B = \frac{-\$19{,}000}{(1+IRR)^0} + \frac{\$6{,}000}{(1+IRR)^1} + \frac{\$6{,}000}{(1+IRR)^2} + \frac{\$6{,}000}{(1+IRR)^3} + \frac{\$6{,}000}{(1+IRR)^4} + \frac{\$6{,}000}{(1+IRR)^5} = 0$$

Solving,

$$IRR_B = 17.45\%$$

Finally, using the crossover rate to calculate the net present value of projects A and B yields

$$NPV_A = \frac{-\$25{,}000}{(1.1172)^0} + \frac{\$7{,}000}{(1.1172)^1} + \frac{\$8{,}000}{(1.1172)^2} + \frac{\$9{,}000}{(1.1172)^3} + \frac{\$9{,}000}{(1.1172)^4} + \frac{\$9{,}000}{(1.1172)^5} = \$5{,}077.91$$

$$NPV_B = \frac{-\$19{,}000}{(1.1172)^0} + \frac{\$6{,}000}{(1.1172)^1} + \frac{\$6{,}000}{(1.1172)^2} + \frac{\$6{,}000}{(1.1172)^3} + \frac{\$6{,}000}{(1.1172)^4} + \frac{\$6{,}000}{(1.1172)^5} = \$2{,}780$$

With this information, the net present value profiles for projects A and B may be illustrated in Figure 12.17.

b. From Figure 12.17, the crossover rate for the two projects is 11.72%.

c. From Figure 12.17, if the cost of capital is greater than 11.72%, but less than 16.17%, project B is preferred to project A because $NPV_B > NPV_A$. This choice of projects is consistent with the IRR method, since $IRR_B >$

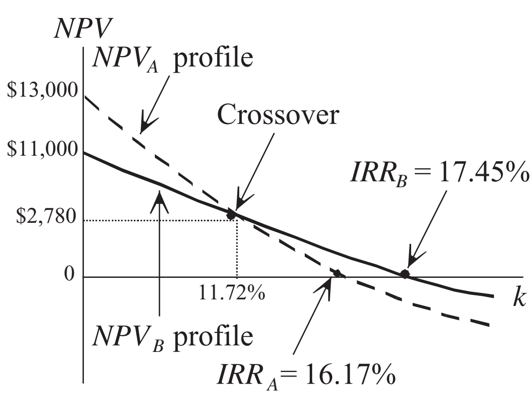

FIGURE 12.17 Diagrammatic solution to problem 12.18, parts b and c.

IRR_A. On the other hand, if the cost of capital is less than 11.72%, project A is preferred to project B, since $NPV_A > NPV_B$. This result conflicts with the choice of projects indicated by the IRR method.

MULTIPLE INTERNAL RATES OF RETURN

In addition to the problems associated with using the IRR method for evaluating capital investment projects, there is yet another potential fly in the ointment: a project may have multiple internal rates of return.

Definition: A project with two or more internal rates of return is said to have multiple internal rates of return.

To illustrate how multiple internal rates of return might occur, consider again Equation (12.28) for calculating the net present value of a project.

$$NPV = CF_0 + \frac{CF_1}{(1+IRR)^1} + \frac{CF_2}{(1+IRR)^2} + \ldots + \frac{CF_n}{(1+IRR)^n} = \frac{\Sigma_{t=1\to n}CF_t}{(1+IRR)^t} = 0 \quad (12.28)$$

The student will immediately recognize that Equation (12.28) is a polynomial of degree n. What this means is that depending on the values of CF_t, Equation (12.28) may have n possible solutions for the internal rate of return! Before discussing the conditions under which multiple internal rates of return are possible, consider Table 12.12, which summarizes the cash flows of a capital investment project.

Substituting the cash flow information from Table 12.12 into Equation (12.28), we obtain

$$NPV = -\$1,000 + \frac{\$6,000}{(1+IRR)^1} - \frac{\$6,000}{(1+IRR)^2} = 0 \quad (12.29)$$

Equation (12.29) is a second-degree polynomial (quadratic) equation, which may have two solution values. To find the solution values, rewrite Equation (12.29) as

$$-\$6,000\left(\frac{1}{1+IRR}\right)^2 + \$6,000\left(\frac{1}{1+IRR}\right) - \$1,000 = 0$$

TABLE 12.12 Net Cash Flows (CF_t) for Project A

Year, t	CF_t
0	–\$1,000
1	6,000
2	–6,000

which is of the general form

$$ax^2 + bx + c = 0 \tag{2.69}$$

The solution values may be found by applying the quadratic equation

$$x_{1,2} = \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} \tag{2.70}$$

Substituting the information provided in Equation (12.29) into Equation (2.70) yields

$$\left(\frac{1}{1+IRR}\right)_{1,2} = \frac{-6{,}000 \pm \left[(6{,}000)^2 - 4(-6{,}000)(-1{,}000)\right]^{0.5}}{2(-6{,}000)}$$
$$= \frac{-6{,}000 \pm [36{,}000{,}000 - 24{,}000{,}000]^{0.5}}{-12{,}000}$$
$$= \frac{-6{,}000 \pm (12{,}000{,}000)^{0.5}}{-12{,}000}$$
$$= \frac{-6{,}000 \pm 3{,}464.10}{-12{,}000}$$

The solution values are

$$\left(\frac{1}{1+IRR}\right)_1 = \frac{-6{,}000 - 3{,}464.10}{-12{,}000} = 0.21$$
$$(1+IRR)_1 = 4.76$$
$$IRR_1 = 3.76$$
$$\left(\frac{1}{1+IRR}\right)_2 = \frac{-6{,}000 + 3{,}464.10}{-12{,}000} = 0.79$$
$$(1+IRR)_2 = 1.27$$
$$IRR_2 = 0.27$$

We find that for the cash flows summarized in Table 12.12, this project has internal rates of return of both 27 and 476%. The *NPV* profile for this project is summarized in Table 12.13 and Figure 12.18.

Under what circumstances are multiple internal rates of return possible? Thus, far we have dealt only with *normal cash flows*. A project has normal cash flows when one or more of the cash outflows are followed by a series of cash inflows. The cash flow depicted in Table 12.12 is an example of an *abnormal cash flow*. A large cash outflow during or toward the end of the life of a project is considered to be abnormal. Projects with abnormal cash flows may exhibit multiple internal rates of return.

Definition: A project has a normal cash flow if one or more cash outflows are followed by a series of cash inflows.

TABLE 12.13 Net Present Value Profile for Project *A*

k	*NPV*
0.00	–$1,000.00
0.25	–40.00
0.27	**0.00**
0.50	333.33
1.00	500.00
1.50	440.00
2.00	333.33
2.50	224.49
3.00	125.00
3.50	37.04
3.76	**0.00**
4.00	–40.00
4.50	–107.44

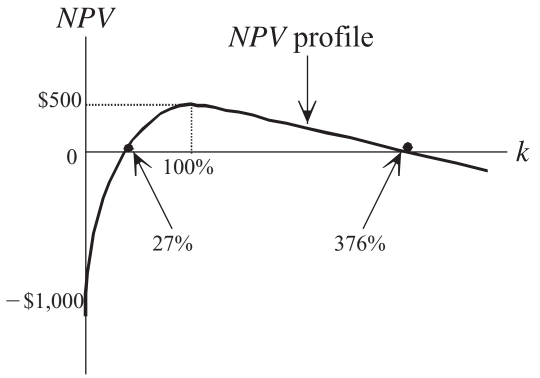

FIGURE 12.18 Multiple internal rates of return.

Definition: A project has an abnormal cash flow when large cash outflows occur during or toward the end of the project's life.

As before, no difficulties arise when the net present value method is used to evaluate capital investment projects. In our example, if the cost of capital is between 27 and 376% independent projects should be accepted because their net present value is positive. On the other hand, project selection is problematic if the internal rate of return method is employed. It may no longer be automatically presumed that if the internal rate of return is greater than the cost of capital, the project should be accepted. Suppose, for example, that the cost of capital is 10%, which is less than both internal rates of return. Using the *IRR* method, which project should be accepted? In general, the approach will be preferred. Using the *NPV* method, however, the project should be clearly rejected.

TABLE 12.14 Net Cash Flows (CF_t) for Project X

Year, t	CF_t
0	–$500
1	4,000
2	–5,000

TABLE 12.15 Net Present Value Profile for Project A

k	NPV
0.00	–$1,500.00
0.10	–995.87
0.25	–500.00
0.50	–55.56
0.56	**0.00**
1.00	250.00
1.50	300.00
2.00	277.78
2.50	234.69
3.00	187.50
3.50	141.98
4.00	100.00
4.50	61.98
5.00	27.78
5.25	**0.00**
5.50	–2.96

Our example illustrates multiple internal rates of return resulting from abnormal cash flows. Abnormal cash flows can also create other problems, such as no internal rate of return at all. Either way, the *NPV* method is a clearly superior method for evaluating capital investment projects.

Problem 12.19. Consider the cash flows for project X, summarized in Table 12.14.

a. Summarize in a table project *X*'s net present value profile for selected costs of capital.
b. Does project *X* have multiple internal rates of return? What are they?
c. Diagram your answer.

Solution

a. Substituting the cash flows provided and alternative costs of capital into Equation (12.28), we obtain Table 12.15.
b. Substituting the cash flow information into Equation (12.28) yields

$$NPV = -\$500 + \frac{\$4,000}{(1+IRR)^1} - \frac{\$5,000}{(1+IRR)^2} = 0$$

Rearranging, we have

$$-\$5,000\left(\frac{1}{1+IRR}\right)^2 + \$4,000\left(\frac{1}{1+IRR}\right)^1 - \$500 = 0$$

which is of the general form

$$a\left(\frac{1}{1+IRR}\right)^2 + b\left(\frac{1}{1+IRR}\right)^1 + c = 0$$

The solution values to this expression may be found by solving the quadratic equation

$$\begin{aligned}\left(\frac{1}{1+IRR}\right)_{1,2} &= \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} \\ &= \frac{-4,000 \pm \left[(4,000)^2 - 4(-5,000)(-500)\right]^{0.5}}{2(-5,000)} \\ &= \frac{-4,000 \pm 2,449.49}{-10,000}\end{aligned}$$

The solution values are

$$\begin{aligned}\left(\frac{1}{1+IRR}\right)_1 &= \frac{-4,000 + 2,449.49}{-10,000} = 0.16 \\ (1+IRR)_1 &= 6.25 \\ IRR_1 &= 5.25, \text{ or } 525\% \\ \left(\frac{1}{1+IRR}\right)_2 &= \frac{-4,000 - 2,449.49}{-10,000} = 0.64 \\ (1+IRR)_2 &= 1.56 \\ IRR_2 &= 0.56, \text{ or } 56\%\end{aligned}$$

Project *X* has internal rates of return of both 56 and 525%.

c. Figure 12.19 shows the *NPV* profile for Project *A*.

MODIFIED INTERNAL RATE OF RETURN (MIRR) METHOD

Earlier we compared the *NPV* and *IRR* methods for evaluating independent and mutually exclusive investment projects. We found that for independent projects, both the *NPV* and the *IRR* methods will yield the same accept/reject decision rules. We also found that for mutually exclusive

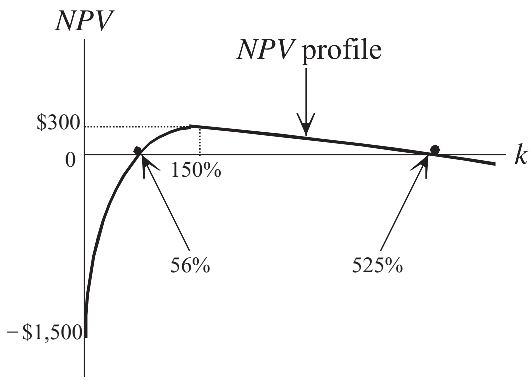

FIGURE 12.19 Diagrammatic solution to problem 12.19.

capital investment projects the *NPV* and the *IRR* methods could result in conflicting accept/reject decision rules.

It was noted that when the net present value profiles of two mutually exclusive projects intersect, the choice of projects should be based on the *NPV* method. This is because the *NPV* method implicitly assumes that net cash inflows are reinvested at the cost of capital, whereas the *IRR* method implicitly assumes that net cash inflows are reinvested at the internal rate of return. In view of its widespread practical application, is it possible to modify the *IRR* method by incorporating into the calculation the assumption that net cash flows are reinvested at the cost of capital? Happily, the answer to this question is yes. What is more, this method also overcomes the problem of multiple internal rates of return.

The *modified internal rate of return* (*MIRR*) method for evaluating capital investment projects is similar to the *IRR* method in that it generates accept/reject decision rules based on interest rate comparisons. But unlike the *IRR* method, the *MIRR* method assumes that cash flows are reinvested at the cost of capital and avoids some of the problems associated with multiple internal rates of return. The modified internal rate of return for a capital investment project may be calculated by using Equation (12.30)

$$\frac{\Sigma_{t=1\to n} O_t}{(1+k)^t} = \frac{\Sigma_{t=1\to n} R_t (1+k)^{n-t}}{(1+MIRR)^n} \tag{12.30}$$

where O_t represents cash outflows (costs), R_t represents the project's cash inflows (revenues), and k is the firm's cost of capital.

The term on the left hand side of Equation (12.30) is simply the present value of the firm's investment outlays *discounted at the firm's cost of capital*. The numerator on the right side of Equation (12.30) is the future value of the project's cash inflows *reinvested at the firm's cost of capital*. The future value of a project's cash inflows is sometimes referred to as the *terminal* *value* (*TV*) of the project. The modified internal rate of return is defined as the discount rate that equates the present value of cash outflows with the present value of the project's terminal value.

Definition: A project's terminal value is the future value of cash inflows compounded at the firm's cost of capital.

Definition: The modified internal rate of return is the discount rate that equates the present value of a project's cash outflows with the present value of the project's terminal value.

Consider, again, the net cash flows summarized in Table 12.1. Assuming a cost of capital of 10%, and substituting the cash flows in Table 12.1 into Equation (12.30), the *MIRR* for project *A* is

$$\frac{\Sigma_{t=1\to n}O_t}{(1+k)^t} = \frac{\Sigma_{t=1\to n}R_t(1+k)^{n-t}}{(1+MIRR_A)^n}$$

$$\frac{\$25,000}{(1.10)^0} = \frac{\$10,000(1.10)^4 + \$8,000(1.10)^3 + \$6,000(1.10)^2 + \$5,000(1.10)^1 + \$4,000(1.10)^0}{(1+MIRR_A)^5}$$

$$= \frac{\$14,641 + \$10,648 + \$7,260 + \$5,500 + \$4,000}{(1+MIRR_A)^5}$$

$$\$25,000 = \frac{\$42,049}{(1+MIRR_A)^5}$$

$$(1+MIRR_A)^5 = \frac{\$42,049}{25,000} = 1.68196$$

$$1+MIRR_A = 1.1096$$

$$MIRR_A = 0.1096, \text{ or } 10.96\%$$

The calculation of *MIRR* for project *A* is illustrated in Figure 12.20. Likewise, the *MIRR* for project *B* is

$$\frac{\Sigma_{t=1\to n}O_t}{(1+k)^t} = \frac{\Sigma_{t=1\to n}R_t(1+k)^{n-t}}{(1+MIRR_B)^n}$$

$$\frac{\$25,000}{(1.10)^0} = \frac{\$3,000(1.10)^4 + \$5,000(1.10)^3 + \$7,000(1.10)^2 + \$9,000(1.10)^1 + \$11,000(1.10)^0}{(1+MIRR_B)^5}$$

$$= \frac{\$3,000(1.4641) + \$5,000(1.331) + \$7,000(1.21) + \$9,000(1.10) + \$11,000}{(1+MIRR_B)^5}$$

$$= \frac{\$4,392.30 + \$6,655.00 + \$8,470.00 + \$9,900 + \$11,000}{(1+MIRR_B)^5}$$

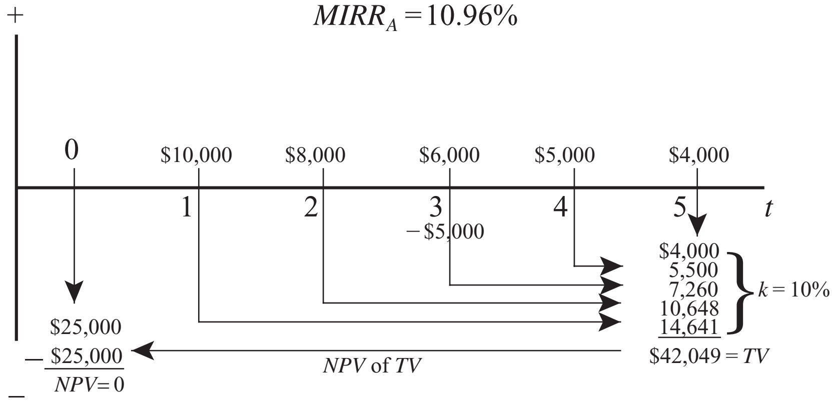

FIGURE 12.20 Modified internal rate of return for project *A*.

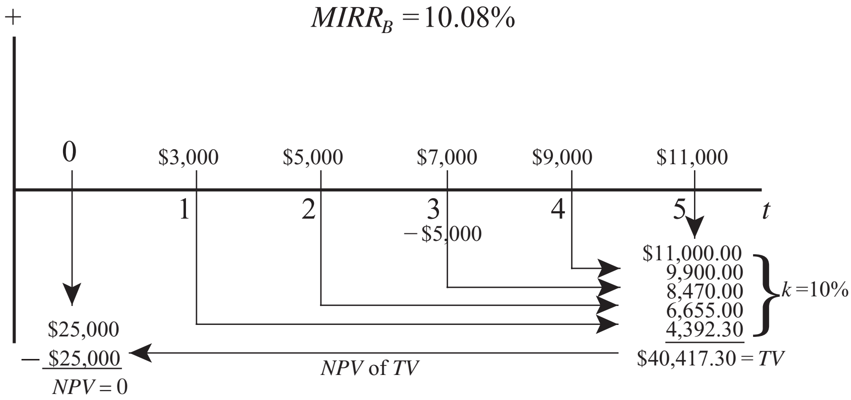

FIGURE 12.21 Modified internal rate of return for project *B*.

$$\$25,000 = \frac{\$40,417.30}{(1 + MIRR_B)^5}$$

$$(1 + MIRR_B)^5 = \frac{\$40,417.30}{\$25,000} = 1.616692$$

$$1 + MIRR_B = 1.1008$$

$$MIRR_A = 0.1008, \text{ or } 10.08\%$$

The calculation of *MIRR* for project *B* is illustrated in Figure 12.21.

Based on the foregoing calculations, project *A* will be preferred to project *B* because $MIRR_A > MIRR_B$. To reiterate, although the *NPV* method should be preferred to both the *IRR* and *MIRR* methods, the *MIRR* method is superior to the *IRR* method for two reasons. Unlike the *IRR* method, the

MIRR method assumes that cash flows are reinvested at the more defensible cost of capital. Recall that the *IRR* method assumes that cash flows are reinvested at the firm's internal rate of return. Moreover, the *MIRR* method is not plagued by the problem of multiple internal rates of return.

CAPITAL RATIONING

In each of the methods of evaluating capital investment projects discussed thus far it was implicitly assumed that the firm had unfettered access to the funds needed to invest in each and every profitable project. If capital markets are efficient, this assumption is approximately true for large, well-established companies with a good record of performance. For smaller, less well-established companies, however, easy access to finance capital may be limited. In some cases, finance capital may be relatively easy to obtain, but for any of a number of reasons senior management may decide to impose a limit on the company's capital expenditures. Senior management may be reluctant to incur higher levels of debt associated with bank borrowing or with issuing corporate bonds. Alternatively, senior management may be unwilling to issue equity shares (stock) to raise the requisite financing because this will dilute ownership and control. For these and other reasons, senior management may decide to reject potentially profitable projects.

The situation of management-imposed cops on capital expenditures may be generally described as a problem of *capital scarcity*. When finance capital is scarce, the firm's investment alternatives are said to be constrained, in which case whatever finance capital is available should be used as efficiently as possible. The process of allocating scarce finance capital as efficiently as possible is called *capital rationing*.

Definition: Capital rationing refers to the efficient allocation of scarce finance capital.

Although details of the procedures involved in efficiently allocating scarce capital are beyond the scope of the present discussion, a simple example will convey the spirit of the capital rationing process. Assume that senior management has $1,000 to invest in six independent projects, each with a life expectancy of 5 years. Assume also that the firm's cost of capital is 5% per year. Table 12.16 summarizes the net present values of six feasible capital investment projects.

It is readily apparent from Table 12.16 that $1,250 in finance capital will be required for the firm to undertake all six projects for a maximum net present value of $945. The problem, of course, is that the firm only has $1,000 to invest. Given this constraint, which projects should the firm undertake to maximize the net present value of $1,000?

The question confronting senior management is this: Which projects should be selected? Table 12.17 ranks from highest to lowest the alterna-

TABLE 12.16 Net Present Values of Alternative Capital Investment Projects

Project	Initial outlay	Net present value
1	$400	$250
2	300	150
3	200	140
4	150	140
5	100	135
6	100	130

TABLE 12.17 Investment Alternatives

Option	Projects	Total outlay	Total net present value	Future value of residual earnings	Total net present value
A	2, 3, 4, 5, 6	$850	$695	$191.44	$886.44
B	1, 3, 4, 5, 6	950	795	63.81	858.81
C	1, 2, 5, 6	900	665	127.63	792.63
D	1, 2, 3, 5	1,000	675	0.00	675.00
E	1, 2, 3, 6	1,000	670	0.00	670.00
F	1, 2, 3	950	540	63.81	603.81

tives available to the firm based on total net present value. Table 12.17 assumes that any residual funds not allocated to a project are invested for 5 years at the firm's cost of capital.

For senior management to generate the highest total net present value, the information presented in Table 12.17 points to investments in projects 2, 3, 4, 5, and 6 for a total net present value of $886.44.

THE COST OF CAPITAL

In each of the methods for evaluating capital investment projects discussed thus far the firm's cost of capital was assumed, almost as an afterthought. The firm's cost of capital, however, is a crucial element in the capital budgeting process. Calculation of the firm's cost of capital is a complicated issue, and a detailed discussion of its derivation is beyond the scope of this chapter. Nevertheless, a brief digression into this important concept is fundamental to an understanding of capital budgeting.

To begin with, it must be recognized that the firm has available several financing options. It must decide whether to satisfy its capital financing requirements by assuming long-term debt, by issuing bonds or by commercial bank borrowing, by selling equity shares, which may dilute ownership and control, by issuing preferred stock, or by some combination of these measures. Moreover, the method of financing may affect the profitability of the firm's operations, the public's perception of the riskiness of the method of financing and its impact on the firm's future ability to raise finance capital, and the impact of the method of financing on the future cost of raising finance capital. When the costs of alternative methods of raising finance capital have been considered, the firm must select the debt/equity mix that results in the lowest, risk-adjusted, cost of capital.

WEIGHTED AVERAGE COST OF CAPITAL (WACC)

The firm's cost of capital is generally taken to be some average of the cost of funds acquired from a variety of sources. Generally, firms can raise finance capital by issuing common stock, by issuing preferred stock, or by borrowing from commercial banks or by selling bonds directly to the public.

Definition: Common stock represents a share of equity ownership in a company. Companies that are owned by a large number of investors who are not actively involved in management are referred to as publicly owned or publicly held corporations. Common stockholders earn dividends that are in proportion to the number of shares owned.

Definition: Dividends are payments to corporate stockholders representing a share of the firm's earnings.

Definition: A bond is a long-term debt instrument in which a borrower agrees to make principal and interest payments at specified time intervals to the holder of the bond.

Definition: Preferred stock is a hybrid financial instrument. Preferred stock is similar to a corporate bond in that it has a par value and fixed dividends per share must be paid to the preferred stockholder before common stockholders receive their dividends. On the other hand, a board of directors that opts to forgo paying preferred dividends will not automatically plunge the firm into bankruptcy.

When a firm raises the entire amount of investment capital by issuing *common stock*, the cost of capital is taken to be the firm's required return on equity. In practice, however, firms raise a substantial portion of their finance capital in the form of long-term debt, or by issuing preferred stock. A discussion of the advantages and disadvantages associated with any of these financing methods is clearly beyond the scope of the present discussion.

It may be argued that for any firm there is an optimal mix of debt and preferred and common stock. This optimal mix is sometimes referred to as the firm's *optimal capital structure*. A firm's optimal capital structure is the mix of financing alternatives that maximizes the firm's stock price.

Definition: The optimal capital structure of a firm is the combination of debt and preferred and common stock that maximizes the firm's share values.

The proportion of debt and preferred and common stock, which define the firm's optimal capital structure, may be used to calculate the firm's *weighted average cost of capital* (*WACC*). The weighed average cost of capital may be calculated by using Equation (12.31)

$$WACC = \omega_d k_d (1-t) + \omega_p k_p + \omega_c k_c \tag{12.31}$$

where ω_d, ω_p, and ω_c are the weights used for the cost of debt, preferred stock, and common stock, respectively.

Definition: The weighted cost of capital is the weighed average of the component sources of capital financing, including common stock, long-term debt, and preferred stock.

The term $\omega_d k_d(1 - t)$ represents the firm's after-tax cost of debt, where t is the firm's marginal tax rate. The after-tax cost of debt recognizes that the financing cost (interest) of debt is tax deductible.

The cost of preferred stock, k_p, is generally taken to be the preferred stock dividend, d_p, divided by the preferred stock price p_p, that is,

$$k_p = \frac{d_p}{p_p} \tag{12.32}$$

In the case of long-term debt and preferred stock, the cost of capital is the rate of return that is required by holders of these securities. As noted earlier, the cost of common stock, k_c, is taken to be the rate of return that stockholders require on the company's common stock. In general, there are two sources of equity capital: retained earnings and capital financing obtained by issuing new shares of common stock.

Corporate profits may be disposed in of in one of two ways. Some or all of the profits may be returned to the owners of the corporation, the stockholders, as distributed corporate profits. Distributed corporate profits are commonly referred to as *dividends*. Corporate profits not returned to the stockholder are referred to as undistributed corporate profits. Undistributed corporate profits are commonly referred to as *retained earnings*.

An important source of finance capital is retained earnings. It is tempting to think of retained earnings as being "free," but this would be a mistake. Retained earnings that are used to finance capital investment projects have opportunity costs. Remember, in the final analysis retained earnings belong to the stockholders but have been held back by senior management to reinvest in the company. Had the stockholders received these undistributed corporate profits, they would have been in a position to reinvest the funds in alternative financial instruments. What then is the cost of funds of retained earnings? This cost should be the rate of return the stockholder could earn on an investment of equivalent risk. In general, a firm that cannot earn at least this equivalent to the rate of return should pay out retained earnings to the stockholders.

CHAPTER REVIEW

Capital budgeting is the application of the principle of profit maximization to multiperiod projects. Capital budgeting involves investment decisions in which expenditures and receipts continue over a significant period of time. In general, capital budgeting projects may be classified into one of several major categories, including *capital expansion*, *replacement*, *new product lines*, *mandated investments*, and *miscellaneous investments*.

Capital budgeting involves the subtraction of cash outflows from cash inflows with adjustments for differences in their values over time. Differences in the values of the flows are based on the *time value of money*, which says that a dollar today is worth more than a dollar tomorrow.

There are five standard methods used to evaluate the value of alternative investment projects: *payback period*, *discounted payback period*, *net present value* (*NPV*), *internal rate of return* (*IRR*), and *modified internal rate of return* (*MIRR*). The payback period is the number of periods required to recover an original investment. In general, risk-averse managers prefer investments with shorter payback periods.

The net present value of a project is calculated by subtracting the discounted present value of all outflows from the discounted present value of all inflows. The discount rate is the interest rate used to evaluate the project and is sometimes referred to as the *cost of capital*, *hurdle rate*, *cutoff rate*, or *required rate of return*. If the net present value of an investment is positive (negative), the project is accepted (rejected). If the net present value of an investment is zero, the manager is indifferent to the project.

The internal rate of return is the interest rate that equates the present values of inflows to the present values of outflows; that is, the rate that causes the net present value of the project to equal zero. If the internal rate of return is greater than the cost of capital, the project is accepted.

There are a number of problems associated with using the *IRR* method for evaluating capital investment projects. One problem is the possibility of multiple internal rates of return. Multiple internal rates of return occur when a project that has two or more internal rates of return.

For independent projects both the *NPV* and the *IRR* methods will yield the same accept/reject decision rules. For mutually exclusive capital investment projects, the *NPV* and the *IRR* methods could result in conflicting accept/reject decision rules. This is because the *NPV* method implicitly assumes that net cash inflows are reinvested at the cost of capital, whereas the *IRR* method assumes that net cash inflows are reinvested at the internal rate of return.

The *modified internal rate of return* (*MIRR*) method for evaluating capital investment projects is similar to the *IRR* method in that it generates accept/reject decision rules based on interest rate comparisons. But unlike the *IRR* method, the *MIRR* method assumes that cash flows are reinvested at the cost of capital and avoids some of the problems associated with multiple internal rates of return.

Categories of cost of capital include the *cost of debt*, *the cost of equity*, and the *weighted cost of capital*. The cost of debt is the interest rate that must be paid on after-tax debt.

The weighed cost of capital is a measure of the overall cost of capital. It is obtained by weighting the various costs by the relative proportion of each component's value in the total capital structure.

KEY TERMS AND CONCEPTS

Abnormal cash flow Large cash outflows that occur during or toward the end of the life of a project.

Annuity A series of equal payments, which are made at fixed intervals for a specified number of periods.

Annuity due An annuity in which the fixed payments are made at the beginning of each period.

Capital budgeting The process whereby senior management analyzes the comparative net revenues from alternative investment projects. In capital budgeting future cash inflows and outflows of different capital investment projects are expressed as a single value at a common point in time, usually at the moment the project is undertaken, so that they may be compared.

Capital rationing The efficient allocation of scarce finance capital.

Cash flow diagram Illustrates the cash inflows and cash outflows expected to arise from a given investment.

Common stock A share of equity ownership in a company. Companies that are owned by a large number of investors who are not actively involved in management are referred to as publicly owned or publicly held corporations. Common stockholders earn dividends that are in proportion to the number of shares owned.

Compounding With an adjective (e.g., annual) indicates how frequently the rate of return on an investment is calculated.

Cost of capital The cost of acquiring funds to finance a capital investment project. It is the minimum rate of return that must be earned to justify a capital investment. The cost of capital is often referred to as the required rate of return, the cutoff rate, or the hurdle rate.

Cost of debt The term $\omega_d k_d(1 - t)$ represents the firm's after-tax cost of debt, with t standing for the firm's marginal tax rate. The after-tax cost of debt recognizes that the financing cost (interest) of debt is tax deductible.

Cost of equity The required rate of return on common stock.

Coupon bond A debt obligations in which the issuer of the bond promises to pay the bearer of the bond fixed dollar interest payments at regular intervals for a specified period of time.

Crossover rate The cost of capital at which the net present values of two projects are equal. Diagrammatically, this is the cost of capital at which the net present value profiles of two projects intersect.

Cutoff rate Another name for the hurdle rate.

Discount rate The rate of interest that is used to discount a cash flow.

Discounted cash flow The present value of an investment, or series of investments.

Discounted payback period Similar to the payback period except that the cost of capital is used in discounting cash flows.

Dividends Payments to corporate stockholders representing a share of the firm's earnings. Commonly referred to as distributed corporate profits.

Future value (*FV*) The final accumulated value of a sum of money at some future time period.

Future value of an annuity due (*FVAD*) The future value of an annuity in which the fixed payments are made at the beginning of each period.

Future value of an ordinary annuity (*FVOA*) The future value of an annuity in which the fixed payments are made at the end of each period.

Hurdle rate The cost of capital that must be covered by the internal rate of return if a project is to be undertaken. The hurdle rate is often referred to as the required rate of return or the cutoff rate.

Independent projects Projects are independent if their cash flows are unrelated.

Internal rate of return (*IRR*) The discount rate that equates the present value of a project's cash inflows to the present value of its cash outflows.

Modified internal rate of return (*MIRR*) The discount rate that equates the present value of a project's cash outflows with the present value of its terminal value.

Multiple internal rates of return Two or more internal rates of return for the same project.

Mutually exclusive projects Projects are mutually exclusive if acceptance of one project means rejection of all other projects.

Net present value (*NPV*) The present value of future net cash flows discounted at the cost of capital.

Normal cash flow One or more cash outflows of a project followed by a series of cash inflows.

Ordinary (deferred) annuity An annuity in which the fixed payments occur at the end of each period.

Operating cash flow The cash flow generated from a company's operations.

Par value of a bond The face value of the bond. It is the amount originally borrowed by the issuer.

Payback period The number of years required to recover the original investment.

Preferred stock Similar to a corporate bond in that it has a par value and that a fixed amount of dividends per share must be paid to the preferred stockholder before dividends can be distributed to common stockholders. A board of directors that opts to forgo paying preferred dividends will not automatically plunge the firm into bankruptcy.

Present value (*PV*) The value of a sum of money at some initial time period.

Present value of an annuity The present value of a series of fixed payments made at fixed intervals for a specified period of time.

Required rate of return Another name for the hurdle rate or the cutoff rate.

Retained earnings The portion of corporate profits not returned to the stockholders. Commonly referred to as undistributed corporate profits.

Salvage value The estimated market value of a capital asset at the end of its life.

Terminal value (*TV*) The future value of a project's cash inflows compounded at the firm's cost of capital.

Time value of money Reflects the understanding that a dollar received today is worth more than a dollar received tomorrow.

Weighted average cost of capital The weighed average of the component sources of capital financing, including common stock, long-term debt, and preferred stock.

Yield to maturity (*YTM*) The rate of return that is earned on a bond when held to maturity.

CHAPTER QUESTIONS

12.1 Define capital budgeting. What are the four main categories of capital budgeting projects? Briefly explain each.

12.2 Explain why assessing the time value of money is important in capital budgeting.

12.3 A dollar received today will never be worth the same as a dollar received tomorrow. Do you agree? If not, then why not?

12.4 Explain the difference between an ordinary annuity and an annuity due.

12.5 Other things being equal, the future value of an ordinary annuity is greater than the future value of an annuity due. Do you agree with this statement? Explain.

12.6 The more frequent the compounding, the greater the present value of a lump-sum investment. Do you agree? If not, then why not?

12.7 Other things being equal, the present value of an ordinary annuity is greater than the present value of an annuity due. Do you agree with this statement? Explain.

12.8 The smallest interest component of an amortization schedule is paid in at the end of the first year; thereafter, as the amount of the principal outstanding declines, the paid interest component increases. Do you agree or disagree? Explain.

12.9 What is the difference between the payback period and discounted payback period methods of evaluating a capital investment project? Assuming that the projects are mutually exclusive, do the two methods result in the same project rankings? What is the main deficiency of these methods? What is the in primary usefulness?

12.10 If two independent projects have positive net present values, the project with the highest net present value should be adopted. Do you agree? If not, then why not?

12.11 Suppose that two mutually exclusive projects have only cash outflows. The project with the highest net present value should be adopted. Do you agree with this statement? Explain.

12.12 The internal rate of return is the minimum rate of interest an investor will pay to finance a capital investment project. Do you agree? If not, then why not?

12.13 The net present value and internal rate of return methods will always result in the same accept and reject decisions for mutually exclusive projects. Do you agree with this statement?

12.14 What is the relationship between changes in the hurdle rate and changes in the net present value of a project?

12.15 The net present value of a project in which the cash flows are received in the near future will decline at a faster rate than the net present value for projects in which the cash flows are generated in the distant future. Do you agree with this statement?

12.16 Why may the net present value profiles of two projects intersectz. Give two reasons.

12.17 For mutually exclusive projects, when the net present value profiles of two projects intersect, should the net present value method or the internal rate of return method be used for selecting one project over the other?

12.18 What are the maximum possible internal rates of return for a single project?

12.19 Under what circumstances is a project likely to exhibit multiple internal rates of return possible?

12.20 What is the difference between the internal rate of return method and the modified internal rate of return method for evaluating capital investment projects? What problem does the second method overcome?

12.21 The modified internal rate of return method is preferable to the net present value method for evaluating capital investment projects because it assumes that cash flows are reinvested at the cost of capital. Do you agree with this statement?

CHAPTER EXERCISES

12.1 What is the present value of a cash inflow of $100,000 in 5 years if the annual interest rate is 8%? What would the present value be if there was an additional cash inflow of $200,000 in 10 years?

12.2 An drew borrows $20,000 for 3 years at an annual rate of 7% compounded monthly to purchase a new car. The first payment is due at the end of the first month.

a. What is the amount of Andrew's automobile payments?

b. What is the total amount of interest paid?

12.3 Suppose that Adam deposits $200,000 in a time deposit that pays 15% interest per year compounded annually. How much will Adam receive when the deposit is redeemed after 7 years? How would your answer have been different for interest compounded quarterly?

12.4 Suppose that Adam borrows $20,000 from the National Central Bank and agrees to repay the loan in 4 years at an interest rate of 8% per year, compounded continuously. How much will Adam have repaid to the bank at the end of 4 years?

12.5 Calculate the future value of a 5-year annuity due with payments of $5,000 a year at 4% compounded semiannually.

12.6 How much should an individual invest today for that investment to be worth $750 in 8 years if the interest rate is 22% per year, compounded annually?

12.7 If the prevailing interest rate on a time deposit is 9% per year compounded annually, how much would Eleanor Rigby have to deposit today to receive $400,000 at the end of 6 years?

12.8 Consider the cash flow diagram in Figure E12.8.

Calculate the terminal value of the cash flow stream at $t = 3$ if interest is compounded quarterly.

12.9 Calculate the present value of $20,000 in 10 years if the interest rate is 7% compounded

a. Annually

b. Quarterly

c. Monthly

d. Continuously

12.10 If the prevailing interest rate on a time deposit is 9% annually, how much would Sam Orez have to deposit today to receive $400,000 at the end of 6 years if the interest rate were compounded quarterly, monthly, and continuously?

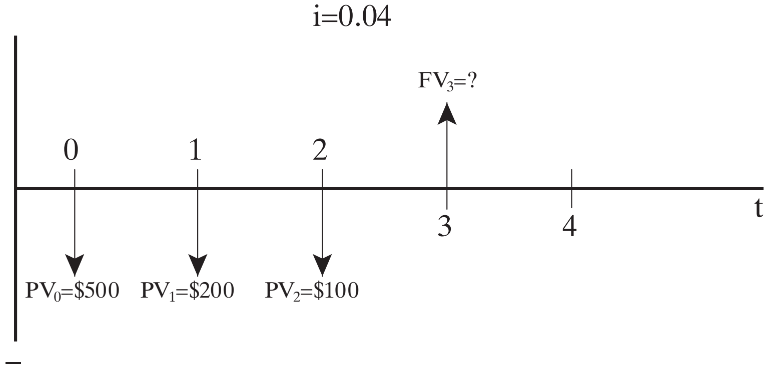

FIGURE E12.8

TABLE E12.12 Net Cash Flows (CF_t) for Projects *A* and *B*

Year, *t*	Project *A*	Project *B*
0	–$20,000	–$20,000
1	10,000	8,000
2	8,000	8,000
3	5,000	8,000
4	3,000	8,000

12.11 Calculate the present value of a 10-year ordinary annuity paying $10,000 a year at 5, 10, and 15%.

12.12 Senior management of Valhaus Entertainment is considering two proposed capital investment projects, *A* and *B*. Each project requires an initial cash outlay of $20,000. The projects' cash flows, which have been adjusted to reflect inflation, taxes, depreciation, and salvage values, are summarized in Table E12.12. Use the payback period method to determine, which project should be selected.

12.13 Suppose that the chief financial officer (CFO) of Orange Company is considering two mutually exclusive investment projects. The projected net cash flows for projects *X* and *Y* are summarized in Table E12.13.

If the discount rate (cost of capital) is expected to be 15%, which project should be undertaken?

12.14 Senior management of Teal Corporation is considering the projected net cash flows for two mutually exclusive projects, which are provided in Table E12.14.

Determine which project should be adopted if the cost of capital is 6%.

12.15 Suppose that an investment project requires an immediate cash

TABLE E12.13 Net Cash Flows for Projects *X* and *Y*

Year, *t*	Project *X*	Project *Y*
0	–$30,000	–$25,000
1	10,000	6,000
2	12,000	10,000
3	14,000	12,000
4	15,000	12,000
5	8,000	10,000

TABLE E12.14 Net Cash Flows for Projects Red and Blue

Year, *t*	Project Red	Project Blue
0	–$5,000	–$10,000
1	3,000	1,000
2	5,500	3,000
3		5,000
4		7,000

outlay of $25,000 and provides for an annual cash inflow of $10,000 for the next 5 years.

a. Estimate the internal rate of return.

b. Should the project be undertaken if the cost of capital (hurdle rate) is 30%?

12.16 Illustrate the net present value profile for alternative interest rates for the cash flow information Projects *A* and *B* in Exercise 12.12. Be sure to include in your answer the internal rate of return for each project.

12.17 Red Lion pays a corporate income tax rate of 38%. Red Lion is planning to build a new factory in the country of Paragon to manufacture primary and secondary school supplies. The new factory will require an immediate cash outlay of $4 million but is expected to generate annual profits of $1 million. According to the Paragon Uniform Tax Code, Red Lion may deduct $250,000 annually as a depreciation expense. The life of the new factory is expected to be 10 years. Assuming that the annual interest rate is 20%, should Red Lion build the new factory? Explain.

12.18 Senior management of Vandaley Enterprises is considering two mutually exclusive investment projects. The projected net cash flows for projects *A* and *B* are summarized in Table E12.18.

If the discount rate (cost of capital) is expected to be 15%, which project should be undertaken?

TABLE E12.18 Net Cash Flows (CF_t) for Projects *A* and *B*

Year, *t*	Project *A*	Project *B*
0	–$27,000	–$21,000
1	8,000	6,500
2	9,000	6,500
3	10,000	6,500
4	10,000	6,500
5	6,000	6,500

TABLE E12.20 Net Cash Flows (CF_t) for Yellow Project

Year, *t*	CF_t
0	–$1,500
1	500,000
2	–400,000

12.19 Suppose that an investment opportunity, which requires an initial outlay of $100,000, is expected to yield a return of $250,000 after 30 years.

a. Will the investment be profitable if the cost of capital is 7%?

b. Will the investment be profitable if the cost of capital is 2%?

c. At what cost of capital will the investor be indifferent to the investment?

12.20 Consider the net cash flows for Yellow Project given in Table E12.20.

a. What is the net present value profile for Yellow Project at selected costs of capital?

b. Does Yellow Project have multiple internal rate of return? What are they?

c. Diagram your answer.

12.21 Calculate the weighted average cost of capital of a project that is 30% debt and 70% equity. Assume that the firm pays 10% on debt and 25% on equity. Assume that the firm's marginal tax rate is 33%.

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